18.175: Lecture 17

Poisson random variables

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Outline

More on random walks and local CLT

Poisson random variable convergence

Extend CLT idea to stable random variables

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- Write $p_n(x) = P(S_n/\sqrt{n} = x)$ for $x \in \mathcal{L}_n := (nb + h\mathbb{Z})/\sqrt{n}$ and $n(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$.

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- Assume X_i are i.i.d. lattice with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0,\infty)$. **Theorem:** As $n \to \infty$,

$$\sup_{x \in \mathcal{L}^n} \left|\frac{n^{1/2}}{h} p_n(x) - n(x)\right| \to 0.$$

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- **Proof idea:** Use characteristic functions, reduce to periodic integral problem. Look up "Fourier series". Note that for Y supported on $a + \theta\mathbb{Z}$, we have $P(Y = x) = \frac{1}{2\pi/\theta} \int_{-\pi/\theta}^{\pi/\theta} e^{-itx} \phi_Y(t) dt$.

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- How about a random walk on $\mathbb{Z}^2$?
- Can one use this to establish when a random walk on $\mathbb{Z}^d$ is recurrent versus transient?

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- **Key idea for all these examples:** Divide time into large number of small increments. Assume that during each increment, there is some small probability of thing happening (independently of other increments).

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- Use Taylor expansion $e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$.

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- Setting $j = k - 1$, this is $\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}e^{-\lambda} = \lambda$.

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- Then $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$.

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- **Theorem:** Let $X_{n,m}$ be independent $\{0,1\}$-valued random variables with $P(X_{n,m} = 1) = p_{n,m}$. Suppose $\sum_{m=1}^{n} p_{n,m} \to \lambda$ and $\max_{1 \le m \le n} p_{n,m} \to 0$. Then $S_n = X_{n,1} + \ldots + X_{n,n} \implies Z$ were Z is Poisson(λ).

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- **Proof idea:** Just write down the log characteristic functions for Bernoulli and Poisson random variables. Check the conditions of the continuity theorem.

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- **Strong continuity theorem:** If $\mu_n \implies \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all t. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions μ_n is tight and converges weakly to a measure μ with characteristic function ϕ.

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- When we zoom in on a twice differentiable function near zero (scaling vertically by n and horizontally by $\sqrt{n}$) the picture looks increasingly like a parabola.

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- We already saw that this should work for Cauchy random variables. What's the characteristic function in that case?

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- We already saw that this should work for Cauchy random variables. What's the characteristic function in that case?
- Let's look up stable distributions.

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- More general constructions are possible via Lévy Khintchine representation.