Call function

Black-Scholes
Outline

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Call function: pretty cool whether you love finance or not

- **Recall**: if $X$ is non-negative random variable with cumulative distribution function $F$, then $\int_0^\infty (1 - F(x)) \, dx = E[X]$.
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Note that $C(0) = E[X]$ and $\lim_{K \to \infty} C(K) = 0$. $C$ is convex with slope increasing from $-1$ to $0$.
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- By translation argument, it is also $E[\max(X - K, 0)]$.
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- Let’s give $C$ a name: we’ll call it the **call function** of $X$.
  1. $C(K)$ is an expectation: $E[\max(X - K, 0)]$.
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- So now any random variable $X$ comes with a pdf $f = f_X$, a cdf $F = F_X$ (an anti-derivative of $f_X$) and this call function $C = C_X$ (an anti-anti-derivative of $f$).
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- Wonder if $C$ is good for anything...
Goals for today

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- **Weird fact:** If $X$ is a real world random quantity (such as the price of gold or euros or stock shares at a future date) and we use risk neutral probability, then sometimes the call function $C$ (or a related “put function”) is what we can look up online. One then uses the quoted $C$ values to work out $F_X$ and $f_X$. 

- **Grand story goal:** Say something about the link between probability and the real world. What is the probability that price of Microsoft stock will rise by more than ten dollars over the next month? What is the probability that price of oil will drop more than ten percent next year? How can I (using internet and math) come up with a reasonable answer?
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Asset price as discounted expectation: $X_0 = E_{RN}(X_T)e^{-rT}$

- If $r$ is risk free interest rate, then by definition, price of a contract paying dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$. 

- Generally, in absence of arbitrage, price of contract that pays $X$ at time $T$ should be $E_{RN}(X_T)e^{-rT}$ where $E_{RN}$ denotes expectation with respect to the risk neutral probability.

- Example: if a non-dividend-paying stock will be worth $X$ at time $T$, then its price today should be $E_{RN}(X_T)e^{-rT}$.

- Risk neutral probability basically defined so price of asset today is $e^{-rT}$ times risk neutral expectation of time $T$ price.

- In particular, the risk neutral expectation of tomorrow's (interest discounted) stock price is today's stock price.

- Implies fundamental theorem of asset pricing, which says discounted price $X(n)A(n)$ (where $A$ is a risk-free asset) is a martingale with respected to risk neutral probability.
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A European call option on a stock at maturity date $T$, strike price $K$, gives the holder the right (but not obligation) to purchase a share of stock for $K$ dollars at time $T$. The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. SS
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- If \( X \) is time \( T \) stock price, then value of option at time \( T \) is \( g(X) = \max\{0, X - K\} \). If we use the risk neutral probability measure, then the price now should be

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e^{-rT}E[g(X)] = e^{-rT}C(K),
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Can look up \( C(K) \) values for stock (say GOOG) at cboe.com, apply smoothing, take derivatives, approximate \( F_X \) and \( f_X \).
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- Analysis is basically the same as for call options except that one replaces the “call function” $C(K) = E[\max(X - K, 0)]$ with the “put function” defined by

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$max(a, 0) - max(-a, 0) = a$. So $C(K) - P(K) = E[X - K]$.

$$P(K) = C(K) - E[X] + K = \int_0^K F(x)dx.$$
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- The put function is an anti-anti-derivative of $f$ (like the call function) but it has a slope that increases from 0 to 1 (instead of from $-1$ to 0) and it satisfies $P(0) = 0$. 

Many trading platforms sell call and put options side by side. For simplicity we focus on call functions in this lecture.
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Assumption:
the log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T\sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.

Observation:
$N$ normal ($\mu, T\sigma^2$) implies $E[e^N] = e^{\mu + T\sigma^2/2}$.

Observation:
If $X_0$ is the current price then $X_0 = E[R_N[X]]e^{-rT} = E[R_N[e^N]]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$.

Observation:
This implies $\mu = \log X_0 + (r - \sigma^2/2)T$.

General Black-Scholes conclusion:
If $g$ is any function then the price of a contract that pays $g(X)$ at time $T$ is $E[g(e^N)]e^{-rT}$ where $N$ is normal with mean $\mu$ and variance $T\sigma^2$.

Surprise:
No need to guess $\mu$. It is fixed by $X_0, r, \sigma, T$. 
Black-Scholes: main assumption and conclusion

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\[ \text{General Black-Scholes conclusion: If } g \text{ is any function then the price of a contract that pays } g(X) \text{ at time } T \text{ is } E\left[g(e^{N})\right] e^{-rT} \text{ where } N \text{ is normal with mean } \mu \text{ and variance } T \sigma^2. \]

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- **Assumption:** the log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T \sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.
- **Observation:** $N$ normal $(\mu, T \sigma^2)$ implies $E[e^N] = e^{\mu + T\sigma^2/2}$.
- **Observation:** If $X_0$ is the current price then $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2-r)T}$.
- **Observation:** This implies $\mu = \log X_0 + (r - \sigma^2/2)T$.
- **General Black-Scholes conclusion:** If $g$ is any function then the price of a contract that pays $g(X)$ at time $T$ is

$$E[g(e^N)]e^{-rT}$$

where $N$ is normal with mean $\mu$ and variance $T\sigma^2$. 

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Black-Scholes for European call option

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  Write this as

  $$e^{-rT}E[\max\{0, e^N - K\}] = e^{-rT}E[(e^N - K)1_{N\geq \log K}]$$

  $$= \frac{e^{-rT}}{\sigma \sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) \, dx.$$
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- Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function $\Phi$. 

- Price of European call is $\Phi(d_1) X_0 - \Phi(d_2) Ke^{-rT}$ where $d_1 = \ln(X_0/K) + (r + \sigma^2/2)(T)^{1/2}$ and $d_2 = \ln(X_0/K) + (r - \sigma^2/2)(T)^{1/2}$. 


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Perspective: implied volatility

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Nonetheless, “implied volatility” has become a standard part of the finance lexicon. When traders want to get a rough sense of how a financial derivative is priced, they often ask for the implied volatility (a number automatically computed in many financial software packages).
Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption**: risk neutral probability densities are lognormal.

- Heuristic support for this assumption:
  - If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large $T$.

- Replicating portfolio point of view:
  - In simple models (e.g., where wealth always goes up or down by fixed factor each day) can transfer money between the stock and the risk free asset to ensure our wealth at time $T$ equals option payout.

- Where arguments for assumption break down:
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