1. (20 points)

(a) Melissa is applying to 20 different out-of-state medical schools. Because of her excellent GPA/MCAT/essays, her chance of being accepted to each school is $\frac{1}{20}$, and the decisions at the 20 schools are independent of each other. Using a Poisson approximation, estimate the probability that Melissa will be accepted to at least two of these schools. **ANSWER:**

Number $X$ of acceptances is roughly Poisson with parameter $\lambda = 20 \cdot \frac{1}{20} = 1$. Thus $P(X \geq 2) = 1 - P(X = 1) - P(X = 0) \approx 1 - e^{-\lambda} \lambda^1/1! - e^{-\lambda} \lambda^0/0! = 1 - \frac{2}{e} \approx .26424$.

**Remark:** If we compute the exact value using a binomial distribution, we get $P(X \geq 2) \approx .26416$, so the approximation is quite good.

(b) Jill is applying to 25 different out-of-state medical schools and has a $\frac{1}{5}$ chance (independently) of being invited for an interview at each school. Let $X$ be the number of medical schools at which she is invited to interview. Compute $E[X]$ and $\text{Var}[X]$.

**ANSWER:** The number of interviews is binomial with parameter $n = 25$ and $p = 1/5$. So $E[X] = np = 5$ and $\text{Var}[X] = np(1 - p) = 4$.

(c) Using a normal approximation, roughly approximate the probability that Jill is invited to interview at fewer than 2.5 schools. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** Since the standard deviation of $X$ is 2, the value 2.5 is $5/4$ standard deviations below the mean. Hence the probability is approximately $\Phi(-5/4) \approx .10565$. **Remark:** The true probability is .098 which is pretty close.

2. (20 points) A room has four lightbulbs, each of which will burn out at a random time. Let $X_1, X_2, X_3, X_4$ be the burnout times, and assume they are independent exponential random variables with parameter $\lambda = 1$. Write

1. $X = X_1 + X_2 + X_3 + X_4$.
2. $Y = \min\{X_1, X_2, X_3, X_4\}$, i.e., $Y$ is time when first bulb burns out.
3. $Z = \max\{X_1, X_2, X_3, X_4\}$, i.e., $Z$ is time when last bulb burns out.

Compute the following:

(a) The probability density function $f_X$. **ANSWER:** This is a Gamma distribution with parameters $\lambda = 1$ and $n = 4$. So $f_X(x) = x^3 e^{-x}/3!$ for $x \in [0, \infty)$.

(b) The probability density function $f_Y$. **ANSWER:** The minimum of four exponentials of parameter 1 is exponential with parameter 4. Hence $f_Y(x) = 4e^{-4x}$ for $x \in [0, \infty)$. 
3. (20 points) Five applicants are applying for a job, and an interviewer gives each applicant a}

(a) The cumulative distribution function \( F_Y(r) \) for \( r \in [0, 1] \). \textbf{ANSWER:} This is basically the radioactive decay problem from

(b) The density function \( f_Y \). \textbf{ANSWER:} \( f_Y(r) = 5r^4 \) for \( r \in [0, 1] \) (and zero if \( r \not\in [0, 1] \)).

(c) The density function \( f_Z \) and the value \( E[Z] \). \textbf{NOTE:} If you remember what this means,

(d) The probability \( P(X_2 > 2X_1) \) (i.e., probability second candidate’s score is more than than double first candidate’s score). \textbf{ANSWER:} Note that joint density \( f_{X_1,X_2}(x,y) \) is 1 on the unit square \([0,1]^2\) and zero elsewhere. Therefore the probability is the area of the subset of \([0,1]^2\) where \( y > 2x \), which comes to 1/4. So the answer is 1/4.

4. (15 points) Let \( X \) and \( Y \) be independent random variables with density function given by

(a) Compute \( P(X < 1) \). \textbf{ANSWER:} \( X \) is a Cauchy random variable, so the answer is 3/4 by

(b) Compute the probability density function for the random variable \( Z = (X - Y)/2 \).

\textbf{ANSWER:} If \( Y \) is Cauchy then \(-Y\) is also Cauchy. The average of two independent

Cauchy random variables it itself Cauchy, so the answer is \( \frac{1}{\pi(1+x^2)} \).
(c) Compute $E[e^{-X^2-Y^2}]$. You can leave your answer as a double integral—no need to evaluate it explicitly. **Answer:** $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \frac{1}{\pi(1+y^2)} e^{-x^2-y^2} \, dx \, dy$

5. (10 points) Let $X_1, X_2, X_3, \ldots, X_{10}$ be the outcomes of independent standard die rolls—so each takes one of the values in $\{1, 2, 3, 4, 5, 6\}$, each with equal probability. Write $S = X_1 + X_2 + \ldots + X_{10}$. Compute the following:

(a) The moment generating function $M_{X_1}(t)$. **Answer:** $M_{X_1}(t) = \frac{1}{6} (e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$.

(b) The moment generating function $M_{S}(t)$. **Answer:** The moment generating function of a sum of independent random variables is the product of the moment generating functions of the individual random variables. Hence $M_{S}(t) = \left(\frac{1}{6} (e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})\right)^{10}$.

6. (15 points) Let $X$ and $Y$ be be random variables with joint density function $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$. Write $Z = X + Y$.

(a) Compute $E[XY]$. **Answer:** $X$ and $Y$ are independent normal random variables, each with mean zero and variance one. Since they are independent we have $E[XY] = E[X]E[Y] = 0$. Alternatively, write $E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \frac{1}{2\pi} e^{-(x^2+y^2)/2} \, dx \, dy$. Then there are various ways to argue by symmetry that this must be zero.

(b) Compute the conditional expectation $E[Y|Z]$. That is, express the random variable $E[Y|Z]$ in terms of $Z$. **Answer:** We have $Z = E[Z|Z] = E[X|Z] + E[Y|Z]$. Since $E[X|Z]$ and $E[Y|Z]$ are the same by symmetry, the answer must be $Z/2$.

(c) Compute the probability $P(X^2 + Y^2 \leq 4)$. **Answer:** This can be computed using polar coordinates. The integral becomes $\int_0^2 \int_0^{2\pi} \frac{1}{2\pi} e^{-r^2/2} r \, \theta \, dr \, d\theta = \int_0^2 e^{-r^2/2} r \, dr = -e^{-r^2/2}\bigg|_0^2 = -e^{-2} - (-1) = 1 - e^{-2} \approx .86466$