18.175: Lecture 16

Conditional expectation, random walks, martingales

Scott Sheffield

MIT
Outline

Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
Say we’re given a probability space $(\Omega, \mathcal{F}_0, P)$ and a $\sigma$-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable $X$ measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The \textbf{conditional expectation of $X$ given $\mathcal{F}$} is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$. 
Conditional expectation

Say we’re given a probability space \((\Omega, \mathcal{F}_0, P)\) and a \(\sigma\)-field \(\mathcal{F} \subset \mathcal{F}_0\) and a random variable \(X\) measurable w.r.t. \(\mathcal{F}_0\), with \(E|X| < \infty\). The **conditional expectation of \(X\) given \(\mathcal{F}\)** is a new random variable, which we can denote by \(Y = E(X|\mathcal{F})\).

We require that \(Y\) is \(\mathcal{F}\) measurable and that for all \(A\) in \(\mathcal{F}\), we have \(\int_A XdP = \int_A YdP\).
Conditional expectation

Say we’re given a probability space \((\Omega, \mathcal{F}_0, P)\) and a \(\sigma\)-field \(\mathcal{F} \subset \mathcal{F}_0\) and a random variable \(X\) measurable w.r.t. \(\mathcal{F}_0\), with \(E|X| < \infty\). The **conditional expectation of \(X\) given \(\mathcal{F}\)** is a new random variable, which we can denote by \(Y = E(X|\mathcal{F})\).

We require that \(Y\) is \(\mathcal{F}\) measurable and that for all \(A\) in \(\mathcal{F}\), we have \(\int_A XdP = \int_A YdP\).

Any \(Y\) satisfying these properties is called a **version** of \(E(X|\mathcal{F})\).
Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a $\sigma$-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable $X$ measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The conditional expectation of $X$ given $\mathcal{F}$ is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.

We require that $Y$ is $\mathcal{F}$ measurable and that for all $A$ in $\mathcal{F}$, we have $\int_A XdP = \int_A YdP$.

Any $Y$ satisfying these properties is called a version of $E(X|\mathcal{F})$.

Is it possible that there exists more than one version of $E(X|\mathcal{F})$ (which would mean that in some sense the conditional expectation is not canonically defined)?
Conditional expectation

Say we’re given a probability space \((\Omega, \mathcal{F}_0, P)\) and a \(\sigma\)-field \(\mathcal{F} \subset \mathcal{F}_0\) and a random variable \(X\) measurable w.r.t. \(\mathcal{F}_0\), with \(E|X| < \infty\). The \textbf{conditional expectation of} \(X\) \textbf{given} \(\mathcal{F}\) is a new random variable, which we can denote by \(Y = E(X|\mathcal{F})\).

We require that \(Y\) is \(\mathcal{F}\) measurable and that for all \(A\) in \(\mathcal{F}\), we have \(\int_A XdP = \int_A YdP\).

Any \(Y\) satisfying these properties is called a \textbf{version} of \(E(X|\mathcal{F})\).

Is it possible that there exists more than one version of \(E(X|\mathcal{F})\) (which would mean that in some sense the conditional expectation is not canonically defined)?

Is there some sense in which \(E(X|\mathcal{F})\) always exists and is always uniquely defined (maybe up to set of measure zero)?
Claim: Assuming $Y = E(X|\mathcal{F})$ as above, and $E|X| < \infty$, we have $E|Y| \leq E|X|$. In particular, $Y$ is integrable.
Claim: Assuming $Y = E(X|\mathcal{F})$ as above, and $E|X| < \infty$, we have $E|Y| \leq E|X|$. In particular, $Y$ is integrable.

Proof: Let $A = \{Y > 0\} \in \mathcal{F}$ and observe:

$$\int_A YdP = \int_AXdP \leq \int_A |X|dP.$$  

By similar argument,

$$\int_{A^c} YdP \leq \int_{A^c} |X|dP.$$  

Uniqueness of $Y$: Suppose $Y'$ is $\mathcal{F}$-measurable and satisfies

$$\int_A Y'dP = \int_A XdP = \int_A YdP$$

for all $A \in \mathcal{F}$. Then consider the set $Y - Y' \geq \epsilon$. Integrating over that gives zero. Must hold for any $\epsilon$. Conclude that $Y = Y'$ almost everywhere.
Claim: Assuming $Y = E(X|\mathcal{F})$ as above, and $E|X| < \infty$, we have $E|Y| \leq E|X|$. In particular, $Y$ is integrable.

Proof: let $A = \{ Y > 0 \} \in \mathcal{F}$ and observe:
\[
\int_A YdP = \int_A XdP \leq \int_A |X|dP.
\]
By similar argument,
\[
\int_{A^c} YdP \leq \int_{A^c} |X|dP.
\]
Uniqueness of $Y$: Suppose $Y'$ is $\mathcal{F}$-measurable and satisfies
\[
\int_A Y'dP = \int_A XdP = \int_A YdP \text{ for all } A \in \mathcal{F}.
\]
Then consider the set $Y - Y' \geq \epsilon$. Integrating over that gives zero. Must hold for any $\epsilon$. Conclude that $Y = Y'$ almost everywhere.
Let $\mu$ and $\nu$ be $\sigma$-finite measures on $(\Omega, \mathcal{F})$. Say $\nu << \mu$ (or $\nu$ is absolutely continuous w.r.t. $\mu$) if $\mu(A) = 0$ implies $\nu(A) = 0$. 

Recall Radon-Nikodym theorem: If $\mu$ and $\nu$ are $\sigma$-finite measures on $(\Omega, \mathcal{F})$ and $\nu$ is absolutely continuous w.r.t. $\mu$, then there exists a measurable $f: \Omega \rightarrow [0, \infty)$ such that $\nu(A) = \int_A f \, d\mu$.

Observe: this theorem implies existence of conditional expectation.
Radon-Nikodym theorem

- Let $\mu$ and $\nu$ be $\sigma$-finite measures on $(\Omega, \mathcal{F})$. Say $\nu \ll \mu$ (or $\nu$ is **absolutely continuous w.r.t.** $\mu$) if $\mu(A) = 0$ implies $\nu(A) = 0$.

- Recall **Radon-Nikodym theorem**: If $\mu$ and $\nu$ are $\sigma$-finite measures on $(\Omega, \mathcal{F})$ and $\nu$ is absolutely continuous w.r.t. $\mu$, then there exists a measurable $f : \Omega \to [0, \infty)$ such that $\nu(A) = \int_A f \, d\mu$. 

Observe: this theorem implies existence of conditional expectation.
Let $\mu$ and $\nu$ be $\sigma$-finite measures on $(\Omega, \mathcal{F})$. Say $\nu \ll \mu$ (or $\nu$ is absolutely continuous w.r.t. $\mu$) if $\mu(A) = 0$ implies $\nu(A) = 0$.

Recall **Radon-Nikodym theorem**: If $\mu$ and $\nu$ are $\sigma$-finite measures on $(\Omega, \mathcal{F})$ and $\nu$ is absolutely continuous w.r.t. $\mu$, then there exists a measurable $f : \Omega \to [0, \infty)$ such that $\nu(A) = \int_A f d\mu$.

Observe: this theorem implies existence of conditional expectation.
Outline

Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
Outline

Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
Two big results

- **Optional stopping theorem**: Can’t make money in expectation by timing sale of asset whose price is non-negative martingale.
Two big results

- **Optional stopping theorem**: Can’t make money in expectation by timing sale of asset whose price is non-negative martingale.

- **Martingale convergence**: A non-negative martingale almost surely has a limit.
Outline

Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
Outline

Conditional expectation

Martingales

Random walks

Stopping times

Arcsine law, other SRW stories
Exchangeable events

Start with measure space $(S, S, \mu)$. Let

$\Omega = \{(\omega_1, \omega_2, \ldots) : \omega_i \in S\}$, let $\mathcal{F}$ be product $\sigma$-algebra and $P$ the product probability measure.
Exchangeable events

- Start with measure space \((S, S, \mu)\). Let \(\Omega = \{(\omega_1, \omega_2, \ldots) : \omega_i \in S\}\), let \(\mathcal{F}\) be product \(\sigma\)-algebra and \(P\) the product probability measure.

- **Finite permutation** of \(\mathbb{N}\) is one-to-one map from \(\mathbb{N}\) to itself that fixes all but finitely many points.
Exchangeable events

- Start with measure space \((S, S, \mu)\). Let \(\Omega = \{(\omega_1, \omega_2, \ldots) : \omega_i \in S\}\), let \(\mathcal{F}\) be product \(\sigma\)-algebra and \(P\) the product probability measure.

- **Finite permutation** of \(\mathbb{N}\) is a one-to-one map from \(\mathbb{N}\) to itself that fixes all but finitely many points.

- Event \(A \in \mathcal{F}\) is permutable if it is invariant under any finite permutation of the \(\omega_i\).
Exchangeable events

- Start with measure space \((S, \mathcal{S}, \mu)\). Let 
  \(\Omega = \{ (\omega_1, \omega_2, \ldots) : \omega_i \in S \}\), let \(\mathcal{F}\) be product \(\sigma\)-algebra and
  \(P\) the product probability measure.

- **Finite permutation** of \(\mathbb{N}\) is one-to-one map from \(\mathbb{N}\) to itself that fixes all but finitely many points.

- Event \(A \in \mathcal{F}\) is permutable if it is invariant under any finite permutation of the \(\omega_i\).

- Let \(\mathcal{E}\) be the \(\sigma\)-field of permutable events.
Exchangeable events

- Start with measure space \((S, S, \mu)\). Let
  \(\Omega = \{(\omega_1, \omega_2, \ldots) : \omega_i \in S\}\), let \(F\) be product \(\sigma\)-algebra and
  \(P\) the product probability measure.

- **Finite permutation** of \(\mathbb{N}\) is one-to-one map from \(\mathbb{N}\) to itself
  that fixes all but finitely many points.

- Event \(A \in F\) is permutable if it is invariant under any finite
  permutation of the \(\omega_i\).

- Let \(E\) be the \(\sigma\)-field of permutable events.

- This is related to the tail \(\sigma\)-algebra we introduced earlier in
  the course. Bigger or smaller?
Hewitt-Savage 0-1 law

- If $X_1, X_2, \ldots$ are i.i.d. and $A \in \mathcal{A}$ then $P(A) \in \{0, 1\}$. 

Idea of proof:
Try to show $A$ is independent of itself, i.e., that $P(A) = P(A \cap A) = P(A)P(A)$. Start with measure-theoretic fact that we can approximate $A$ by a set $A_n$ in the $\sigma$-algebra generated by $X_1, \ldots, X_n$, so that symmetric difference of $A$ and $A_n$ has very small probability. Note that $A_n$ is independent of event $A_n'$ that $A_n$ holds when $X_1, \ldots, X_n$ and $X_1', \ldots, X_2'$ are swapped. Symmetric difference between $A$ and $A_n$ is also small, so $A$ is independent of itself up to this small error. Then make error arbitrarily small.
Hewitt-Savage 0-1 law

- If $X_1, X_2, \ldots$ are i.i.d. and $A \in \mathcal{A}$ then $P(A) \in \{0, 1\}$.

- **Idea of proof:** Try to show $A$ is independent of itself, i.e., that $P(A) = P(A \cap A) = P(A)P(A)$. Start with measure theoretic fact that we can approximate $A$ by a set $A_n$ in $\sigma$-algebra generated by $X_1, \ldots, X_n$, so that symmetric difference of $A$ and $A_n$ has very small probability. Note that $A_n$ is independent of event $A'_n$ that $A_n$ holds when $X_1, \ldots, X_n$ and $X_{n_1}, \ldots, X_{2n}$ are swapped. Symmetric difference between $A$ and $A'_n$ is also small, so $A$ is independent of itself up to this small error. Then make error arbitrarily small.
Application of Hewitt-Savage:

- If $X_i$ are i.i.d. in $\mathbb{R}^n$ then $S_n = \sum_{i=1}^{n} X_i$ is a random walk on $\mathbb{R}^n$. 

Theorem: if $S_n$ is a random walk on $\mathbb{R}$ then one of the following occurs with probability one:

- $S_n = 0$ for all $n$
- $S_n \to \infty$
- $S_n \to -\infty$
- $-\infty = \lim inf S_n < \lim sup S_n = \infty$

Idea of proof: Hewitt-Savage implies the $\lim sup S_n$ and $\lim inf S_n$ are almost sure constants in $[-\infty, \infty]$. Note that if $X_1$ is not a.s. constant, then both values would depend on $X_1$ if they were not in $\pm \infty$. 

18.175 Lecture 16
Application of Hewitt-Savage:

- If $X_i$ are i.i.d. in $\mathbb{R}^n$ then $S_n = \sum_{i=1}^n X_i$ is a **random walk** on $\mathbb{R}^n$.

- **Theorem:** if $S_n$ is a random walk on $\mathbb{R}$ then one of the following occurs with probability one:
  
  - $S_n = 0$ for all $n$
  - $S_n \to \infty$
  - $S_n \to -\infty$
  - $-\infty = \lim \inf S_n < \lim \sup S_n = \infty$

- **Idea of proof:** Hewitt-Savage implies the $\lim \sup S_n$ and $\lim \inf S_n$ are almost sure constants in $[-\infty, \infty]$. Note that if $X_1$ is not a.s. constant, then both values would depend on $X_1$ if they were not in $\pm \infty$. 
Application of Hewitt-Savage:

- If $X_i$ are i.i.d. in $\mathbb{R}^n$ then $S_n = \sum_{i=1}^{n} X_i$ is a random walk on $\mathbb{R}^n$.
- **Theorem:** if $S_n$ is a random walk on $\mathbb{R}$ then one of the following occurs with probability one:
  - $S_n = 0$ for all $n$
Application of Hewitt-Savage:

- If $X_i$ are i.i.d. in $\mathbb{R}^n$ then $S_n = \sum_{i=1}^n X_i$ is a **random walk** on $\mathbb{R}^n$.

- **Theorem:** if $S_n$ is a random walk on $\mathbb{R}$ then one of the following occurs with probability one:
  - $S_n = 0$ for all $n$
  - $S_n \to \infty$
  - $S_n \to -\infty$
  - $-\infty = \lim \inf S_n < \lim \sup S_n = \infty$
Application of Hewitt-Savage:

- If $X_i$ are i.i.d. in $\mathbb{R}^n$ then $S_n = \sum_{i=1}^{n} X_i$ is a **random walk** on $\mathbb{R}^n$.

- **Theorem:** if $S_n$ is a random walk on $\mathbb{R}$ then one of the following occurs with probability one:
  - $S_n = 0$ for all $n$
  - $S_n \to \infty$
  - $S_n \to -\infty$

  Idea of proof: Hewitt-Savage implies the lim sup $S_n$ and lim inf $S_n$ are almost sure constants in $[-\infty, \infty]$. Note that if $X_1$ is not a.s. constant, then both values would depend on $X_1$ if they were not in $\pm \infty$. 
Application of Hewitt-Savage:

- If $X_i$ are i.i.d. in $\mathbb{R}^n$ then $S_n = \sum_{i=1}^{n} X_i$ is a **random walk** on $\mathbb{R}^n$.

- **Theorem**: if $S_n$ is a random walk on $\mathbb{R}$ then one of the following occurs with probability one:
  - $S_n = 0$ for all $n$
  - $S_n \to \infty$
  - $S_n \to -\infty$
  - $-\infty = \lim inf S_n < \lim sup S_n = \infty$
Application of Hewitt-Savage:

- If $X_i$ are i.i.d. in $\mathbb{R}^n$ then $S_n = \sum_{i=1}^{n} X_i$ is a random walk on $\mathbb{R}^n$.

- **Theorem:** if $S_n$ is a random walk on $\mathbb{R}$ then one of the following occurs with probability one:
  - $S_n = 0$ for all $n$
  - $S_n \to \infty$
  - $S_n \to -\infty$
  - $-\infty = \lim \inf S_n < \lim \sup S_n = \infty$

- **Idea of proof:** Hewitt-Savage implies the lim sup $S_n$ and lim inf $S_n$ are almost sure constants in $[-\infty, \infty]$. Note that if $X_1$ is not a.s. constant, then both values would depend on $X_1$ if they were not in $\pm \infty$. 
Outline

Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
▶ Say that $T$ is a **stopping time** if the event that $T = n$ is in $\mathcal{F}_i$ for $i \leq n$. 
Say that $T$ is a **stopping time** if the event that $T = n$ is in $\mathcal{F}_n$ for $i \leq n$.

In finance applications, $T$ might be the time one sells a stock. Then this states that the decision to sell at time $n$ depends only on prices up to time $n$, not on (as yet unknown) future prices.
Let $A_1, \ldots$ be i.i.d. random variables equal to $-1$ with probability $0.5$ and $1$ with probability $0.5$ and let $X_0 = 0$ and $X_n = \sum_{i=1}^{n} A_i$ for $n \geq 0$.

Which of the following is a stopping time?

1. The smallest $T$ for which $|X_T| = 50$
2. The smallest $T$ for which $X_T \in \{-10, 100\}$
3. The smallest $T$ for which $X_T = 0$.
4. The $T$ at which the $X_n$ sequence achieves the value 17 for the 9th time.
5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T$ is largest.
6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$.

Answer: first four, not last two.
Let $A_1, \ldots$ be i.i.d. random variables equal to $-1$ with probability $0.5$ and $1$ with probability $0.5$ and let $X_0 = 0$ and $X_n = \sum_{i=1}^n A_i$ for $n \geq 0$.

Which of the following is a stopping time?

1. The smallest $T$ for which $|X_T| = 50$
2. The smallest $T$ for which $X_T \in \{-10, 100\}$
3. The smallest $T$ for which $X_T = 0$.
4. The $T$ at which the $X_n$ sequence achieves the value 17 for the 9th time.
5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T$ is largest.
6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$. 

Answer: first four, not last two.
Let $A_1, \ldots$ be i.i.d. random variables equal to $-1$ with probability $0.5$ and 1 with probability $0.5$ and let $X_0 = 0$ and $X_n = \sum_{i=1}^{n} A_i$ for $n \geq 0$.

Which of the following is a stopping time?

1. The smallest $T$ for which $|X_T| = 50$
2. The smallest $T$ for which $X_T \in \{-10, 100\}$
3. The smallest $T$ for which $X_T = 0$.
4. The $T$ at which the $X_n$ sequence achieves the value 17 for the 9th time.
5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T$ is largest.
6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$. 

Answer: first four, not last two.
Let $A_1, \ldots$ be i.i.d. random variables equal to $-1$ with probability $0.5$ and $1$ with probability $0.5$ and let $X_0 = 0$ and $X_n = \sum_{i=1}^{n} A_i$ for $n \geq 0$.

Which of the following is a stopping time?

1. The smallest $T$ for which $|X_T| = 50$
2. The smallest $T$ for which $X_T \in \{-10, 100\}$
3. The smallest $T$ for which $X_T = 0$.
4. The $T$ at which the $X_n$ sequence achieves the value $17$ for the $9$th time.
5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T$ is largest.
6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$.

Answer: first four, not last two.
Theorem: Let $X_1, X_2, \ldots$ be i.i.d. and $N$ a stopping time with $N < \infty$. 
Theorem: Let $X_1, X_2, \ldots$ be i.i.d. and $N$ a stopping time with $N < \infty$.

Conditioned on stopping time $N < \infty$, conditional law of $\{X_{N+n}, n \geq 1\}$ is independent of $\mathcal{F}_n$ and has same law as original sequence.
Stopping time theorems

- **Theorem:** Let $X_1, X_2, \ldots$ be i.i.d. and $N$ a stopping time with $N < \infty$.

- Conditioned on stopping time $N < \infty$, conditional law of $\{X_{N+n}, n \geq 1\}$ is independent of $\mathcal{F}_n$ and has same law as original sequence.

- **Wald’s equation:** Let $X_i$ be i.i.d. with $E|X_i| < \infty$. If $N$ is a stopping time with $EN < \infty$ then $ES_N = EX_1 EN$. 
**Theorem:** Let $X_1, X_2, \ldots$ be i.i.d. and $N$ a stopping time with $N < \infty$.

Conditioned on stopping time $N < \infty$, conditional law of \{\(X_{N+n}, n \geq 1\)\} is independent of $\mathcal{F}_n$ and has same law as original sequence.

**Wald’s equation:** Let $X_i$ be i.i.d. with $E|X_i| < \infty$. If $N$ is a stopping time with $EN < \infty$ then $ES_N = EX_1 EN$.

**Wald’s second equation:** Let $X_i$ be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If $N$ is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$. 
Wald’s equation: Let $X_i$ be i.i.d. with $E|X_i| < \infty$. If $N$ is a stopping time with $EN < \infty$ then $ES_N = EX_1EN$. 
Wald

- **Wald’s equation:** Let $X_i$ be i.i.d. with $E|X_i| < \infty$. If $N$ is a stopping time with $EN < \infty$ then $ES_N = EX_1EN$.

- **Wald’s second equation:** Let $X_i$ be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If $N$ is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$. 

18.175 Lecture 16
Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step $S_j$ independently changes by $\pm 1$ according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}\}$. What is $\mathbb{E}S_N$?
S_0 = a \in \mathbb{Z} \text{ and at each time step } S_j \text{ independently changes by } \pm 1 \text{ according to a fair coin toss. Fix } A \in \mathbb{Z} \text{ and let } N = \inf\{ k : S_k \in \{0, A\} \}. \text{ What is } \mathbb{E} S_N? \\
\text{What is } \mathbb{E} N?
Outline

Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
Outline

Conditional expectation

Martingales

Random walks

Stopping times

Arcsin law, other SRW stories
Reflection principle

How many walks from \((0, x)\) to \((n, y)\) that don’t cross the horizontal axis?
Reflection principle

- How many walks from \((0, x)\) to \((n, y)\) that don’t cross the horizontal axis?
- Try counting walks that \textit{do} cross by giving bijection to walks from \((0, -x)\) to \((n, y)\).
Suppose that in election candidate $A$ gets $\alpha$ votes and $B$ gets $\beta < \alpha$ votes. What’s probability that $A$ is a head throughout the counting?

Answer: $\frac{\alpha - \beta}{\alpha + \beta}$. Can be proved using reflection principle.
Suppose that in election candidate $A$ gets $\alpha$ votes and $B$ gets $\beta < \alpha$ votes. What’s probability that $A$ is a head throughout the counting?

Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.
Theorem for last hitting time.
Arcsin theorem

- Theorem for last hitting time.
- Theorem for amount of positive positive time.