18.175: Lecture 26

Last lecture

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Outline

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Recollections

Strong Markov property

More σ-algebra thoughts

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- Write $\mathcal{F}_s^+ = \cap_{t>s}\mathcal{F}_t^o$
- Note right continuity: $\cap_{t>s}\mathcal{F}_t^+ = \mathcal{F}_s^+$.
- $\mathcal{F}_s^+$ allows an "infinitesimal peek at future"

Looking ahead

- **Expectation equivalence theorem** If Z is bounded and measurable then for all $s \geq 0$ and $x \in \mathbb{R}^d$ have

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- **Proof idea:** Consider case that $Z = \sum_{i=1}^m f_m(B(t_m))$ and the f_m are bounded and measurable. Kind of obvious in this case. Then use same measure theory as in Markov property proof to extend general Z.

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- **Proof idea:** Consider case that $Z = \sum_{i=1}^m f_m(B(t_m))$ and the f_m are bounded and measurable. Kind of obvious in this case. Then use same measure theory as in Markov property proof to extend general Z.
- **Observe:** If $Z \in \mathcal{F}_s^+$ then $Z = E_x(Z|\mathcal{F}_s^o)$. Conclude that $\mathcal{F}_s^+$ and $\mathcal{F}_s^o$ agree up to null sets.

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- There's nothing you can learn from infinitesimal neighborhood of future.
- **Proof:** If we have $A \in \mathcal{F}_0^+$, then previous theorem implies

$$1_A = E_x(1_A | \mathcal{F}_0^+) = E_x(1_A | \mathcal{F}_0^o) = P_x(A) \quad P_x \text{a.s.}$$

Markov property

- If $s \geq 0$ and Y is bounded and $\mathcal{C}$-measurable, then for all $x \in \mathbb{R}^d$, we have

$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_{B_s} Y,$$

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where the RHS is function $\phi(x) = E_x Y$ evaluated at $x = B_s$.
- **Proof idea:** First establish this for some simple functions Y (depending on finitely many time values) and then use measure theory (monotone class theorem) to extend to general case.

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- If $T_0 = \inf\{t > 0 : B_t = 0\}$ then $P_0(T_0 = 0) = 1$.
- If B_t is Brownian motion started at 0, then so is process defined by $X_0 = 0$ and $X_t = tB(1/t)$. (Proved by checking $E(X_s X_t) = stE(B(1/s)B(1/t)) = s$ when $s < t$. Then check continuity at zero.)

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- Distinction between $\{S < t\}$ and $\{S \leq t\}$ doesn't make a difference for a right continuous filtration.
- Example: let $S = \inf\{t : B_t \in A\}$ for some open (or closed) set A.

Strong Markov property

- Let $(s, \omega) \to Y_s(\omega)$ be bounded and $\mathcal{R} \times \mathcal{C}$-measurable. If S is a stopping time, then for all $x \in \mathbb{R}^d$

$$E_x(Y_S \circ \theta_S | \mathcal{F}_S) = E_{B(S)} Y_S \text{ on } \{S < \infty\},$$

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- Extend optional stopping to continuous martingales similarly.

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- **Question:** Suppose B_t is a one dimensional Brownian motion, and $g_t : \mathbb{C} \to \mathbb{C}$ is determined by solving the ODE

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - 2B_t}, \quad g_0(z) = z.$$

Is $\arg(g_t(z) - W_t)$ a martingale?

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- Thanks for taking the class!