18.175: Lecture 17

Martingales

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Outline

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Conditional expectation

Regular conditional probabilities

Martingales

Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of X given $\mathcal{F}$** is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.

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- This follows from Radon-Nikodym theorem.

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- Second is kind of interesting: says, after I learn $\mathcal{F}_1$, my best guess of what my best guess for X will be after learning $\mathcal{F}_2$ is simply my current best guess for X.
- Deduce that $E(X|\mathcal{F}_i)$ is a martingale if $\mathcal{F}_i$ is an increasing sequence of σ-algebras and $E(|X|) < \infty$.

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  - For each A, $\omega \to \mu(\omega, A)$ is a version of $P(X \in A|\mathcal{G})$.
  - For a.e. ω, $A \to \mu(\omega, A)$ is a probability measure on $(S, \mathcal{S})$.
- **Theorem:** Regular conditional probabilities exist if $(S, \mathcal{S})$ is nice.

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- A sequence X_n is **adapted** to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ for all n. If X_n is an adapted sequence (with $E|X_n| < \infty$) then it is called a **martingale** if
$$E(X_{n+1}|\mathcal{F}_n) = X_n$$
for all n. It's a **supermartingale** (resp., **submartingale**) if same thing holds with $=$ replaced by $\leq$ (resp., $\geq$).

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- Example: take $\phi(x) = \max\{x, 0\}$.

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- Example: take $H_n = 1_{N \geq n}$ for stopping time N.

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- **Proof:** Just a special case of statement about $(H \cdot X)$.
- **Martingale convergence:** A non-negative martingale almost surely has a limit.
- **Idea of proof:** Count upcrossings (times martingale crosses a fixed interval) and devise gambling strategy that makes lots of money if the number of these is not a.s. finite.