# 18.175: Lecture 3

# Integration

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# Outline

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Random variables

Integration

Expectation

# Recall definitions

- **Probability space** is triple $(\Omega, \mathcal{F}, P)$ where $\Omega$ is sample space, $\mathcal{F}$ is set of events (the $\sigma$-algebra) and $P : \mathcal{F} \to [0, 1]$ is the probability function.

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- Measure $\mu$ is **probability measure** if $\mu(\Omega) = 1$.
- The **Borel** $\sigma$-**algebra** $\mathcal{B}$ on a topological space is the smallest $\sigma$-algebra containing all open sets.

# Defining random variables

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- Example of random variable: indicator function of a set. Or sum of finitely many indicator functions of sets.
- Let $F(x) = F_X(x) = P(X \leq x)$ be **distribution function** for $X$. Write $f = f_X = F'_X$ for **density function** of $X$.

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- Can also define distribution functions for random variables that are a.s. integers (like Poisson or geometric or binomial random variables, say). How about for a ratio of two independent Poisson random variables? (This is a random rational with a dense support on $[0,\infty)$.)
- Higher dimensional density functions analogously defined.

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- Given infinite sequence of random variables, consider the event that they converge to a limit. Is this a measurable event?
- Yes. If it has measure one, we say sequence converges almost surely.

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  - $f$ is any measurable function (hint: treat positive/negative parts separately, difference makes sense if both integrals finite).

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  - $|\int f d\mu| \leq \int |f| d\mu$.
- When $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{R}^d, \lambda)$, write $\int_E f(x)dx = \int 1_E f d\lambda$.

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- $EX^k$ is called $k$**th moment of** $X$. Also, if $m = EX$ then $E(X - m)^2$ is called the **variance** of $X$.

# Properties of expectation/integration

- **Jensen's inequality:** If $\mu$ is probability measure and $\phi : \mathbb{R} \to \mathbb{R}$ is convex then $\phi(\int f d\mu) \leq \int \phi(f) d\mu$. If $X$ is random variable then $E\phi(X) \geq \phi(EX)$.

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- **Cauchy-Schwarz inequality:** Special case $p = q = 2$. Gives $\int |fg| d\mu \leq \|f\|_2 \|g\|_2$. Says that dot product of two vectors is at most product of vector lengths.

# Bounded convergence theorem

- **Bounded convergence theorem:** Consider *probability* measure $\mu$ and suppose $|f_n| \leq M$ a.s. for all $n$ and some fixed $M > 0$, and that $f_n \to f$ in probability (i.e., $\lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0$ for all $\epsilon > 0$). Then

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

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- **Main idea of proof:** for any $\epsilon$, $\delta$ can take $n$ large enough so $\int |f_n - f| d\mu < M\delta + \epsilon$.

# Fatou's lemma

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(Counterexample for opposite-direction inequality using thin and tall rectangles?)
- **Main idea of proof:** first reduce to case that the $f_n$ are increasing by writing $g_n(x) = \inf_{m\geq n} f_m(x)$ and observing that $g_n(x) \uparrow g(x) = \liminf_{n\to\infty} f_n(x)$. Then truncate, used bounded convergence, take limits.

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- **Dominated convergence:** If $f_n \to f$ a.e. and $|f_n| \leq g$ for all $n$ and $g$ is integrable, then $\int f_n d\mu \to \int f d\mu$.

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- **Main idea of proof:** Fatou for functions $g + f_n \geq 0$ gives one side. Fatou for $g - f_n \geq 0$ gives other.

# Computing expectations

- Change of variables. Measure space $(\Omega, \mathcal{F}, P)$. Let $X$ be random variable in $(S, \mathcal{S})$ with distribution $\mu$. Then if $f(S, \mathcal{S}) \to (R, \mathcal{R})$ is measurable we have $Ef(X) = \int_S f(y)\mu(dy)$.

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- Prove by checking for indicators, simple functions, non-negative functions, integrable functions.
- Examples: normal, exponential, Bernoulli, Poisson, geometric...