1. (20 points) Roll six standard six-sided dice independently, and let \( X \) be the number of dice that show the number 6.

(a) Compute the expectation \( E[X] \). **ANSWER:** The number of heads is binomial with \( n = 6 \) and \( p = 1/6 \), so \( E[X] = np = 1 \).

(b) Compute the variance \( \text{Var}(X) \). **ANSWER:** \( \text{Var}(X) = npq = 5/6 \) (where \( q = 1 - p \)).

(c) Compute \( P(X = 5) \). **ANSWER:**
\[
P(X = k) = \binom{6}{k} p^k (1 - p)^{6-k} = \binom{6}{5} (1/6)^5 (5/6) = 6 \cdot 5/6^6 = 5/6^5
\]

(d) Compute \( P(X = 6 | X \geq 5) \). **ANSWER:**
\[
P(X = 6 | X \geq 5) = \frac{P(X = 6)}{P(X = 6) + P(X = 5)} = \frac{1/6^6}{30/6^6 + 1/6^6} = 1/31
\]

2. (10 points) Suppose that \( E, F \) and \( G \) are events such that
\[
P(E) = P(F) = P(G) = .4
\]

and
\[
P(EF) = P(EG) = P(FG) = .2
\]

and
\[
P(EFG) = .1.
\]

Compute \( P(E \cup F \cup G) \). **ANSWER:** Inclusion-exclusion tells us
\[
P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)
\]
\[
= 3(.4) - 3(.2) + .1 = 1.2 - .6 + .1 = .7
\]

3. (10 points) Compute the following

(a) \[
\lim_{n \to \infty} (1 + 5/n)^n
\]

**ANSWER:** General formula is \( e^x = \lim_{n \to \infty} (1 + x/n)^n \). Plug in \( x = 5 \) and answer is \( e^5 \). Alternatively, one can just use definition of \( e \) (the special case \( x = 1 \)). Write \( n = 5m \) (which implies \( 5/n = 1/m \)) and note that
\[
\lim_{n \to \infty} (1 + 5/n)^n = \lim_{m \to \infty} (1 + 1/m)^{5m} = \lim_{m \to \infty} ((1 + 1/m)^m)^5 = \left( \lim_{m \to \infty} (1 + 1/m)^m \right)^5 = e^5.
\]
4. (20 points) Suppose that 10000 people visit Alice’s new restaurant during its first few months of operation. Each person independently chooses to leave a positive Yelp review (e.g., “Great samosas!”) with probability $5/10000$, a negative Yelp review (e.g., “Rude servers!”) with probability $1/10000$ or no review at all with probability $9994/10000$. Let $X$ be total number of positive reviews received and $Y$ the total number of negative reviews received.

(a) Compute $E(Y)$ and $\text{Var}(Y)$. (Give exact values, not approximations.)

**ANSWER:** This is binomial with $n = 10000$ and $p = 1/10000$ so $E[Y] = np = 1$ and $\text{Var}(Y) = np(1-p) = (1-p) = 9999/10000$.

(b) Use a Poisson random variable to approximate $P(X = 3)$.

**ANSWER:** $E[X] = 5$, so $X$ is approximately Poisson with parameter $\lambda = 5$. This suggests

$$P(X = k) \approx e^{-\lambda} \lambda^k / k! = e^{-5} 5^3 / 3! = \frac{125}{6e^5}.$$  

(c) Use a Poisson random variable to approximate $P(Y = 0)$.

**ANSWER:** $e^{-\lambda} \lambda^k / k!$ with $k = 1$ and $\lambda = 1$ is $1/e$. Alternatively, just note directly that $P(Y = 0) = (1 - 1/10000)^{10000} \approx e^{-1} = 1/e$.

5. (10 points) Suppose that a deck of cards contains 120 cards: 30 red cards, 40 black cards, and 50 blue cards. A random collection of 12 cards is chosen (with all possible 12-card subsets being equally likely). What is the probability that this collection contains three red cards, four black cards, and five blue cards? **ANSWER:** Total number of ways to choose 12 cards is $\binom{120}{12}$. The number of ways to choose cards with desired color breakdown is $\binom{30}{3} \binom{40}{4} \binom{50}{5}$. So the ratio is

$$\frac{\binom{30}{3} \binom{40}{4} \binom{50}{5}}{\binom{120}{12}}.$$  

6. (15 points) Ten people toss their hats in a bin and have them randomly shuffled and returned, one hat to each person. Let $X_i$ be 1 if $i$th person
gets own hat back, 0 otherwise. Let \( X = \sum_{i=1}^{10} X_i \) be the total number of people who get their own hat back. Compute the following:

(a) The expectation \( E[X] \). **ANSWER:** \( 10 \cdot \frac{1}{10} = 1 \)

(b) The expectation \( E[X_3X_7] \). **ANSWER:** \( X_3X_7 \) is 1 on the event that 3rd and 7th people both get own hats, and zero otherwise. So \( E[X_3X_7] \) is the probability that both 3 and 7 their own hats. There are 8! permutations in which 3 and 7 get own hats, so answer is \( 8!/10! = 1/90 \).

(c) The expectation \( E[X_1^2 + X_2^2 + X_3^2] \). **ANSWER:** Note that \( X_j^2 = X_j \) for each \( j \), so this is just \( E[X_1 + X_2 + X_3] = 3E[X_1] = 3/10 \).

7. (15 points) Bob is at an airport kiosk considering the purchase of a bottle of water with no listed price. Bob is thirsty but is too shy to ask about the price. He happens to know that the price in dollars (denoted \( X \)) is an integer between 3 and 7 and he considers each of the values in \{3, 4, 5, 6, 7\} to be equally likely (probability \( 1/5 \) for each). According to Bob’s probability measure, find the following:

(a) \( E[X] \) **ANSWER:** \( \frac{1}{5}(3 + 4 + 5 + 6 + 7) = 5 \)

(b) \( \text{Var}[X] \) **ANSWER:** \( \text{Var}(X) = E[(X - 5)^2] \). Note that \( (X - 5)^2 \) is 4 with probability 2/5 and 1 with probability 2/5 and zero otherwise. So \( \text{Var}[X] = E[(X - 5)^2] = \frac{2}{5} \cdot 4 + \frac{2}{5} \cdot 1 = 2 \).

(c) \( \text{Var}[1.05X] \) (That is, the variance of the sales-tax-inclusive price assuming the airport is in a state with five percent sales tax.)

**ANSWER:** \( \text{Var}(1.05X) = 1.05^2 \text{Var}(X) = (21/20)^2 \cdot 2 = 441/200 \)