18.600: Lecture 2

Multinomial coefficients and more counting problems

Scott Sheffield

MIT

Outline

Multinomial coefficients

Integer partitions

More problems

Outline

Multinomial coefficients

Integer partitions

More problems

Partition problems

- You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?

Partition problems

- You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?
- Answer: $8!/(3!2!3!)$

Partition problems

- You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?
- Answer: $8!/(3!2!3!)$
- One way to think of this: given any permutation of eight elements (e.g., 12435876 or 87625431) declare first three as breakfast, second two as lunch, last three as dinner. This maps set of $8!$ permutations on to the set of food-meal divisions in a many-to-one way: each food-meal division comes from $3!2!3!$ permutations.

Partition problems

- You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?
- Answer: $8!/(3!2!3!)$
- One way to think of this: given any permutation of eight elements (e.g., 12435876 or 87625431) declare first three as breakfast, second two as lunch, last three as dinner. This maps set of $8!$ permutations on to the set of food-meal divisions in a many-to-one way: each food-meal division comes from $3!2!3!$ permutations.
- How many 8-letter sequences with 3 A's, 2 B's, and 3 C's?

Partition problems

- You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?
- Answer: $8!/(3!2!3!)$
- One way to think of this: given any permutation of eight elements (e.g., 12435876 or 87625431) declare first three as breakfast, second two as lunch, last three as dinner. This maps set of $8!$ permutations on to the set of food-meal divisions in a many-to-one way: each food-meal division comes from $3!2!3!$ permutations.
- How many 8-letter sequences with 3 A's, 2 B's, and 3 C's?
- Answer: $8!/(3!2!3!)$. Same as other problem. Imagine 8 "slots" for the letters. Choose 3 to be A's, 2 to be B's, and 3 to be C's.

Partition problems

- In general, if you have n elements you wish to divide into r distinct piles of sizes $n_1, n_2 \ldots n_r$, how many ways to do that?

Partition problems

- In general, if you have n elements you wish to divide into r distinct piles of sizes $n_1, n_2 \dots n_r$, how many ways to do that?
- Answer $\binom{n}{n_1, n_2, \dots, n_r} := \frac{n!}{n_1! n_2! \dots n_r!}$.

One way to understand the binomial theorem

- Expand the product $(A_1 + B_1)(A_2 + B_2)(A_3 + B_3)(A_4 + B_4)$.

One way to understand the binomial theorem

- Expand the product $(A_1 + B_1)(A_2 + B_2)(A_3 + B_3)(A_4 + B_4)$.
- 16 terms correspond to 16 length-4 sequences of A's and B's.

$$A_1A_2A_3A_4 + A_1A_2A_3B_4 + A_1A_2B_3A_4 + A_1A_2B_3B_4 +$$

$$A_1B_2A_3A_4 + A_1B_2A_3B_4 + A_1B_2B_3A_4 + A_1B_2B_3B_4 +$$

$$B_1A_2A_3A_4 + B_1A_2A_3B_4 + B_1A_2B_3A_4 + B_1A_2B_3B_4 +$$

$$B_1B_2A_3A_4 + B_1B_2A_3B_4 + B_1B_2B_3A_4 + B_1B_2B_3B_4$$

One way to understand the binomial theorem

- Expand the product $(A_1 + B_1)(A_2 + B_2)(A_3 + B_3)(A_4 + B_4)$.
- 16 terms correspond to 16 length-4 sequences of A's and B's.

$$A_1A_2A_3A_4 + A_1A_2A_3B_4 + A_1A_2B_3A_4 + A_1A_2B_3B_4 +$$

$$A_1B_2A_3A_4 + A_1B_2A_3B_4 + A_1B_2B_3A_4 + A_1B_2B_3B_4 +$$

$$B_1A_2A_3A_4 + B_1A_2A_3B_4 + B_1A_2B_3A_4 + B_1A_2B_3B_4 +$$

$$B_1B_2A_3A_4 + B_1B_2A_3B_4 + B_1B_2B_3A_4 + B_1B_2B_3B_4$$

- What happens to this sum if we erase subscripts?

One way to understand the binomial theorem

- Expand the product $(A_1 + B_1)(A_2 + B_2)(A_3 + B_3)(A_4 + B_4)$.
- 16 terms correspond to 16 length-4 sequences of A's and B's.

$$A_1A_2A_3A_4 + A_1A_2A_3B_4 + A_1A_2B_3A_4 + A_1A_2B_3B_4 +$$

$$A_1B_2A_3A_4 + A_1B_2A_3B_4 + A_1B_2B_3A_4 + A_1B_2B_3B_4 +$$

$$B_1A_2A_3A_4 + B_1A_2A_3B_4 + B_1A_2B_3A_4 + B_1A_2B_3B_4 +$$

$$B_1B_2A_3A_4 + B_1B_2A_3B_4 + B_1B_2B_3A_4 + B_1B_2B_3B_4$$

- What happens to this sum if we erase subscripts?
- $(A + B)^4 = B^4 + 4AB^3 + 6A^2B^2 + 4A^3B + A^4$. Coefficient of A^2B^2 is 6 because 6 length-4 sequences have 2 A's and 2 B's.

One way to understand the binomial theorem

- Expand the product $(A_1 + B_1)(A_2 + B_2)(A_3 + B_3)(A_4 + B_4)$.
- 16 terms correspond to 16 length-4 sequences of A's and B's.

$$A_1A_2A_3A_4 + A_1A_2A_3B_4 + A_1A_2B_3A_4 + A_1A_2B_3B_4 +$$

$$A_1B_2A_3A_4 + A_1B_2A_3B_4 + A_1B_2B_3A_4 + A_1B_2B_3B_4 +$$

$$B_1A_2A_3A_4 + B_1A_2A_3B_4 + B_1A_2B_3A_4 + B_1A_2B_3B_4 +$$

$$B_1B_2A_3A_4 + B_1B_2A_3B_4 + B_1B_2B_3A_4 + B_1B_2B_3B_4$$

- What happens to this sum if we erase subscripts?
- $(A + B)^4 = B^4 + 4AB^3 + 6A^2B^2 + 4A^3B + A^4$. Coefficient of A^2B^2 is 6 because 6 length-4 sequences have 2 A's and 2 B's.
- Generally, $(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$, because there are $\binom{n}{k}$ sequences with k A's and $(n - k)$ B's.

How about trinomials?

- Expand
  $(A_1 + B_1 + C_1)(A_2 + B_2 + C_2)(A_3 + B_3 + C_3)(A_4 + B_4 + C_4)$.
  How many terms?

How about trinomials?

- Expand
  $(A_1 + B_1 + C_1)(A_2 + B_2 + C_2)(A_3 + B_3 + C_3)(A_4 + B_4 + C_4)$.
  How many terms?
- Answer: 81, one for each length-4 sequence of A's and B's and C's.

How about trinomials?

- Expand $(A_1 + B_1 + C_1)(A_2 + B_2 + C_2)(A_3 + B_3 + C_3)(A_4 + B_4 + C_4)$. How many terms?
- Answer: 81, one for each length-4 sequence of A's and B's and C's.
- We can also compute $(A + B + C)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4 + 4A^3C + 12A^2BC + 12AB^2C + 4B^3C + 6A^2C^2 + 12ABC^2 + 6B^2C^2 + 4AC^3 + 4BC^3 + C^4$

How about trinomials?

- Expand $(A_1 + B_1 + C_1)(A_2 + B_2 + C_2)(A_3 + B_3 + C_3)(A_4 + B_4 + C_4)$. How many terms?
- Answer: 81, one for each length-4 sequence of A's and B's and C's.
- We can also compute $(A + B + C)^4 = A^4+4A^3B+6A^2B^2+4AB^3+B^4+4A^3C+12A^2BC+12AB^2C+4B^3C + 6A^2C^2 + 12ABC^2 + 6B^2C^2 + 4AC^3 + 4BC^3 + C^4$
- What is the sum of the coefficients in this expansion? What is the combinatorial interpretation of coefficient of, say, ABC^2?

How about trinomials?

- Expand $(A_1 + B_1 + C_1)(A_2 + B_2 + C_2)(A_3 + B_3 + C_3)(A_4 + B_4 + C_4)$. How many terms?
- Answer: 81, one for each length-4 sequence of A's and B's and C's.
- We can also compute $(A + B + C)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4 + 4A^3C + 12A^2BC + 12AB^2C + 4B^3C + 6A^2C^2 + 12ABC^2 + 6B^2C^2 + 4AC^3 + 4BC^3 + C^4$
- What is the sum of the coefficients in this expansion? What is the combinatorial interpretation of coefficient of, say, ABC^2?
- Answer $81 = (1 + 1 + 1)^4$. ABC^2 has coefficient 12 because there are 12 length-4 words have one A, one B, two C's.

Multinomial coefficients

- Is there a higher dimensional analog of binomial theorem?

Multinomial coefficients

- Is there a higher dimensional analog of binomial theorem?
- Answer: yes.

Multinomial coefficients

- Is there a higher dimensional analog of binomial theorem?
- Answer: yes.
- Then what is it?

Multinomial coefficients

- Is there a higher dimensional analog of binomial theorem?
- Answer: yes.
- Then what is it?
- $$(x_1+x_2+\ldots+x_r)^n = \sum_{n_1,\ldots,n_r : n_1+\ldots+n_r=n} \binom{n}{n_1,\ldots,n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

Multinomial coefficients

- Is there a higher dimensional analog of binomial theorem?
- Answer: yes.
- Then what is it?
- $$(x_1+x_2+\ldots+x_r)^n = \sum_{n_1,\ldots,n_r:n_1+\ldots+n_r=n} \binom{n}{n_1,\ldots,n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$
- The sum on the right is taken over all collections $(n_1, n_2, \ldots, n_r)$ of r non-negative integers that add up to n.

Multinomial coefficients

- Is there a higher dimensional analog of binomial theorem?
- Answer: yes.
- Then what is it?
- $$(x_1+x_2+\ldots+x_r)^n = \sum_{n_1,\ldots,n_r : n_1+\ldots+n_r=n} \binom{n}{n_1,\ldots,n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$
- The sum on the right is taken over all collections $(n_1, n_2, \ldots, n_r)$ of r non-negative integers that add up to n.
- Pascal's triangle gives coefficients in binomial expansions. Is there something like a "Pascal's pyramid" for trinomial expansions?

Multinomial coefficients

- Is there a higher dimensional analog of binomial theorem?
- Answer: yes.
- Then what is it?
-

$$(x_1+x_2+\ldots+x_r)^n = \sum_{n_1,\ldots,n_r:n_1+\ldots+n_r=n} \binom{n}{n_1,\ldots,n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

- The sum on the right is taken over all collections $(n_1, n_2, \ldots, n_r)$ of r non-negative integers that add up to n.
- Pascal's triangle gives coefficients in binomial expansions. Is there something like a "Pascal's pyramid" for trinomial expansions?
- Yes (look it up) but it is a bit tricker to draw and visualize than Pascal's triangle.

By the way...

- If $n!$ is the product of all integers in the interval with endpoints 1 and n, then $0! = 0$.

By the way...

- If $n!$ is the product of all integers in the interval with endpoints 1 and n, then $0! = 0$.
- Actually, we say $0! = 1$. What are the reasons for that?

By the way...

- If $n!$ is the product of all integers in the interval with endpoints 1 and n, then $0! = 0$.
- Actually, we say $0! = 1$. What are the reasons for that?
- **Because** there is one map from the empty set to itself.

By the way...

- If $n!$ is the product of all integers in the interval with endpoints 1 and n, then $0! = 0$.
- Actually, we say $0! = 1$. What are the reasons for that?
- **Because** there is one map from the empty set to itself.
- **Because** we want the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to still make sense when $k = 0$ and $k = n$. There is clearly 1 way to choose n elements from a group of n elements. And 1 way to choose 0 elements from a group of n elements so $\frac{n!}{n!0!} = \frac{n!}{0!n!} = 1$.

By the way...

- If $n!$ is the product of all integers in the interval with endpoints 1 and n, then $0! = 0$.
- Actually, we say $0! = 1$. What are the reasons for that?
- **Because** there is one map from the empty set to itself.
- **Because** we want the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to still make sense when $k = 0$ and $k = n$. There is clearly 1 way to choose n elements from a group of n elements. And 1 way to choose 0 elements from a group of n elements so $\frac{n!}{n!0!} = \frac{n!}{0!n!} = 1$.
- **Because** we want the recursion $n(n-1)! = n!$ to hold for $n = 1$. (We won't define factorials of negative integers.)

By the way...

- If $n!$ is the product of all integers in the interval with endpoints 1 and n, then $0! = 0$.
- Actually, we say $0! = 1$. What are the reasons for that?
- **Because** there is one map from the empty set to itself.
- **Because** we want the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to still make sense when $k = 0$ and $k = n$. There is clearly 1 way to choose n elements from a group of n elements. And 1 way to choose 0 elements from a group of n elements so $\frac{n!}{n!0!} = \frac{n!}{0!n!} = 1$.
- **Because** we want the recursion $n(n-1)! = n!$ to hold for $n = 1$. (We won't define factorials of negative integers.)
- **Because** we want $n! = \int_0^\infty t^n e^{-t} dt$ to hold for all non-negative integers. (Check for positive integers by integration by parts.) This is one of those formulas you should just know. Can use it to define $n!$ for non-integer n.

By the way...

- If $n!$ is the product of all integers in the interval with endpoints 1 and n, then $0! = 0$.
- Actually, we say $0! = 1$. What are the reasons for that?
- **Because** there is one map from the empty set to itself.
- **Because** we want the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to still make sense when $k = 0$ and $k = n$. There is clearly 1 way to choose n elements from a group of n elements. And 1 way to choose 0 elements from a group of n elements so $\frac{n!}{n!0!} = \frac{n!}{0!n!} = 1$.
- **Because** we want the recursion $n(n-1)! = n!$ to hold for $n = 1$. (We won't define factorials of negative integers.)
- **Because** we want $n! = \int_0^\infty t^n e^{-t} dt$ to hold for all non-negative integers. (Check for positive integers by integration by parts.) This is one of those formulas you should just know. Can use it to define $n!$ for non-integer n.
- Another common notation: write $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$ and define $n! := \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt$, so that $\Gamma(n) = (n-1)!$.

Outline

Multinomial coefficients

Integer partitions

More problems

Outline

Multinomial coefficients

Integer partitions

More problems

Integer partitions

- How many sequences $a_1, \ldots, a_k$ of non-negative integers satisfy $a_1 + a_2 + \ldots + a_k = n$?

Integer partitions

- How many sequences $a_1, \ldots, a_k$ of non-negative integers satisfy $a_1 + a_2 + \ldots + a_k = n$?
- Answer: $\binom{n+k-1}{n}$. Represent partition by $k-1$ bars and n stars, e.g., as $**|**||****|*$.

Outline

Multinomial coefficients

Integer partitions

More problems

Outline

Multinomial coefficients

Integer partitions

More problems

More counting problems

- In 18.821, a class of 27 students needs to be divided into 9 teams of three students each? How many ways are there to do that?

More counting problems

- In 18.821, a class of 27 students needs to be divided into 9 teams of three students each? How many ways are there to do that?
- $\frac{27!}{(3!)^9 9!}$

More counting problems

- In 18.821, a class of 27 students needs to be divided into 9 teams of three students each? How many ways are there to do that?
- $\frac{27!}{(3!)^9 9!}$
- You teach a class with 90 students. In a rather severe effort to combat grade inflation, your department chair insists that you assign the students exactly 10 A's, 20 B's, 30 C's, 20 D's, and 10 F's. How many ways to do this?

More counting problems

- In 18.821, a class of 27 students needs to be divided into 9 teams of three students each? How many ways are there to do that?
- $\frac{27!}{(3!)^9 9!}$
- You teach a class with 90 students. In a rather severe effort to combat grade inflation, your department chair insists that you assign the students exactly 10 A's, 20 B's, 30 C's, 20 D's, and 10 F's. How many ways to do this?
- $\binom{90}{10,20,30,20,10} = \frac{90!}{10!20!30!20!10!}$

More counting problems

- In 18.821, a class of 27 students needs to be divided into 9 teams of three students each? How many ways are there to do that?
- $\frac{27!}{(3!)^9 9!}$
- You teach a class with 90 students. In a rather severe effort to combat grade inflation, your department chair insists that you assign the students exactly 10 A's, 20 B's, 30 C's, 20 D's, and 10 F's. How many ways to do this?
- $\binom{90}{10,20,30,20,10} = \frac{90!}{10!20!30!20!10!}$
- You have 90 (indistinguishable) pieces of pizza to divide among the 90 (distinguishable) students. How many ways to do that (giving each student a non-negative integer number of slices)?

More counting problems

- In 18.821, a class of 27 students needs to be divided into 9 teams of three students each? How many ways are there to do that?
- $\frac{27!}{(3!)^9 9!}$
- You teach a class with 90 students. In a rather severe effort to combat grade inflation, your department chair insists that you assign the students exactly 10 A's, 20 B's, 30 C's, 20 D's, and 10 F's. How many ways to do this?
- $\binom{90}{10,20,30,20,10} = \frac{90!}{10!20!30!20!10!}$
- You have 90 (indistinguishable) pieces of pizza to divide among the 90 (distinguishable) students. How many ways to do that (giving each student a non-negative integer number of slices)?
- $\binom{179}{90} = \binom{179}{89}$

More counting problems

- How many 13-card bridge hands have 4 of one suit, 3 of one suit, 5 of one suit, 1 of one suit?

More counting problems

- How many 13-card bridge hands have 4 of one suit, 3 of one suit, 5 of one suit, 1 of one suit?
- $4!\binom{13}{4}\binom{13}{3}\binom{13}{5}\binom{13}{1}$

More counting problems

- How many 13-card bridge hands have 4 of one suit, 3 of one suit, 5 of one suit, 1 of one suit?
- $4!\binom{13}{4}\binom{13}{3}\binom{13}{5}\binom{13}{1}$
- How many bridge hands have at most two suits represented?

More counting problems

- How many 13-card bridge hands have 4 of one suit, 3 of one suit, 5 of one suit, 1 of one suit?
- $4!\binom{13}{4}\binom{13}{3}\binom{13}{5}\binom{13}{1}$
- How many bridge hands have at most two suits represented?
- $\binom{4}{2}\binom{26}{13} - 8$

More counting problems

- How many 13-card bridge hands have 4 of one suit, 3 of one suit, 5 of one suit, 1 of one suit?
- $4!\binom{13}{4}\binom{13}{3}\binom{13}{5}\binom{13}{1}$
- How many bridge hands have at most two suits represented?
- $\binom{4}{2}\binom{26}{13} - 8$
- How many hands have either 3 or 4 cards in each suit?

More counting problems

- How many 13-card bridge hands have 4 of one suit, 3 of one suit, 5 of one suit, 1 of one suit?
- $4!\binom{13}{4}\binom{13}{3}\binom{13}{5}\binom{13}{1}$
- How many bridge hands have at most two suits represented?
- $\binom{4}{2}\binom{26}{13} - 8$
- How many hands have either 3 or 4 cards in each suit?
- Need three 3-card suits, one 4-card suit, to make 13 cards total. Answer is $4\binom{13}{3}^3\binom{13}{4}$