

# **18.600: Lecture 39**

## **Review: practice problems**

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- ▶ When Alice showers, she first checks to see if at least one towel is present. If a towel is present, she dries off with that towel and returns it to the bathroom towel rack. Otherwise, she cheerfully retrieves both towels from the walk-in closet, then showers, dries off and leaves both towels on the rack.

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- ▶ **Problem:** describe towel-distribution evolution as a Markov chain and determine (over the long term) on what fraction of days Bob emerges from the shower to find no towel.

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$$M = \begin{pmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{pmatrix}$$

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- ▶ Bob finds no towel only if morning starts in state zero and Bob goes first. Over long term Bob finds no towel  $\frac{2}{9} \times \frac{1}{2} = \frac{1}{9}$  fraction of the time.

## Optional stopping, martingales, central limit theorem

Suppose that  $X_1, X_2, X_3, \dots$  is an infinite sequence of independent random variables which are each equal to 1 with probability  $1/2$  and  $-1$  with probability  $1/2$ . Let  $Y_n = \sum_{i=1}^n X_i$ . Answer the following:

- ▶ What is the the probability that  $Y_n$  reaches  $-25$  before the first time that it reaches  $5$ ?

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- ▶ What is the the probability that  $Y_n$  reaches  $-25$  before the first time that it reaches  $5$ ?
- ▶ Use the central limit theorem to approximate the probability that  $Y_{9000000}$  is greater than  $6000$ .



# Optional stopping, martingales, central limit theorem — answers

- ▶  $p_{-25}25 + p_55 = 0$  and  $p_{-25} + p_5 = 1$ . Solving, we obtain  $p_{-25} = 1/6$  and  $p_5 = 5/6$ .

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- ▶ One standard deviation is  $\sqrt{9000000} = 3000$ . We want probability to be 2 standard deviations above mean. Should be about  $\int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ .

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  - ▶  $Y_n = \prod_{i=1}^n (X_i - 1)$

# Martingales

- ▶ Yes, no, yes, no.



# Calculations like those needed for Black-Scholes derivation

- ▶ Let  $X$  be a normal random variable with mean 0 and variance 1. Compute the following (you may use the function  $\Phi(a) := \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$  in your answers):

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  - ▶  $E[e^{3X-3}]$ .
  - ▶  $E[e^X 1_{X \in (a,b)}]$  for fixed constants  $a < b$ .

# Calculations like those needed for Black-Scholes derivation

– answers

$$\begin{aligned} E[e^{3X-3}] &= \int_{-\infty}^{\infty} e^{3x-3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-6x+6}{2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-6x+9}{2}} e^{3/2} dx \\ &= e^{3/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-3)^2}{2}} dx \\ &= e^{3/2} \end{aligned}$$

# Calculations like those needed for Black-Scholes derivation

– answers

$$\begin{aligned} E[e^X 1_{X \in (a,b)}] &= \int_a^b e^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= \int_a^b e^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 - 2x + 1 - 1}{2}} dx \\ &= e^{1/2} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx \\ &= e^{1/2} \int_{a-1}^{b-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= e^{1/2} (\Phi(b-1) - \Phi(a-1)) \end{aligned}$$

If you want *more* probability and statistics...

► **UNDERGRADUATE:**

- (a) 18.615 Introduction to Stochastic Processes
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## ► **OUTSIDE OF MATH DEPARTMENT**

- (a) Look up MIT minor in statistics and data sciences.
- (b) Courses by title keywords: probability, statistics, inference, econometrics, data mining, stochastic, machine learning, time series, Markov, finance, analytics, biostatistics, modeling....

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- ▶ And may the odds be ever in your favor.