18.175: Lecture 18

More on martingales

Scott Sheffield

MIT

Outline

Conditional expectation

Regular conditional probabilities

Martingales

Arcsin law, other SRW stories

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Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of** X **given** $\mathcal{F}$ is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.

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- **Theorem:** Up to redefinition on a measure zero set, the random variable $E(X|\mathcal{F})$ exists and is unique.
- This follows from Radon-Nikodym theorem.

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- Second is kind of interesting: says, after I learn $\mathcal{F}_1$, my best guess of what my best guess for X will be after learning $\mathcal{F}_2$ is simply my current best guess for X.
- Deduce that $E(X|\mathcal{F}_i)$ is a martingale if $\mathcal{F}_i$ is an increasing sequence of σ-algebras and $E(|X|) < \infty$.

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  - For each A, $\omega \to \mu(\omega, A)$ is a version of $P(X \in A|\mathcal{G})$.
  - For a.e. ω, $A \to \mu(\omega, A)$ is a probability measure on $(S, \mathcal{S})$.
- **Theorem:** Regular conditional probabilities exist if $(S, \mathcal{S})$ is nice.

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- A sequence X_n is **adapted** to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ for all n. If X_n is an adapted sequence (with $E|X_n| < \infty$) then it is called a **martingale** if
$$E(X_{n+1}|\mathcal{F}_n) = X_n$$
for all n. It's a **supermartingale** (resp., **submartingale**) if same thing holds with $=$ replaced by $\leq$ (resp., $\geq$).

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- Example: take $\phi(x) = \max\{x, 0\}$.

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- Example: take $H_n = 1_{N \geq n}$ for stopping time N.

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- **Stronger convergence statement:** If X_n is a submartingale with $\sup EX_n^+ < \infty$ then as $n \to \infty$, X_+n converges a.s. to a limit X with $E|X| < \infty$.

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- **Proof idea:** Just let M_n be sum of "surprises" (i.e., the values $X_n - E(X_n|\mathcal{F}_{n-1})$).
- A martingale with bounded increments a.s. either converges to limit or oscillates between $\pm\infty$. That is, a.s. either $\lim X_n < \infty$ exists or $\limsup X_n = +\infty$ and $\liminf X_n = -\infty$.

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- Polya's urn: r red and g green balls. Repeatedly sample randomly and add extra ball of sampled color. Ratio of red to green is martingale, hence a.s. converges to limit.

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- Suppose that you expect to get married once during your life How many people do you expect will reach the point that you would say you have a twenty five percent chance to marry them?
- Compute probability of having a continuously updated conditional probability reach a before b.

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- **Wald's second equation:** Let X_i be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$.

Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}\}$. What is $\mathbb{E}S_N$?

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- Try counting walks that *do* cross by giving bijection to walks from $(0, -x)$ to (n, y).

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- Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.

Arcsin theorem

- Theorem for last hitting time.

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- Theorem for amount of positive positive time.