18.177: Lecture 4

Critical percolation

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Outline

Recollections

Intuition when d is very large

What about possible infinite cluster when $p = p_c$?

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- **BK inequality, Russo's formula, plus work:** Probability radius of C exceeds R decays exponentially in R when $p < p_c$.
- **Consequence of that:** No infinite sequence of nested cycles when $p < p_c$. So $p_c = 1/2$ when $d = 2$.

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- Thus $\frac{\partial}{\partial p} P_p(A)$ is $p^{-1} E_p(N(A); A)$.

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- Get approximately a critical Galton-Watson tree with Poisson offspring numbers.
- Expect to have lots of large tree like clusters intersecting the n^d box.
- Heuristically, tree with k vertices should have a longest path of length $\sqrt{k}$. Is distance of tip from origin about $k^{1/4}$?

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- Consider S_K where K is so large that it is likely (probability at least $p_E = 1 - 10^40$, say) that infinite cluster hits S_K.
- Now consider an N large enough so that it is extremely likely (prob at least p_E) that all vertices in S_K are either in finite clusters of diameter at most N or can be joined together by a path within a box of size S_N.

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- Peierls argument: if so, could decrease p a bit and still have infinite cluster.