18.175: Lecture 7
Zero-one laws and maximal inequalities

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Outline

Borel-Cantelli applications

Strong law of large numbers

Kolmogorov zero-one law and three-series theorem
Outline

Borel-Cantelli applications

Strong law of large numbers

Kolmogorov zero-one law and three-series theorem
First Borel-Cantelli lemma: If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n \text{ i.o.}) = 0$. 

Second Borel-Cantelli lemma: If $A_n$ are independent, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$. 

Borel-Cantelli lemmas
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Second Borel-Cantelli lemma: If $A_n$ are independent, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$. 
Theorem: $X_n \to X$ in probability if and only if for every subsequence of the $X_n$ there is a further subsequence converging a.s. to $X$. 
Theorem: $X_n \rightarrow X$ in probability if and only if for every subsequence of the $X_n$ there is a further subsequence converging a.s. to $X$.

Main idea of proof: Consider event $E_n$ that $X_n$ and $X$ differ by $\epsilon$. Do the $E_n$ occur i.o.? Use Borel-Cantelli.
Pairwise independence example

- **Theorem:** Suppose $A_1, A_2, \ldots$ are pairwise independent and $\sum P(A_n) = \infty$, and write $S_n = \sum_{i=1}^{n} 1_{A_i}$. Then the ratio $S_n/ES_n$ tends a.s. to 1.
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Main idea of proof: First, pairwise independence implies that variances add. Conclude (by checking term by term) that $\text{Var}S_n \leq ES_n$. Then Chebyshev implies
\[
P(|S_n - ES_n| > \delta ES_n) \leq \frac{\text{Var}(S_n)}{(\delta ES_n)^2} \to 0,
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which gives us convergence in probability.
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Second, take a smart subsequence. Let $n_k = \inf\{n : ES_n \geq k^2\}$. Use Borel Cantelli to get a.s. convergence along this subsequence. Check that convergence along this subsequence deterministically implies the non-subsequential convergence.
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Strong law of large numbers

Kolmogorov zero-one law and three-series theorem
Theorem (strong law): If $X_1, X_2, \ldots$ are i.i.d. real-valued random variables with expectation $m$ and $A_n := n^{-1} \sum_{i=1}^{n} X_i$ are the empirical means then $\lim_{n \to \infty} A_n = m$ almost surely.
Proof of strong law assuming $E[X^4] < \infty$

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- Note: $\text{Var}[X^2] = E[X^4] - E[X^2]^2 \geq 0$, so $E[X^2]^2 \leq K$. 

$\sum_{n=1}^{\infty} A_n^4 < \infty$ (and hence $A_n \to 0$) with probability 1.
Proof of strong law assuming \( E[X^4] < \infty \)

- Assume \( K := E[X^4] < \infty \). Not necessary, but simplifies proof.
- Note: \( \text{Var}[X^2] = E[X^4] - E[X^2]^2 \geq 0 \), so \( E[X^2]^2 \leq K \).
- The strong law holds for i.i.d. copies of \( X \) if and only if it holds for i.i.d. copies of \( X - \mu \) where \( \mu \) is a constant.
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- So we may as well assume $E[X] = 0$.
- Key to proof is to bound fourth moments of $A_n$. 

\[ E[A_n^4] = n - 4E[S_n^4] = n - 4E[(X_1 + \cdots + X_n)^4]. \]

Expand $(X_1 + \cdots + X_n)^4$. Five kinds of terms: $X_i X_j X_k X_l$ and $X_i X_j X^2_k$ and $X^3_i X_j$ and $X^2_i X^2_j$ and $X^4_i$. The first three terms all have expectation zero. There are $n^2$ of the fourth type and $n$ of the last type, each equal to at most $K$. So $E[A_n^4] \leq n - 4(n^2/6 + n)K$. Thus $E[\sum_{n=1}^{\infty} A_n^4] = \sum_{n=1}^{\infty} E[A_n^4] < \infty$. So $\sum_{n=1}^{\infty} A_n^4 < \infty$ (and hence $A_n \to 0$) with probability 1.
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- Expand $(X_1 + \ldots + X_n)^4$. Five kinds of terms: $X_i X_j X_k X_l$ and $X_i X_j X_k^2$ and $X_i X_j^3$ and $X_i^2 X_j^2$ and $X_i^4$. 
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- The first three terms all have expectation zero. There are $\binom{n}{2}$ of the fourth type and $n$ of the last type, each equal to at most $K$. So $E[A_n^4] \leq n^{-4} \left( 6 \binom{n}{2} + n \right) K$.
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- The first three terms all have expectation zero. There are $\binom{n}{2}$ of the fourth type and $n$ of the last type, each equal to at most $K$. So $E[A_n^4] \leq n^{-4}\left(6\left(\binom{n}{2}\right) + n\right)K$.
- Thus $E[\sum_{n=1}^{\infty} A_n^4] = \sum_{n=1}^{\infty} E[A_n^4] < \infty$. So $\sum_{n=1}^{\infty} A_n^4 < \infty$ (and hence $A_n \to 0$) with probability 1.
Suppose $X_k$ are i.i.d. with finite mean. Let $Y_k = X_k 1_{|X_k| \leq k}$. Write $T_n = Y_1 + \ldots + Y_n$. **Claim:** $X_k = Y_k$ all but finitely often a.s. so suffices to show $T_n/n \to \mu$. (Borel Cantelli, expectation of positive r.v. is area between cdf and line $y = 1$)
General proof of strong law

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- **Claim:** $\sum_{k=1}^{\infty} \text{Var}(Y_k)/k^2 \leq 4E|X_1| < \infty$. How to prove it?
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- **Claim:** $\sum_{k=1}^{\infty} \text{Var}(Y_k)/k^2 \leq 4E|X_1| < \infty$. How to prove it?

- **Observe:** $\text{Var}(Y_k) \leq E(Y_k^2) = \int_0^\infty 2yP(|Y_k| > y)dy \leq \int_0^k 2yP(|X_1| > y)dy$. Use Fubini (interchange sum/integral, since everything positive)

\[
\sum_{k=1}^{\infty} E(Y_k^2)/k^2 \leq \sum_{k=1}^{\infty} k^{-2} \int_0^\infty 1_{(y<k)} 2yP(|X_1| > y)dy = \\
\int_0^\infty \left( \sum_{k=1}^{\infty} k^{-2} 1_{(y<k)} \right) 2yP(|X_1| > y)dy.
\]

Since $E|X_1| = \int_0^\infty P(|X_1| > y)dy$, complete proof of claim by showing that if $y \geq 0$ then $2y \sum_{k>y} k^{-2} \leq 4$. 

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Claim: $\sum_{k=1}^{\infty} \text{Var}(Y_k)/k^2 \leq 4E|X_1| < \infty$. How to use it?
General proof of strong law

- **Claim:** \( \sum_{k=1}^{\infty} \frac{\text{Var}(Y_k)}{k^2} \leq 4E|X_1| < \infty \). How to use it?
- Consider subsequence \( k(n) = \lfloor \alpha^n \rfloor \) for arbitrary \( \alpha > 1 \). Using Chebyshev, if \( \epsilon > 0 \) then

\[
\sum_{n=1}^{\infty} P(|T_{k(n)} - ET_{k(n)}| > \epsilon k(n)) \leq \epsilon^{-1} \sum_{n=1}^{\infty} \frac{\text{Var}(T_{k(n)})}{k(n)^2}
\]

\[
= \epsilon^{-2} \sum_{n=1}^{\infty} k(n)^{-2} \sum_{m=1}^{k(n)} \text{Var}(Y_m) = \epsilon^{-2} \sum_{m=1}^{\infty} \text{Var}(Y_m) \sum_{n:k(n) \geq m} k(n)^{-2}.
\]
Claim: $\sum_{k=1}^{\infty} \frac{\text{Var}(Y_k)}{k^2} \leq 4E|X_1| < \infty$. How to use it?

Consider subsequence $k(n) = [\alpha^n]$ for arbitrary $\alpha > 1$. Using Chebyshev, if $\epsilon > 0$ then

$$\sum_{n=1}^{\infty} P\left( | T_{k(n)} - ET_{k(n)} | > \epsilon k(n) \right) \leq \epsilon^{-1} \sum_{n=1}^{\infty} \frac{\text{Var}(T_{k(n)})}{k(n)^2}$$

$$= \epsilon^{-2} \sum_{n=1}^{\infty} k(n)^{-2} \sum_{m=1}^{k(n)} \text{Var}(Y_m) = \epsilon^{-2} \sum_{m=1}^{\infty} \text{Var}(Y_m) \sum_{n: k(n) \geq m} k(n)^{-2}.$$ 

Sum series:

$$\sum_{n: \alpha^n \geq m} [\alpha^n]^{-2} \leq 4 \sum_{n: \alpha^n \geq m} \alpha^{-2n} \leq 4(1 - \alpha^{-2})^{-1} m^{-2}.$$
Claim: $\sum_{k=1}^{\infty} \text{Var}(Y_k)/k^2 \leq 4E|X_1| < \infty$. How to use it?

Consider subsequence $k(n) = \lceil \alpha^n \rceil$ for arbitrary $\alpha > 1$. Using Chebyshev, if $\epsilon > 0$ then

$$\sum_{n=1}^{\infty} P\left( |T_{k(n)} - ET_{k(n)}| > \epsilon k(n) \right) \leq \epsilon^{-1} \sum_{n=1}^{\infty} \text{Var}(T_{k(n)})/k(n)^2$$

$$= \epsilon^{-2} \sum_{n=1}^{\infty} k(n)^{-2} \sum_{m=1}^{k(n)} \text{Var}(Y_m) = \epsilon^{-2} \sum_{m=1}^{\infty} \text{Var}(Y_m) \sum_{n: k(n) \geq m} k(n)^{-2}.$$

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Combine computations (observe RHS below is finite):

$$\sum_{n=1}^{\infty} P\left( |T_{k(n)} - ET_{k(n)}| > \epsilon k(n) \right) \leq 4(1-\alpha^{-2})^{-1} \epsilon^{-2} \sum_{m=1}^{\infty} E(Y_m^2) m^{-2}.$$
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$$\sum_{n=1}^{\infty} P\left( |T_{k(n)} - ET_{k(n)}| > \epsilon k(n) \right) \leq \epsilon^{-1} \sum_{n=1}^{\infty} \text{Var}(T_{k(n)})/k(n)^2$$

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$$\sum_{n:\alpha^n \geq m} [\alpha^n]^{-2} \leq 4 \sum_{n:\alpha^n \geq m} \alpha^{-2n} \leq 4(1 - \alpha^{-2})^{-1} m^{-2}.$$

Combine computations (observe RHS below is finite):
$$\sum_{n=1}^{\infty} P\left( |T_{k(n)} - ET_{k(n)}| > \epsilon k(n) \right) \leq 4(1 - \alpha^{-2})^{-1} \epsilon^{-2} \sum_{m=1}^{\infty} E(Y_m^2)m^{-2}.$$

Since $\epsilon$ is arbitrary, get $(T_{k(n)} - ET_{k(n)})/k(n) \to 0$ a.s.
Conclude by taking $\alpha \to 1$. This finishes the case that the $X_1$ are a.s. positive.
General proof of strong law

- Conclude by taking $\alpha \to 1$. This finishes the case that the $X_1$ are a.s. positive.
- Can extend to the case that $X_1$ is a.s. positive with infinite mean.
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Can extend to the case that $X_1$ is a.s. positive with infinite mean.

Generally, can consider $X_1^+$ and $X_1^-$, and it is enough if one of them has a finite mean.
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Kolmogorov zero-one law

Consider sequence of random variables $X_n$ on some probability space. Write $\mathcal{F}_n' = \sigma(X_n, X_{n_1}, \ldots)$ and $\mathcal{T} = \cap_n \mathcal{F}_n'$.

$\mathcal{T}$ is called the tail $\sigma$-algebra. It contains the information you can observe by looking only at stuff arbitrarily far into the future. Intuitively, membership in tail event doesn’t change when finitely many $X_n$ are changed.

Event that $X_n$ converge to a limit is example of a tail event. Other examples?

Theorem: If $X_1, X_2, \ldots$ are independent and $A \in \mathcal{T}$ then $P(A) \in \{0, 1\}$. 
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Consider sequence of random variables $X_n$ on some probability space. Write $F'_n = \sigma(X_n, X_{n1}, \ldots)$ and $T = \cap_n F'_n$.

$T$ is called the **tail $\sigma$-algebra**. It contains the information you can observe by looking only at stuff arbitrarily far into the future. Intuitively, membership in tail event doesn’t change when finitely many $X_n$ are changed.

Event that $X_n$ converge to a limit is example of a tail event. Other examples?

**Theorem:** If $X_1, X_2, \ldots$ are independent and $A \in T$ then $P(A) \in \{0, 1\}$. 
Theorem: If $X_1, X_2, \ldots$ are independent and $A \in \mathcal{T}$ then $P(A) \in \{0, 1\}$. 

Main idea of proof: Statement is equivalent to saying that $A$ is independent of itself, i.e., $P(A) = P(A \cap A) = P(A)^2$. How do we prove that?

Recall theorem that if $A_i$ are independent $\pi$-systems, then $\sigma(A_i)$ are independent.

Deduce that $\sigma(X_1, X_2, \ldots, X_n)$ and $\sigma(X_n+1, X_n+2, \ldots)$ are independent. Then deduce that $\sigma(X_1, X_2, \ldots)$ and $\mathcal{T}$ are independent, using fact that $\bigcup_k \sigma(X_1, \ldots, X_k)$ and $\mathcal{T}$ are $\pi$-systems.
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Kolmogorov zero-one law proof idea

**Theorem:** If \( X_1, X_2, \ldots \) are independent and \( A \in \mathcal{T} \) then \( P(A) \in \{0, 1\} \).

**Main idea of proof:** Statement is equivalent to saying that \( A \) is independent of itself, i.e., \( P(A) = P(A \cap A) = P(A)^2 \). How do we prove that?

Recall theorem that if \( A_i \) are independent \( \pi \)-systems, then \( \sigma A_i \) are independent.

Deduce that \( \sigma(X_1, X_2, \ldots, X_n) \) and \( \sigma(X_{n+1}, X_{n+1}, \ldots) \) are independent. Then deduce that \( \sigma(X_1, X_2, \ldots) \) and \( \mathcal{T} \) are independent, using fact that \( \cup_k \sigma(X_1, \ldots, X_k) \) and \( \mathcal{T} \) are \( \pi \)-systems.
Theorem: Suppose $X_i$ are independent with mean zero and finite variances, and $S_n = \sum_{i=1}^{n} X_n$. Then

$$P\left( \max_{1 \leq k \leq n} |S_k| \geq x \right) \leq x^{-2} \text{Var}(S_n) = x^{-2} E|S_n|^2.$$
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Main idea of proof: Consider first time maximum is exceeded. Bound below the expected square sum on that event.
Theorem: Let $X_1, X_2, \ldots$ be independent and fix $A > 0$. Write $Y_i = X_i1(|X_i| \leq A)$. Then $\sum X_i$ converges a.s. if and only if the following are all true:
Theorem: Let $X_1, X_2, \ldots$ be independent and fix $A > 0$. Write $Y_i = X_i 1(\{|X_i| \leq A\})$. Then $\sum X_i$ converges a.s. if and only if the following are all true:

- $\sum_{n=1}^{\infty} P(|X_n| > A) < \infty$
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1. $\sum_{n=1}^{\infty} P(|X_n| > A) < \infty$
2. $\sum_{n=1}^{\infty} EY_n$ converges
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1. $\sum_{n=1}^{\infty} P(|X_n| > A) < \infty$
2. $\sum_{n=1}^{\infty} EY_n$ converges
3. $\sum_{n=1}^{\infty} \text{Var}(Y_n) < \infty$

Main ideas behind the proof: Kolmogorov zero-one law implies that $\sum X_i$ converges with probability $p \in \{0, 1\}$. We just have to show that $p = 1$ when all hypotheses are satisfied (sufficiency of conditions) and $p = 0$ if any one of them fails (necessity).
Kolmogorov three-series theorem

Theorem: Let $X_1, X_2, \ldots$ be independent and fix $A > 0$. Write $Y_i = X_i 1(|X_i| \leq A)$. Then $\sum X_i$ converges a.s. if and only if the following are all true:

- $\sum_{n=1}^{\infty} P(|X_n| > A) < \infty$
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- $\sum_{n=1}^{\infty} \text{Var}(Y_n) < \infty$

Main ideas behind the proof: Kolmogorov zero-one law implies that $\sum X_i$ converges with probability $p \in \{0, 1\}$. We just have to show that $p = 1$ when all hypotheses are satisfied (sufficiency of conditions) and $p = 0$ if any one of them fails (necessity).

To prove sufficiency, apply Borel-Cantelli to see that probability that $X_n \neq Y_n$ i.o. is zero. Subtract means from $Y_n$, reduce to case that each $Y_n$ has mean zero. Apply Kolmogorov maximal inequality.