Universal random structures in 2D

Introduction to 18.177, Fall 2015

Scott Sheffield

Massachusetts Institute of Technology

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THE NUMBERS

24 lectures (after this one)

3 problem sets (to be assigned)

1 written project (research or expository) of about

5 pages per student, collaboration allowed
THE GOALS

Introduce some fundamental objects

Explain how they are related to each other

Explore some open problems
THE GOAL TODAY

Colloquium-style overview of major objects and relationships
Overview

Prologue:
1. **Universality**: physics intuition, examples
2. **Discrete-continuum interplay**: scaling limits, discretizations
3. **Fractals and complex dynamics**: Julia sets, fractal dimensions, Mandelbrot, etc.

Part I: Cast of Characters: *What are the most fundamental 2D random objects?*
1. **Universal random trees**: Brownian motion, continuum random tree
2. **Universal random surfaces**: quantum gravity, planar maps, string theory, CFT
3. **Universal random paths**: walks, interfaces, Schramm-Loewner evolution, CFT
4. **Universal random growth**: Eden model, DLA, DBM

Part II: Drama: *How are the characters related to each other?*
1. **Welding random surfaces**: a calculus of random surfaces and SLE seams
2. **Mating random trees**: tree plus tree (conformally mated) equals surface plus path
3. **Random growth on random surfaces**: dendrites, dragons, surprising tractability
4. **Mating random trees produced by a snake**: metric spaces and the Brownian map
5. **Two “universal random surfaces” are the same**: Brownian map equals Liouville quantum gravity with parameter $\gamma = \sqrt{8/3}$ (a.k.a. “pure quantum gravity”).
PROLOGUE: UNIVERSALITY
Universality in physics (per Wikipedia)

In statistical mechanics, **universality** is the observation that there are properties for a large class of systems that are independent of the dynamical details of the system. Systems display universality in a **scaling limit**, when a large number of interacting parts come together. The modern meaning of the term was introduced by Leo Kadanoff in the 1960s, but a simpler version of the concept was already implicit in the van der Waals equation and in the earlier Landau theory of **phase transitions**, which did not incorporate scaling correctly. The term is slowly gaining a broader usage in several fields of **mathematics**, including **combinatorics and probability theory**, whenever the quantitative features of a structure (such as asymptotic behaviour) can be deduced from a few global parameters appearing in the definition, **without requiring knowledge of the details of the system**. The **renormalization group** explains universality. It classifies operators in a statistical field theory into relevant and irrelevant. Relevant operators are those responsible for perturbations to the free energy, the imaginary time Lagrangian, that will affect the continuum limit, and can be seen at long distances. Irrelevant operators are those that only change the short-distance details. The collection of scale-invariant statistical theories define the **universality classes**, and the finite-dimensional list of coefficients of relevant operators parametrize the near critical behavior.
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Example: percolation (Cardy 1992; Smirnov 2001).
Percolation interface
PROLOGUE:
DISCRETE-CONTINUUM
INTERPLAY
Discrete world vs. continuum world: more stories

- **Statistical physics**: argue that your (simple) continuum theory approximates your (not so simple) atomic model when the number of atoms is very large.

- **Particle physics**: argue that your (well defined) discrete lattice models approximate your (maybe complicated, maybe ill defined) continuum field theory when the lattice is very fine.

- **One mathematical goal**: develop continuum theories to help you understand scaling limits of beloved discrete models.

- **Another mathematical goal**: develop discrete approximations to help you understand beloved continuum theories (like Navier-Stokes and Yang-Mills).

- **Interplay** between the discrete and continuum is at the heart of many fields within physics and mathematics.

- **Mathematically rigorous** connections between discrete and continuum are sometimes hard to prove, which leads to...

- **Non-rigorous approach**: (common in physics) just assume you can pass from discrete to continuum and back whenever you need to. Then check whether end result seems to match experiments or simulations.

- **Conformal symmetry**: plays special role in 2D, following work by Belavin, Polyakov, Zamolodchikov and others in 1980's.
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PROLOGUE:
NON-RANDOM FRACTALS
FROM COMPLEX DYNAMICS
Google search for Julia sets
Julia sets (Julia, 1918), popularized in 1980’s

- Consider map $\phi(z) = z^2$.
- Maps $\mathbb{C} \setminus \mathbb{D}$ conformally to self (2 to 1) where $\mathbb{D}$ is unit disc. Repeated iteration takes points in $\mathbb{C} \setminus \mathbb{D}$ to $\infty$, leaves others bounded.
- If $K$ is another compact set with connected hull, can construct a similar (2 to 1) conformal map $\phi_K$ from $\mathbb{C} \setminus K$ to itself.
- Might expect more intricate sets $K$ to yield more intricate maps. But suppose we take $\phi_K(z) = z^2 + c$ and let $K$ be set of points remaining bounded under repeated iteration.
- $K$ is a (filled) Julia set. Can “mate” Julia sets to form sphere (Douady 1983, Milnor 1994, see Arnaud Ch´eritat’s animations).
- Popular lexicon: chaos theory, butterfly effect, fractal, self-similar. What about random fractals, only self similar in law?

Published 1989, by Roger T. Stevens
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Part I:
CAST OF CHARACTERS

A Trees
B Simple curves, non-simple curves, space-filling curves
C Surfaces
D Growth
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Discrete analog: Consider a tree embedded in the plane with $n$ edges and a distinguished root. As one traces the outer boundary of the tree clockwise, distance from root performs a simple walk on $\mathbb{Z}_+$ with $2n$ steps, starting and ending at 0.
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CRT is in some sense the “uniformly random planar tree” of a given size.
RANDOM PATHS

Given a simply connected planar domain $D$ with boundary points $a$ and $b$ and a parameter $\kappa \in [0, \infty)$, the **Schramm-Loewner evolution** $\text{SLE}_\kappa$ is a random non-self-crossing path in $\overline{D}$ from $a$ to $b$.

The parameter $\kappa$ roughly indicates how “windy” the path is. Would like to argue that SLE is in some sense the “canonical” random non-self-crossing path. What symmetries characterize SLE?
Conformal Markov property of SLE

If $\phi$ conformally maps $D$ to $\tilde{D}$ and $\eta$ is an $\text{SLE}_\kappa$ from $a$ to $b$ in $D$, then $\phi \circ \eta$ is an $\text{SLE}_\kappa$ from $\phi(a)$ to $\phi(b)$ in $\tilde{D}$. 
Markov Property

Given $\eta$ up to a stopping time $t$...

Law of remainder is SLE in $D \setminus \eta[0, t]$ from $\eta(t)$ to $b$. 
Chordal Schramm-Loewner evolution (SLE)

- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$. 

Explicit construction: An SLE path $\gamma$ from 0 to $\infty$ in the complex upper half plane $\mathbb{H}$ can be defined in an interesting way: given path $\gamma$ one can construct conformal maps $g_t: \mathbb{H} \setminus \gamma(0,t) \to \mathbb{H}$ (normalized to look like identity near infinity, i.e., $\lim_{z \to \infty} g_t(z) - z = 0$). In SLE$_\kappa$, one defines $g_t$ via an ODE (which makes sense for each fixed $z$):

$$\frac{\partial}{\partial t} g_t(z) = 2g_t(z) - W_t,$$ 

$g_0(z) = z$, where $W_t = \sqrt{\kappa} B_t$ is the law of Brownian motion.
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$$\partial_t g_t(z) = \frac{2}{g_t(z) - \mathcal{W}_t}, \quad g_0(z) = z,$$

where $\mathcal{W}_t = \sqrt{\kappa} B_t =_{law} B_{\kappa t}$ and $B_t$ is ordinary Brownian motion.
SLE phases [Rohde, Schramm]

\( \kappa \leq 4 \)

\( \kappa \in (4, 8) \)

\( \kappa \geq 8 \)
Radial Schramm-Loewner evolution (SLE)

- In radial SLE path grows from boundary of domain to center.

Radial measure-driven Loewner evolution:
\[
\partial_t g_t(z) = \int g_t(z) x + g_t(z)x - g_t(z)dm_t(x)
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where, for each $g_t$, $m_t$ is a measure on the complex unit circle.
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- **Radial measure-driven Loewner evolution**: \( \partial_t g_t(z) = \int g_t(z) \frac{x + g_t(z)}{x - g_t(z)} dm_t(x) \) where, for each \( g \), \( m_t \) is a measure on the complex unit circle.
Continuum space-filling path
Uniform spanning tree
Start out with a sheet of paper
Get out pen and ruler
Measure and mark squares of equal size
Get out scissors
Cut into squares
Get out bottle of glue
Attach squares along boundaries with glue to form a surface “without holes.”
What is the structure of a typical quadrangulation when the number of faces is large?
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Random quadrangulation with 25,000 faces

(Simulation due to J.F. Marckert)
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5. Important tool: Bijections encoding surface via pair of trees.
Random quadrangulation

Packed with Stephenson's CirclePack.

Scott Sheffield (MIT)

Universal structure

September 15, 2015
Red tree

Scott Sheffield (MIT)

Universal structure

September 15, 2015
Red and blue trees

Packed with Stephenson's CirclePack.
Red and blue trees alone do not determine the map structure
Random quadrangulation with red and blue trees
Path snaking between the trees. Encodes the trees and how they are glued together.
How was the graph embedded into $\mathbb{R}^2$?
Can subdivide each quadrilateral to obtain a triangulation without multiple edges.
Circle pack the resulting triangulation.

Packed with Stephenson's CirclePack.
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What is the “limit” of this embedding? Circle packings are related to conformal maps.

Packed with Stephenson’s CirclePack.
Conformal maps (from David Gu’s web gallery)

Riemann Uniformization

All metric surfaces can be conformally mapped to three canonical spaces, the sphere, the plane and the hyperbolic plane.

Genus zero closed surface
Picking a surface at random in the continuum

**Uniformization theorem:** every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $S^2$ in $\mathbb{R}^3$.
Picking a surface at random in the continuum

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**Isothermal coordinates:** Metric for the surface takes the form \( e^{\rho(z)} \, dz \) for some smooth function \( \rho \) where \( dz \) is the Euclidean metric.
Picking a surface at random in the continuum

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- If $\Delta \rho = 0$, i.e. if $\rho$ is harmonic, the surface described is flat
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- If $\Delta \rho = 0$, i.e. if $\rho$ is harmonic, the surface described is flat

**Question:** Which measure on $\rho$? If we want our surface to be a perturbation of a flat metric, natural to choose $\rho$ as the canonical perturbation of a harmonic function.
The Gaussian free field

- The discrete Gaussian free field (DGFF) is a Gaussian random surface model.
The Gaussian free field

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- Measure on functions $h: D \rightarrow \mathbb{R}$ for $D \subseteq \mathbb{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbb{R}^{|D|}$:

\[
\frac{1}{\mathcal{Z}} \exp \left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)
\]
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- Natural perturbation of a harmonic function

- Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the **Dirichlet inner product**

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- Continuum GFF not a function — only a generalized function
Liouville quantum gravity

- Liouville quantum gravity: \( e^{\gamma h(z)} \, dz \)
  where \( h \) is a GFF and \( \gamma \in [0, 2) \)

\( \gamma = 0.5 \)

(Number of subdivisions)
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Does not make literal sense since $h$ takes values in the space of distributions. Can make sense of random area measure using a regularization procedure. Areas of regions and lengths of curves are well defined. 

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\[\gamma = 2.0\]
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)

Consider case that graph is $\mathbb{Z}^2$.

Question: Large scale behavior of shape of ball wrt perturbed metric?

Cox and Durrett (1981) showed that the macroscopic shape is convex.

Computer simulations show that it is not a Euclidean disk.

$\mathbb{Z}^2$ is not isotropic enough.

Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if $\mathbb{Z}^2$ is replaced by the Voronoi tesselation associated with a Poisson process.
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![Graph with edge weights](image)

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![Diagram of a graph with edge weights]

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Scott Sheffield (MIT)

Universal structure

September 15, 2015
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Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
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Euclidean Diffusion Limited Aggregation (DLA) introduced by Witten-Sander 1981.
DLA in nature: “A DLA cluster grown from a copper sulfate solution in an electrodeposition cell” (from Wikipedia)
DLA in nature: Magnese oxide patterns on the surface of a rock. (Halsey, Physics Today 2000)
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DLA in art: “High-voltage dielectric breakdown within a block of plexiglas” (from Wikipedia)
DLA in physics

Introduced by Witten and Sander in 1981 as a model for crystal growth. (Mineral deposits, Hele-Shaw flow, electrodeposition, lichen growth, lightning paths, coral, etc.)

An active area of research in physics for the last 33 years:

Google search results for "diffusion limited aggregation"

1. Diffusion-limited aggregation
   TA Witten, LM Sander - Physical Review B, 1983 - APS
   Diffusion-limited aggregation (DLA) is an idealization of the process by which matter irreversibly combines to form dust, soot, dendrites, and other random objects in the case where the rate-limiting step is diffusion of matter to the aggregate. We study the process ...
   Cited by 1472 Related articles All 7 versions Cite Save

2. Diffusion-limited aggregation, a kinetic critical phenomenon
   TA Witten Jr, LM Sander - Physical review letters, 1981 - APS
   A model for random aggregate is studied by computer simulation. The model is applicable to a metal-particle aggregation process whose correlations have been measured previously. Density correlations within the model aggregates fall off with distance with a fractional ...
   Cited by 4469 Related articles All 6 versions Cite Save

3. Formation of fractal clusters and networks by irreversible diffusion-limited aggregation
   P Meakin - Physical Review Letters, 1983 - APS
   In addition to the simulations used to obtain the results shown in Figs. 1 and 2, simulations have also been carried out at a lower concentration (5000 particles on a 400x 400 lattice or p= 0.031-25). From seven such simulations 1 find that n= 0.516+ 0.029*(1 & x & 25 lattice ...
   Cited by 1436 Related articles All 3 versions Cite Save
Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

▶ Does DLA have a “scaling limit”?
▶ Is the shape random at large scales?
▶ Does the macroscopic shape look like a tree?
▶ What is its asymptotic dimension? Simulation prediction: \( \approx 1.71 \) on \( \mathbb{Z}^2 \).

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Part II: DRAMA
STORY A:
SURFACE PLUS SURFACE = SURFACE PLUS CURVE

independence on both sides
WELDING RANDOM SURFACES

Can “weld” and “slice” special quantum surfaces called quantum wedges (with “weight” parameters indicating thickness) to obtain wedges (with other weights).

- Weight parameter $W = \gamma (\gamma + \frac{2}{\gamma} - \alpha)$ is additive under the welding operation.
- Interface between welding of independent wedges $W_1, W_2$ of weight $W_1$ and $W_2$ is an $\text{SLE}_\kappa(W_1 - 2; W_2 - 2)$ on combined surface.
- Glue canonical random surfaces, seam becomes canonical random path.
STORY B:

TREE PLUS TREE = SURFACE PLUS SPACE-FILLING CURVE

LHS independent or correlated, RHS independent
MATING RANDOM TREES

$X$, $Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

$C - Y_t$

$X_t$

Identify points on the graph of $X_t$ if they are connected by a horizontal line which is below the graph; yields a continuum random tree (CRT)

Same for $C - Y_t$ yields an independent CRT

Glue the CRTs together by declaring points on the vertical lines to be equivalent

Q: What is the resulting structure?

A: Sphere with a space-filling path. Apéneansphere.
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$\overset{\text{Same for } C - Y \text{ yields an independent CRT}}{\longrightarrow} X_t$

$\overset{\text{Glue the CRTs together by declaring points on the vertical lines to be equivalent}}{\longrightarrow}$

Q: What is the resulting structure?

A: Sphere with a space-filling path. Apeanosphere.
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Scott Sheffield (MIT)

Universal structure

September 15, 2015 47 / 67
Surface is topologically a sphere by Moore’s theorem

Theorem (Moore 1925)

Let $\cong$ be any topologically closed equivalence relation on the sphere $S^2$. Assume that each equivalence class is connected and not equal to all of $S^2$. Then the quotient space $S^2/\cong$ is homeomorphic to $S^2$ if and only if no equivalence class separates the sphere into two or more connected components.

- An equivalence relation is topologically closed iff for any two sequences $(x_n)$ and $(y_n)$ with
  - $x_n \cong y_n$ for all $n$
  - $x_n \to x$ and $y_n \to y$
- we have that $x \cong y$. 
STORY C:
SURFACE TREE PLUS
SURFACE TREE =
SURFACE PLUS
SELF-HITTING CURVE

independence on both sides
Gluing independent Lévy trees

Can view $\text{SLE}_{\kappa'}$ process, $\kappa' \in (4, 8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.
Gluing independent Lévy trees

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The two trees of quantum disks almost surely determine both the $\text{SLE}_{\kappa'}$ and the $\text{LQG}$ surface on which it is drawn.

Can convert questions about $\text{SLE}_{\kappa'}$ into questions about $\kappa'$-stable processes.

Scaling limit of "exploration path" on random planar map should be $\text{SLE}_6$ on a $\sqrt{8/3}$-LQG. Using welding machinery, we can understand well the "bubbles" cut out by such an exploration process. We can understand conditional law of unexplored region given what we have seen.
Gluing independent Lévy trees

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- Scaling limit of “exploration path” on random planar map should be $\text{SLE}_6$ on a $\sqrt{8/3}$-LQG. Using welding machinery, we can understand well the “bubbles” cut out by such an exploration process. We can understand conditional law of unexplored region given what we have seen.
STORY D:

GROWTH ON SURFACE = “RESHUFFLED” CURVE ON SURFACE
Can we make sense of $\eta$-DBM on a $\gamma$-LQG? We have shown how to tile an LQG surface with diadic squares of “about the same size” so we could run a DLA on this set of squares and try to take a fine mesh limit.
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Question: Are there coral reefs, snowflakes, lichen, crystals, plants, lightning bolts, etc. whose growth rates are affected by a random medium (something like LQG)? The simulations look similar but have a bit more personality when $\gamma$ is larger (as we will see). They look like Chinese dragons.
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We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) quantum Loewner evolution: $\text{QLE}(\gamma^2, \eta)$. 
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We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) **quantum Loewner evolution:** $\text{QLE}(\gamma^2, \eta)$.

But first let’s look at some computer generated images (and some animations), starting with an Eden exploration.
Eden model on $\sqrt{8/3}$-LQG
DLA on a $\sqrt{2}$-LQG
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$. 

Important observations:
- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.
- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.

Belief:
- Isotropic enough so that at large scales this is close to a ball in the graph metric.
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Scott Sheffield (MIT)
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First passage percolation on random planar maps III

**Variant:**

- Pick two **edges** on outer boundary of cluster

This exploration also respects the Markovian structure of the map. If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length. Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
First passage percolation on random planar maps III

**Variant:**

- Pick two *edges* on outer boundary of cluster
- Color vertices between edges blue and yellow

This exploration also respects the Markovian structure of the map.

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Continuum limit ansatz

- Sample a random planar map

\[\text{Ansatz: Image of random map converges to a } \sqrt{\frac{8}{3}}\text{-LQG surface and the image of the interface converges to an independent SLE } 6.\]
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $\frac{1}{2}$ and draw percolation interface
- Conformally map to the sphere
  - The image of the random map converges to a $\sqrt{8}/3$-LQG surface and the image of the interface converges to an independent SLE$_6$. 
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
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Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
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Sample a random planar map and two edges uniformly at random

Color vertices blue/yellow with probability $1/2$ and draw percolation interface

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Sample a random planar map and two edges uniformly at random

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Conformally map to the sphere

**Ansatz** Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE$_6$. 
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of SLE$_6$
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of $\text{SLE}_6$
- Resample the tip according to boundary length
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Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $\times$
- Draw $\delta$ units of SLE$_6$
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- Repeat
- Know the conditional law of the LQG surface at each stage, using exploration results
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
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$\text{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion.
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
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$\text{QLE}(8/3, 0)$ is SLE$_6$ with **tip re-randomization**. It can be understood as a “reshuffling” of the exploration procedure associated to the peanosphere.
What is $\text{QLE}(\gamma^2, \eta)$?

$\text{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\text{QLE}(\gamma^2, \eta)$

- $\gamma$ is the type of LQG surface on which the process grows
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Let $\mu_{\text{HARM}}$ (resp. $\mu_{\text{LEN}}$) be harmonic (resp. length) measure on a $\gamma$-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$\left( \frac{d\mu_{\text{HARM}}}{d\mu_{\text{LEN}}} \right)^\eta d\mu_{\text{LEN}}.$$
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- **First passage percolation**: $\eta = 0$
What is QLE($\gamma^2, \eta$)?

QLE(8/3, 0) is a member of a two-parameter family of processes called QLE($\gamma^2, \eta$)

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- First passage percolation: $\eta = 0$
- Diffusion limited aggregation: $\eta = 1$
What is $\text{QLE}(\gamma^2, \eta)$?

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- **First passage percolation**: $\eta = 0$
- **Diffusion limited aggregation**: $\eta = 1$
- **$\eta$-dieletric breakdown model**: general values of $\eta$
Discrete approximation of QLE(8/3, 0). Metric ball on a $\sqrt{8/3}$-LQG
Discrete approximation of QLE(2, 1). DLA on a $\sqrt{2}$-LQG
QLE($\gamma^2, \eta$) processes we can construct

Each of the QLE($\gamma^2, \eta$) processes with $(\gamma^2, \eta)$ on the orange curves is built from an \text{SLE}_\kappa process using tip re-randomization.
STORY E:

BROWNIAN MAP = \[ \sqrt{\frac{8}{3}} - \text{LIOUVILLE QUANTUM GRAVITY} \]
Dancing snake: a natural random walk on the space of discrete “snakes.”
1. The dancing snake has a scaling limit called the Brownian snake.
2. The $x$ and $y$ coordinates of the Brownian snake's head are two functions.
3. Each of these describes a tree (via the same construction we used to make CRT from Brownian motion).
4. Gluing these two trees together gives a random surface called the Brownian map.
Some QLE-based results

- Existence of $\text{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a $\gamma$-LQG surface.
Some QLE-based results

- Existence of $\text{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a $\gamma$-LQG surface.
- A proof that when $\gamma^2 = 8/3$ and $\eta = 0$, QLE describes the growth of metric balls in Liouville quantum gravity.
Some QLE-based results

- Existence of $\text{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a $\gamma$-LQG surface.
- A proof that when $\gamma^2 = 8/3$ and $\eta = 0$, QLE describes the growth of metric balls in Liouville quantum gravity.
- A proof that, under the metric defined by QLE, Liouville quantum gravity is equivalent (as a random metric measure space) to the Brownian map.
Some QLE-based results

- Existence of $\text{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a $\gamma$-LQG surface.
- A proof that when $\gamma^2 = \frac{8}{3}$ and $\eta = 0$, QLE describes the growth of metric balls in Liouville quantum gravity.
- A proof that, under the metric defined by QLE, Liouville quantum gravity is equivalent (as a random metric measure space) to the Brownian map.
- An understanding of a continuum analog of DLA on a random surface corresponding to $\gamma^2 = 2$. 
Thanks!