18.600: Lecture 16

More discrete random variables

Scott Sheffield

MIT

Outline

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- Say X is a **geometric random variable with parameter** p.

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- Thus $E[X] - 1 = E[X-1] = p \cdot 0 + qE[X] = qE[X]$ and solving for $E[X]$ gives $E[X] = 1/(1-q) = 1/p$.

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- $\mathrm{Var}[X] = (2-p)/p^2 - 1/p^2 = (1-p)/p^2 = 1/p^2 - 1/p = q/p^2$.

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- Takes $1/p$ coin tosses on average to see a heads.

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- Call X **negative binomial random variable with parameters** (r, p).

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- How about $\mathrm{Var}[X]$?
- Turns out that $\mathrm{Var}[X] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \ldots + \mathrm{Var}[X_r]$. So $\mathrm{Var}[X] = rq/p^2$.

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- **Poisson approximation:** Approximate the probability there are exactly five cries during the night.
- **Exponential random variable approximation:** Approximate probability baby quiet all night.

More fun problems

- Suppose two soccer teams play each other. One team's number of points is Poisson with parameter λ_1 and other's is independently Poisson with parameter λ_2. (You can google "soccer" and "Poisson" to see the academic literature on the use of Poisson random variables to model soccer scores.) Using Mathematica (or similar software) compute the probability that the first team wins if $\lambda_1 = 2$ and $\lambda_2 = 1$. What if $\lambda_1 = 2$ and $\lambda_2 = .5$?

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- Imagine you start with the number 60. Then you toss a fair coin to decide whether to add 5 to your number or subtract 5 from it. Repeat this process with independent coin tosses until the number reaches 100 or 0. What is the *expected* number of tosses needed until this occurs?