18.175: Lecture 11
Central limit theorem variants

Scott Sheffield

MIT
Outline

CLT idea

CLT variants

More on random walks and local CLT

Poisson random variable convergence

Extend CLT idea to stable random variables
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Extend CLT idea to stable random variables
Recall Fourier inversion formula

- If $f : \mathbb{R} \to \mathbb{C}$ is in $L^1$, write $\hat{f}(t) := \int_{-\infty}^{\infty} f(x)e^{-itx} \, dx$. 

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Recall Fourier inversion formula

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- **Fourier inversion:** If $f$ is nice: $f(x) = \frac{1}{2\pi} \int \hat{f}(t)e^{itx} \, dt$. 
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- **Fourier inversion**: If $f$ is nice: $f(x) = \frac{1}{2\pi} \int \hat{f}(t)e^{itx} \, dt$.
- Easy to check this when $f$ is density function of a Gaussian. Use linearity of $f \to \hat{f}$ to extend to linear combinations of Gaussians, or to convolutions with Gaussians.

Convolution theorem:

If $h(x) = (f \ast g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y) \, dy$, then $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$.

Observation: can define Fourier transforms of generalized functions. Can interpret finite measure as generalized function.
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- Show $f \to \hat{f}$ is an isometry of Schwartz space (endowed with $L^2$ norm). Extend definition to $L^2$ completion.
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Recall Bochner’s theorem

- Given function $\phi$ and points $t_1, \ldots, t_n$, consider matrix with $i, j$ entry given by $\phi(t_i - t_j)$. Call $\phi$ positive definite if this matrix is always positive semidefinite Hermitian.
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- Bochner’s theorem: a continuous function from \( \mathbb{R} \) to \( \mathbb{C} \) with \( \phi(0) = 1 \) is a characteristic function of a some probability measure on \( \mathbb{R} \) if and only if it is positive definite.
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- Bochner’s theorem: a continuous function from $\mathbb{R}$ to $\mathbb{C}$ with $\phi(0) = 1$ is a characteristic function of a some probability measure on $\mathbb{R}$ if and only if it is positive definite.
- Why positive definite?
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Why positive definite?

Write $Y = \sum_{j=1}^{n} a_j e^{t_j X}$ and observe

$$Y \overline{Y} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j \overline{a_k} e^{(t_i - t_j)X},$$

$$\mathbb{E} Y \overline{Y} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j \overline{a_k} \phi(t_i - t_j).$$
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- **Fourier transform**: natural one-to-one map from set of probability measures on $\mathbb{R}$ (describable by distribution functions $F$) to set of possible characteristic functions.
Recall continuity theorem

▶ **Strong continuity theorem:** If $\mu_n \nrightarrow \mu_\infty$ then $\phi_n(t) \rightarrow \phi_\infty(t)$ for all $t$. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions $\mu_n$ is tight and converges weakly to a measure $\mu$ with characteristic function $\phi$. 

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- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi'_X(0) = 0$ and $\phi''_X(0) = -1$. 

If $V_n = \frac{n - 1}{2} \sum_{i=1}^{n} X_i$ where $X_i$ are i.i.d. with law of $X$, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$. 

When we zoom in on a twice differentiable function near zero (scaling vertically by $n$ and horizontally by $\sqrt{n}$), the picture looks increasingly like a parabola.
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- Write $L_X := -\log \phi_X$. Then $L_X(0) = 0$ and
  \[ L'_X(0) = -\phi'_X(0)/\phi_X(0) = 0 \]
  \[ L''_X = -(\phi''_X(0)\phi_X(0) - \phi'_X(0)^2)/\phi_X(0)^2 = 1. \]
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- If $V_n = n^{-1/2} \sum_{i=1}^n X_i$ where $X_i$ are i.i.d. with law of $X$, then $L_{V_n}(t) = nL_X(n^{-1/2} t)$.
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Lindeberg-Feller theorem

- CLT is pretty special. What other kinds of sums are approximately Gaussian?

Triangular arrays:
Suppose $X_n, m$ are independent expectation-zero random variables when $1 \leq m \leq n$.

Suppose $\sum_{n,m=1}^{n,m=n} \mathbb{E}X_n^2, m \to \sigma^2 > 0$ and for all $\epsilon$, $\lim_{n \to \infty} \mathbb{E}(|X_n, m|^2; |X_n, m| > \epsilon) = 0$.

Then $S_n = X_n, 1 + X_n, 2 + \ldots + X_n, n \Rightarrow \sigma \chi$ (where $\chi$ is standard normal) as $n \to \infty$.

Proof idea: Use characteristic functions $\phi_n, m = \phi X_n, m$. Try to get some uniform handle on how close they are to their quadratic approximations.
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Berry-Esseen theorem

- If $X_i$ are i.i.d. with mean zero, variance $\sigma^2$, and $E|X_i|^3 = \rho < \infty$, and $F_n(x)$ is distribution of $(X_1 + \ldots + X_n)/(\sigma \sqrt{n})$ and $\Phi(x)$ is standard normal distribution, then $|F_n(x) - \Phi(x)| \leq 3\rho/(\sigma^3 \sqrt{n})$. 

Provided one has a third moment, CLT convergence is very quick.

Proof idea: You can convolve with something that has a characteristic function with compact support. Play around with Fubini, error estimates.
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Local limit theorems for walks on $\mathbb{Z}$

- Suppose $X \in b + h\mathbb{Z}$ a.s. for some fixed constants $b$ and $h$. 

- Write $p_n(x) = P(S_n/\sqrt{n} = x)$ for $x \in L_n := (nb + h\mathbb{Z})/\sqrt{n}$ and $n(x) = (2\pi\sigma^2)^{-1/2}e^{-x^2/2\sigma^2}$.

- Assume $X_i$ are i.i.d. lattice with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$. Theorem: As $n \to \infty$, $\sup_{x \in L_n} |n^{1/2}p_n(x) - n(x)| \to 0$.

- Proof idea: Use characteristic functions, reduce to periodic integral problem. Note that for $Y$ supported on $a + \theta \mathbb{Z}$, we have $P(Y = x) = \frac{1}{2\pi/\theta} \int_{-\pi/\theta}^{\pi/\theta} e^{-itx} \phi_Y(t) dt$. 

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Observe that if $\phi_X(\lambda) = 1$ for some $\lambda \neq 0$ then $X$ is supported on (some translation of) $(2\pi/\lambda)\mathbb{Z}$. If this holds for all $\lambda$, then $X$ is a.s. some constant. When the former holds but not the latter (i.e., $\phi_X$ is periodic but not identically 1) we call $X$ a **lattice random variable**.
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Extend CLT idea to stable random variables
Recall local CLT for walks on \( \mathbb{Z} \)

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- Write $p_n(x) = P(S_n/\sqrt{n} = x)$ for $x \in L_n := (nb + h\mathbb{Z})/\sqrt{n}$ and $n(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$. 
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\left| \sup_{x \in L^n} n^{1/2} / hp_n(x) - n(x) \right| \to 0.
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Proof idea: Use characteristic functions, reduce to periodic integral problem. Look up “Fourier series”. Note that for $Y$ supported on $a + \theta \mathbb{Z}$, we have

$$P(Y = x) = \frac{1}{2\pi/\theta} \int_{-\pi/\theta}^{\pi/\theta} e^{-itx} \phi_Y(t) dt.$$
Extending this idea to higher dimensions

Example: suppose we have random walk on \( \mathbb{Z} \) that at each step tosses fair 4-sided coin to decide whether to go 1 unit left, 1 unit right, 2 units left, or 2 units right?

One could compute this in Mathematica by writing out the characteristic function \( \phi_X \) for one-step increment \( X \) and calculating \( \int_{-\pi}^{\pi} \phi^k_X(t) \frac{dt}{2\pi} \).

How about a random walk on \( \mathbb{Z}^2 \)?

Can one use this to establish when a random walk on \( \mathbb{Z}^d \) is recurrent versus transient?
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Example: suppose we have random walk on $\mathbb{Z}$ that at each step tosses fair 4-sided coin to decide whether to go 1 unit left, 1 unit right, 2 units left, or 2 units right?

What is the probability that the walk is back at the origin after one step? Two steps? Three steps?
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How about a random walk on $\mathbb{Z}^2$?
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How about a random walk on $\mathbb{Z}^2$?

Can one use this to establish when a random walk on $\mathbb{Z}^d$ is recurrent versus transient?
Outline

CLT idea

CLT variants

More on random walks and local CLT

Poisson random variable convergence

Extend CLT idea to stable random variables
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Extend CLT idea to stable random variables
Poisson random variables: motivating questions

- How many raindrops hit a given square inch of sidewalk during a ten minute period?
- How many people fall down the stairs in a major city on a given day?
- How many plane crashes in a given year?
- How many radioactive particles emitted during a time period in which the expected number emitted is 5?
- How many calls to call center during a given minute?
- How many goals scored during a 90 minute soccer game?
- How many notable gaffes during 90 minute debate?

Key idea for all these examples: Divide time into a large number of small increments. Assume that during each increment, there is some small probability of the thing happening (independently of other increments).
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**Key idea for all these examples:** Divide time into large number of small increments. Assume that during each increment, there is some small probability of thing happening (independently of other increments).
Let $\lambda$ be some moderate-sized number. Say $\lambda = 2$ or $\lambda = 3$. Let $n$ be a huge number, say $n = 10^6$. 

A Poisson random variable $X$ with parameter $\lambda$ satisfies $P\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda}$ for integer $k \geq 0$. 

Suppose I have a coin that comes up heads with probability $\lambda/n$ and I toss it $n$ times. How many heads do I expect to see? Answer: $np = \lambda$. Let $k$ be some moderate sized number (say $k = 4$). What is the probability that I see exactly $k$ heads?

Binomial formula: $
\binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} p^k (1-p)^{n-k}.$ 

This is approximately $\frac{\lambda^k}{k!}e^{-\lambda} \approx \frac{\lambda^k}{k!}e^{-\lambda}$. 

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Bernoulli random variable with $n$ large and $np = \lambda$
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A Poisson random variable $X$ with parameter $\lambda$ satisfies
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Probabilities sum to one

- A Poisson random variable $X$ with parameter $\lambda$ satisfies $p(k) = P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$.
- How can we show that $\sum_{k=0}^{\infty} p(k) = 1$?
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How can we show that $\sum_{k=0}^{\infty} p(k) = 1$?

Use Taylor expansion $e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$.  

Probabilities sum to one
A Poisson random variable $X$ with parameter $\lambda$ satisfies
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Expectation

- A **Poisson random variable** $X$ with parameter $\lambda$ satisfies $P\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda}$ for integer $k \geq 0$.

- What is $E[X]$?
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By definition of expectation

$$E[X] = \sum_{k=0}^{\infty} P\{X = k\} k = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k - 1)!} e^{-\lambda}.$$
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Setting $j = k - 1$, this is $\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} e^{-\lambda} = \lambda$. 

18.175 Lecture 11
Given $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$, what is $\text{Var}[X]$?
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This suggests $\text{Var}[X] \approx npq \approx \lambda$ (since $np \approx \lambda$ and $q = 1 - p \approx 1$). Can we show directly that $\text{Var}[X] = \lambda$?
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Variance

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- Compute

\[
E[X^2] = \sum_{k=0}^{\infty} P\{X = k\} k^2 = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}.
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Compute $$E[X^2] = \sum_{k=0}^{\infty} P\{X = k\} k^2 = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}.$$ Setting $j = k - 1$, this is $$\lambda \left( \sum_{j=0}^{\infty} (j + 1) \frac{\lambda^j}{j!} e^{-\lambda} \right) = \lambda E[X + 1] = \lambda(\lambda + 1).$$
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Then \( \text{Var}[X] = E[X^2] - E[X]^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda \).
Poisson convergence

- Idea: if we have lots of independent random events, each with very small probability to occur, and expected number to occur is $\lambda$, then total number that occur is roughly Poisson $\lambda$. 

\[ S_n = X_{n,1} + \ldots + X_{n,n} \Rightarrow Z \text{ were } Z \text{ is Poisson}(\lambda). \]
Poisson convergence

- **Idea:** if we have lots of independent random events, each with very small probability to occur, and expected number to occur is $\lambda$, then total number that occur is roughly Poisson $\lambda$.

- **Theorem:** Let $X_{n,m}$ be independent $\{0, 1\}$-valued random variables with $P(X_{n,m} = 1) = p_{n,m}$. Suppose $\sum_{m=1}^{n} p_{n,m} \to \lambda$ and $\max_{1 \leq m \leq n} p_{n,m} \to 0$. Then $S_n = X_{n,1} + \ldots + X_{n,n} \implies Z$ were $Z$ is Poisson($\lambda$).
Idea: if we have lots of independent random events, each with very small probability to occur, and expected number to occur is $\lambda$, then total number that occur is roughly Poisson $\lambda$.

**Theorem:** Let $X_{n,m}$ be independent $\{0, 1\}$-valued random variables with $P(X_{n,m} = 1) = p_{n,m}$. Suppose $\sum_{m=1}^{n} p_{n,m} \to \lambda$ and $\max_{1 \leq m \leq n} p_{n,m} \to 0$. Then $S_n = X_{n,1} + \ldots + X_{n,n} \implies Z$ were $Z$ is Poisson($\lambda$).

**Proof idea:** Just write down the log characteristic functions for Bernoulli and Poisson random variables. Check the conditions of the continuity theorem.
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Recall continuity theorem

**Strong continuity theorem:** If $\mu_n \Rightarrow \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all $t$. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions $\mu_n$ is tight and converges weakly to a measure $\mu$ with characteristic function $\phi$. 
Recall CLT idea

- Let $X$ be a random variable.
Recall CLT idea

- Let $X$ be a random variable.
- The **characteristic function** of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. 

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Recall CLT idea

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- And if $X$ has an $m$th moment then $E[X^m] = im\phi_X^{(m)}(0)$.
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- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi_X'(0) = 0$ and $\phi_X''(0) = -1$. 
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In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi'_X(0) = 0$ and $\phi''_X(0) = -1$.

Write $L_X := -\log \phi_X$. Then $L_X(0) = 0$ and 
$L'_X(0) = -\phi'_X(0)/\phi_X(0) = 0$ and  
$L''_X = - (\phi''_X(0)\phi_X(0) - \phi'_X(0)^2)/\phi_X(0)^2 = 1$. 

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- If $V_n = n^{-1/2} \sum_{i=1}^n X_i$ where $X_i$ are i.i.d. with law of $X$, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$. 

When we zoom in on a twice differentiable function near zero (scaling vertically by $n$ and horizontally by $\sqrt{n}$) the picture looks increasingly like a parabola.
Recall CLT idea

- Let $X$ be a random variable.
- The **characteristic function** of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.
- And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
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- Write $L_X := -\log\phi_X$. Then $L_X(0) = 0$ and $L_X'(0) = -\phi_X'(0)/\phi_X(0) = 0$ and $L_X'' = -(\phi_X''(0)\phi_X(0) - \phi_X'(0)^2)/\phi_X(0)^2 = 1$.
- If $V_n = n^{-1/2} \sum_{i=1}^n X_i$ where $X_i$ are i.i.d. with law of $X$, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$.
- When we zoom in on a twice differentiable function near zero (scaling vertically by $n$ and horizontally by $\sqrt{n}$) the picture looks increasingly like a parabola.
Question? Is it possible for something like a CLT to hold if $X$ has infinite variance? Say we write $V_n = n^{-a} \sum_{i=1}^{n} X_i$ for some $a$. Could the law of these guys converge to something non-Gaussian?
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We already saw that this should work for Cauchy random variables. What’s the characteristic function in that case?
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We already saw that this should work for Cauchy random variables. What’s the characteristic function in that case?

Let’s look up stable distributions.
Say a random variable $X$ is \textbf{infinitely divisible}, for each $n$, there is a random variable $Y$ such that $X$ has the same law as the sum of $n$ i.i.d. copies of $Y$. 

What random variables are infinitely divisible? 

Poisson, Cauchy, normal, stable, etc.

Let's look at the characteristic functions of these objects. 

What about compound Poisson random variables (linear combinations of Poisson random variables)? What are their characteristic functions like? 

More general constructions are possible via Lévy Khintchine representation.
Say a random variable $X$ is **infinitely divisible**, for each $n$, there is a random variable $Y$ such that $X$ has the same law as the sum of $n$ i.i.d. copies of $Y$.

What random variables are infinitely divisible?
Infinitely divisible laws

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- What random variables are infinitely divisible?
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Let’s look at the characteristic functions of these objects. What about compound Poisson random variables (linear combinations of Poisson random variables)? What are their characteristic functions like?
Say a random variable $X$ is **infinitely divisible**, for each $n$, there is a random variable $Y$ such that $X$ has the same law as the sum of $n$ i.i.d. copies of $Y$.

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More general constructions are possible via Lévy Khintchine representation.