18.175: Lecture 17

Poisson random variables

Scott Sheffield

MIT
Outline

More on random walks and local CLT

Poisson random variable convergence

Extend CLT idea to stable random variables
Outline

More on random walks and local CLT

Poisson random variable convergence

Extend CLT idea to stable random variables
Recall local CLT for walks on $\mathbb{Z}$

- Suppose $X \in b + h\mathbb{Z}$ a.s. for some fixed constants $b$ and $h$. 

Assume $X_i$ are i.i.d. lattice with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$.
Recall local CLT for walks on $\mathbb{Z}$

- Suppose $X \in b + h\mathbb{Z}$ a.s. for some fixed constants $b$ and $h$.
- Observe that if $\phi_X(\lambda) = 1$ for some $\lambda \neq 0$ then $X$ is supported on (some translation of) $(2\pi/\lambda)\mathbb{Z}$. If this holds for all $\lambda$, then $X$ is a.s. some constant. When the former holds but not the latter (i.e., $\phi_X$ is periodic but not identically 1) we call $X$ a **lattice random variable**.
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- Write $p_n(x) = P(S_n/\sqrt{n} = x)$ for $x \in \mathcal{L}_n := (nb + h\mathbb{Z})/\sqrt{n}$ and $n(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$.
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- Assume $X_i$ are i.i.d. lattice with $E X_i = 0$ and $E X_i^2 = \sigma^2 \in (0, \infty)$. **Theorem:** As $n \to \infty$,

$$\sup_{x \in \mathcal{L}^n} \left| \frac{n^{1/2}}{h} p_n(x) - n(x) \right| \to 0.$$
Recall local CLT for walks on $\mathbb{Z}$

**Proof idea:** Use characteristic functions, reduce to periodic integral problem. Look up “Fourier series”. Note that for $Y$ supported on $a + \theta \mathbb{Z}$, we have

$$P(Y = x) = \frac{1}{2\pi/\theta} \int_{-\pi/\theta}^{\pi/\theta} e^{-itx} \phi_Y(t) dt.$$
Extending this idea to higher dimensions

Example: suppose we have random walk on $\mathbb{Z}$ that at each step tosses fair 4-sided coin to decide whether to go 1 unit left, 1 unit right, 2 units left, or 2 units right?
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What is the probability that the walk is back at the origin after one step? Two steps? Three steps?
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Let’s compute this in Mathematica by writing out the characteristic function $\phi_X$ for one-step increment $X$ and calculating $\int_0^{2\pi} \phi_X^k(t)dt/2\pi$. 

How about a random walk on $\mathbb{Z}^2$?

Can one use this to establish when a random walk on $\mathbb{Z}^d$ is recurrent versus transient?
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Poisson random variables: motivating questions

- How many raindrops hit a given square inch of sidewalk during a ten minute period?
- How many people fall down the stairs in a major city on a given day?
- How many plane crashes in a given year?
- How many radioactive particles emitted during a time period in which the expected number emitted is 5?
- How many calls to call center during a given minute?
- How many goals scored during a 90 minute soccer game?
- How many notable gaffes during 90 minute debate?

Key idea for all these examples: Divide time into large number of small increments. Assume that during each increment, there is some small probability of thing happening (independently of other increments).
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**Key idea for all these examples:** Divide time into large number of small increments. Assume that during each increment, there is some small probability of thing happening (independently of other increments).
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Binomial formula:
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\binom{n}{k} p^k (1 - p)^{n-k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} p^k (1 - p)^{n-k}.
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A Poisson random variable $X$ with parameter $\lambda$ satisfies $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$. 
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How can we show that $\sum_{k=0}^{\infty} p(k) = 1$?

Use Taylor expansion $e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$.
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$$P\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda}$$

for integer $k \geq 0$. 

What is $E[X]$?

We think of a Poisson random variable as being (roughly) a Bernoulli ($n, p$) random variable with $n$ very large and $p = \lambda/n$.

This would suggest $E[X] = \lambda$. Can we show this directly from the formula for $P\{X = k\}$?

By definition of expectation $E[X] = \sum_{k=0}^{\infty} P\{X = k\} k = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}e^{-\lambda}k$.

Setting $j = k - 1$, this is

$$\lambda\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}e^{-\lambda}(j - 1)! = \lambda.$$
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Setting $j = k - 1$, this is $\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} e^{-\lambda} = \lambda$. 
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This suggests $\text{Var}[X] \approx npq \approx \lambda$ (since $np \approx \lambda$ and $q = 1 - p \approx 1$). Can we show directly that $\text{Var}[X] = \lambda$?
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Compute

$$E[X^2] = \sum_{k=0}^{\infty} P\{X = k\} k^2 = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}.$$
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Setting $j = k - 1$, this is

$$\lambda \left( \sum_{j=0}^{\infty} (j + 1) \frac{\lambda^j}{j!} e^{-\lambda} \right) = \lambda E[X + 1] = \lambda(\lambda + 1).$$
Variance

Given \( P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda} \) for integer \( k \geq 0 \), what is \( \text{Var}[X] \)?

Think of \( X \) as (roughly) a Bernoulli \((n, p)\) random variable with \( n \) very large and \( p = \lambda/n \).

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Then \( \text{Var}[X] = E[X^2] - E[X]^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda. \)
Poisson convergence

- Idea: if we have lots of independent random events, each with very small probability to occur, and expected number to occur is $\lambda$, then total number that occur is roughly Poisson $\lambda$. 

**Theorem:**

Let $X_n, m$ be independent $\{0, 1\}$-valued random variables with $P(X_n, m = 1) = p_{n, m}$. Suppose $\sum_{n, m = 1}^{\infty} p_{n, m} \rightarrow \lambda$ and $\max_{1 \leq m \leq n} p_{n, m} \rightarrow 0$. Then $S_n = X_{n, 1} + \ldots + X_{n, n} \Rightarrow Z$ were $Z$ is Poisson ($\lambda$).

**Proof idea:** Just write down the log characteristic functions for Bernoulli and Poisson random variables. Check the conditions of the continuity theorem.
Poisson convergence

- **Idea**: if we have lots of independent random events, each with very small probability to occur, and expected number to occur is \(\lambda\), then total number that occur is roughly Poisson \(\lambda\).

- **Theorem**: Let \(X_{n,m}\) be independent \(\{0, 1\}\)-valued random variables with \(P(X_{n,m} = 1) = p_{n,m}\). Suppose \(\sum_{m=1}^{n} p_{n,m} \to \lambda\) and \(\max_{1 \leq m \leq n} p_{n,m} \to 0\). Then \(S_n = X_{n,1} + \ldots + X_{n,n} \Rightarrow Z\) were \(Z\) is Poisson\((\lambda)\).
Poisson convergence

- **Idea:** if we have lots of independent random events, each with very small probability to occur, and expected number to occur is $\lambda$, then total number that occur is roughly Poisson $\lambda$.

- **Theorem:** Let $X_{n,m}$ be independent $\{0, 1\}$-valued random variables with $P(X_{n,m} = 1) = p_{n,m}$. Suppose $\sum_{m=1}^{n} p_{n,m} \to \lambda$ and $\max_{1 \leq m \leq n} p_{n,m} \to 0$. Then $S_n = X_{n,1} + \ldots + X_{n,n} \implies Z$ were $Z$ is Poisson($\lambda$).

- **Proof idea:** Just write down the log characteristic functions for Bernoulli and Poisson random variables. Check the conditions of the continuity theorem.
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Strong continuity theorem: If $\mu_n \Rightarrow \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all $t$. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions $\mu_n$ is tight and converges weakly to a measure $\mu$ with characteristic function $\phi$. 
Recall CLT idea

Let $X$ be a random variable.
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The **characteristic function** of $X$ is defined by

$$\phi(t) = \phi_X(t) := E[e^{itX}].$$
Recall CLT idea

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  \[ \phi(t) = \phi_X(t) := E[e^{itX}] \].
- And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
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- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi'_X(0) = 0$ and $\phi''_X(0) = -1$. 


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- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi_X'(0) = 0$ and $\phi_X''(0) = -1$.
- Write $L_X := -\log \phi_X$. Then $L_X(0) = 0$ and $L_X'(0) = -\frac{\phi_X'(0)}{\phi_X(0)} = 0$ and $L_X'' = -\frac{(\phi_X''(0)\phi_X(0) - \phi_X'(0)^2)}{\phi_X(0)^2} = 1$. 
Recall CLT idea

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- The **characteristic function** of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- And if $X$ has an $m$th moment then $E[X^m] = im \phi_X^{(m)}(0)$.
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- Write $L_X := - \log \phi_X$. Then $L_X(0) = 0$ and $L_X'(0) = -\phi_X'(0)/\phi_X(0) = 0$ and $L_X'' = -(\phi_X''(0)\phi_X(0) - \phi_X'(0)^2)/\phi_X(0)^2 = 1$.
- If $V_n = n^{-1/2} \sum_{i=1}^n X_i$ where $X_i$ are i.i.d. with law of $X$, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$.
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  $$L_X'' = -(\phi_X''(0)\phi_X(0) - \phi_X'(0)^2)/\phi_X(0)^2 = 1.$$
- If $V_n = n^{-1/2} \sum_{i=1}^n X_i$ where $X_i$ are i.i.d. with law of $X$, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$.
- When we zoom in on a twice differentiable function near zero (scaling vertically by $n$ and horizontally by $\sqrt{n}$) the picture looks increasingly like a parabola.
Question? Is it possible for something like a CLT to hold if $X$ has infinite variance? Say we write $V_n = n^{-a} \sum_{i=1}^{n} X_i$ for some $a$. Could the law of these guys converge to something non-Gaussian?
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What if the $L_{V_n}$ converge to something else as we increase $n$, maybe to some other power of $|t|$ instead of $|t|^2$?

The appropriately normalized sum should be converge in law to something with characteristic function $e^{-|t|^\alpha}$ instead of $e^{-|t|^2}$.

We already saw that this should work for Cauchy random variables. What's the characteristic function in that case?

Let's look up stable distributions.
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Let’s look up stable distributions.
Say a random variable $X$ is **infinitely divisible**, for each $n$, there is a random variable $Y$ such that $X$ has the same law as the sum of $n$ i.i.d. copies of $Y$. 

What random variables are infinitely divisible? Poisson, Cauchy, normal, stable, etc.

Let's look at the characteristic functions of these objects. What about compound Poisson random variables (linear combinations of Poisson random variables)? What are their characteristic functions like?

More general constructions are possible via Lévy Khintchine representation.
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Infinitely divisible laws

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