18.177: Lecture 6

Critical percolation

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Outline

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Recollections

Robust clusters: more on percolation in slabs and half spaces

Recall

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- What tools to we have for showing existence of "robust" infinite cluster?
- What tools do we have for building large holes?

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- Generally, if F is any infinite connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$, then each $\eta > 0$ there exists an integer k such that $p_c(2kF + B(k)) \leq p_c + \eta$.

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- Idea: just start building things out of blocks and "seeds".
- Need sprinkling to compensate for "negative information".

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- Idea: again, start building things out of blocks again, but now we don't need sprinkling.