

18.600: Lecture 17

Continuous random variables

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Outline

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Expectation and variance of continuous random variables

Uniform random variable on $[0, 1]$

Uniform random variable on $[\alpha, \beta]$

Measurable sets and a famous paradox

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- ▶ Define **cumulative distribution function**
$$F(a) = F_X(a) := P\{X < a\} = P\{X \leq a\} = \int_{-\infty}^a f(x)dx.$$

Simple example

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▶ We say that X is **uniformly distributed on** $[0, 2]$.

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- ▶ This formula is often useful for calculations.

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$$\text{function } f(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & x \notin [0, 1], \end{cases} \text{ which implies}$$

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- ▶ $\text{Var}E[X^2] - E[X]^2 = 1/3 - 1/4 = 1/12$.

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- ▶ Intuitively, we'd guess the midpoint $\frac{\alpha + \beta}{2}$.

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- ▶ Answer: $\text{Var}[X] = \text{Var}[(\beta - \alpha)Y + \alpha] = \text{Var}[(\beta - \alpha)Y] = (\beta - \alpha)^2 \text{Var}[Y] = (\beta - \alpha)^2 / 12$.

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Continuous random variables

Expectation and variance of continuous random variables

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Measurable sets and a famous paradox

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- ▶ How do we mathematically define the volume of an arbitrary set B ?

Idea behind paradox

- ▶ **Hypothetical:** Consider the interval $[0, 1)$ with the two endpoints glued together (so it looks like a circle). *What if* we could partition $[0, 1)$ into a countably infinite collection of disjoint sets that all looked the same (up to a rotation of the circle) and thus had to have the same probability?

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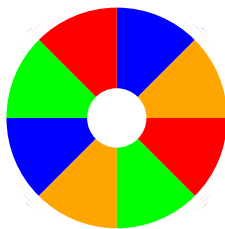
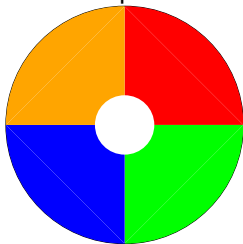
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- ▶ **Related problem:** *if* (in a non-atomic world, where mass was infinitely divisible) you could cut a donut into countably infinitely many pieces all of the same weight, how much would each piece weigh?
- ▶ **Question:** Is it really possible to partition $[0, 1)$ into countably many identical (up to rotation) pieces?

Cutting donut into countably many identical “pieces”

- ▶ Call two points “equivalent” if you can get from one to the other by a 0, 90, 180, or 270 degree rotation.

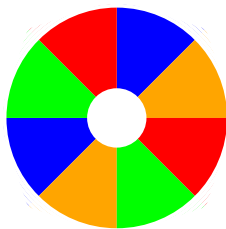
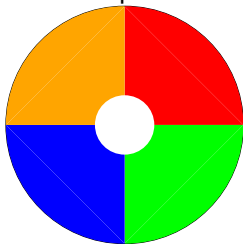
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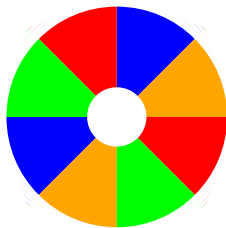
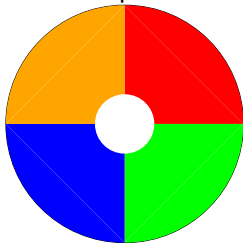
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- ▶ Whole donut is disjoint union of the four sets obtained as 0/90/180/270 degree rotations of red set.
- ▶ What if we replace “0/90/180/270-degree rotations” by “rational-degree-number rotations”? If red set has one point from each equivalence class, whole donut is disjoint union of *countably* many sets obtained as rational rotations of red set.

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- ▶ If $P(A) = 0$, then $P(S) = \sum_r P(\tau_r(A)) = 0$. If $P(A) > 0$ then $P(S) = \sum_r P(\tau_r(A)) = \infty$. Contradicts $P(S) = 1$ axiom.

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- ▶ 1. **Re-examine axioms of mathematics:** the very *existence* of a set A with one element from each equivalence class is consequence of so-called **axiom of choice**. Removing that axiom makes paradox goes away, since one can just suppose (pretend?) these kinds of sets don't exist.

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- ▶ Most mainstream probability and analysis takes the third approach.
- ▶ In practice, sets we care about (e.g., countable unions of points and intervals) tend to be measurable.

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- ▶ Riemann integration is a mathematically rigorous theory. It's just not as robust as Lebesgue integration.