Outline

Covariance and correlation

Paradoxes: getting ready to think about conditional expectation
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A property of independence

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\[ E[g(X)h(Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f(x,y)\,dx\,dy. \]
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- Since $f(x, y) = f_X(x)f_Y(y)$ this factors as
  \[ \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy \int_{-\infty}^{\infty} g(x)f_X(x)\,dx = E[h(Y)]E[g(X)]. \]
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Covariance (like variance) can also written a different way. Write \( \mu_X = E[X] \) and \( \mu_Y = E[Y] \). If laws of \( X \) and \( Y \) are known, then \( \mu_X \) and \( \mu_Y \) are just constants.
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- Covariance formula \( E[XY] - E[X]E[Y] \), or “expectation of product minus product of expectations” is frequently useful.

- Note: if $X$ and $Y$ are independent then \( \text{Cov}(X, Y) = 0 \).
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- **General statement of bilinearity of covariance:**

  $$\text{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \text{Cov}(X_i, Y_j).$$
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- **Special case:**

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\text{Var}\left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{(i,j):i<j} \text{Cov}(X_i, X_j).
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- If $a$ and $b$ are constants and $a > 0$ then $\rho(aX + b, X) = 1$.
- If $a$ and $b$ are constants and $a < 0$ then $\rho(aX + b, X) = -1$. 


Important point

- Say $X$ and $Y$ are uncorrelated when $\rho(X, Y) = 0$. 

- Are independent random variables $X$ and $Y$ always uncorrelated? Yes, assuming variances are finite (so that correlation is defined).

- Are uncorrelated random variables always independent? No. Uncorrelated just means $E[(X - E[X])(Y - E[Y])] = 0$, i.e., the outcomes where $(X - E[X])(Y - E[Y])$ is positive (the upper right and lower left quadrants, if axes are drawn centered at $(E[X], E[Y])$) balance out the outcomes where this quantity is negative (upper left and lower right quadrants). This is a much weaker statement than independence.
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Examples

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The text contains examples of computing covariance and correlation between sums of i.i.d. random variables, illustrating how to calculate these statistical measures when the variables are independent and identically distributed.
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- Recall formula
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Reduces problem to computing $\text{Cov}(X_i, X_j)$ (for $i \neq j$) and $\text{Var}(X_i)$. 
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Famous paradox

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Standard punch line: this is actually what banker deserved.

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Variant without probability: Stay in hell for \( n \) (of your choice) days, and thereafter on days that are multiples of \( 2^n \).

In both stories, make infinitely many good trades and end up with less than I started with. "Paradox" is existence of 2-to-1 map from (smaller set) \{2, 3, ...\} to (bigger set) \{1, 2, ...\}.
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Lets me get arbitrarily rich. But if I go on forever, I return every sack given to me. If \(n\)th sack confers right to spend \(n\)th day in heaven, leads to hell-forever paradox.
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Money pile paradox

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- Banker seemed to make you richer (every pile got bigger) but really just reshuffled your infinite wealth.
Two envelope paradox

- $X$ is geometric with parameter $1/2$. One envelope has $10^X$ dollars, one has $10^{X-1}$ dollars. Envelopes shuffled.
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Kind of a disguised version of money pile paradox. But more subtle. One has to replace "$j$th pile of money" with "restriction of expectation sum to scenario that first chosen envelope has $10^j$". Switching indeed makes each pile bigger.

However, "Higher expectation given amount in first envelope" may not be right notion of "better." If $S$ is payout with switching, $T$ is payout without switching, then $S$ has same law as $T - 1$. In that sense $S$ is worse.
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Two envelope paradox

- X is geometric with parameter 1/2. One envelope has $10^X$ dollars, one has $10^{X-1}$ dollars. Envelopes shuffled.

- You choose an envelope and, after seeing contents, are allowed to choose whether to keep it or switch. (Maybe you have to pay a dollar to switch.)

- Maximizing conditional expectation, it seems it’s always better to switch. But if you always switch, why not just choose second-choice envelope first and avoid switching fee?

- Kind of a disguised version of money pile paradox. But more subtle. One has to replace “$j$th pile of money” with “restriction of expectation sum to scenario that first chosen envelop has $10^j$”. Switching indeed makes each pile bigger.

- However, “Higher expectation given amount in first envelope” may not be right notion of “better.” If $S$ is payout with switching, $T$ is payout without switching, then $S$ has same law as $T - 1$. In that sense $S$ is worse.
Two envelope paradox

VALUE OF ENVELOPE ONE

- ($1 with prob. 1/4) ∼ $.25
- ($10 with prob. 3/8) ∼ $3.75
- ($100 with prob. 3/16) ∼ $18.75
- ($1000 with prob. 3/32) ∼ $93.75
- ($10000 with prob. 3/64) ∼ $468.75

VALUE OF ENVELOPE TWO

- ($1 with prob. 1/4) ∼ $.25
- ($10 with prob. 3/8) ∼ $3.75
- ($100 with prob. 3/16) ∼ $18.75
- ($1000 with prob. 3/32) ∼ $93.75
- ($10000 with prob. 3/64) ∼ $468.75

$12.50
$6.25
$2.50
$1.25
$.25
$312.50
$31.25
$62.50
$156.25
Beware infinite expectations.
▶ Beware infinite expectations.
▶ Beware unbounded utility functions.
Moral

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- They can lead to strange conclusions, sometimes related to “reshuffling infinite (actual or expected) wealth to create more” paradoxes.
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- Beware infinite expectations.
- Beware unbounded utility functions.
- They can lead to strange conclusions, sometimes related to “reshuffling infinite (actual or expected) wealth to create more” paradoxes.
- Paradoxes can arise even when total transaction is finite with probability one (as in envelope problem).