MATH 177 PROBLEM SET 2

A. Read the Wikipedia article on Ito's formula (Ito's lemma). It states that if $dX_t = \sigma_t dB_t + \mu_t dt$ and f is a twice differentiable real function of two variables, then

$$df(t, X_t) = (\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2}) dt + \sigma_t \frac{\partial f}{\partial x} dB_t. \quad (1)$$

Answer the following using the Ito formalism:

1. Compute $d(B_t)^2$ and $d(B_t^2 t^3)$ and de^{B_t}.
2. Find the constant a for which $e^{\gamma B_t - at}$ is a martingale. Check your result by applying Ito's formula and showing that the drift term (the dt term) vanishes. Check also directly that the expectation at time $t = 1$ is 1.
3. Find an a for which $X_t = B_t^2 - at$ is a martingale. Find an adapted process $s = s(t)$, an increasing function of t, such that $X_{s(t)}$ is a Brownian motion.

B. Suppose that $dX_t = \sigma_t dB_t + \mu_t dt$, where σ is ordinary Brownian motion but X_t, σ_t, and μ_t are complex valued. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic. Show (e.g., using Ito's formula for real-valued functions f of higher dimensional drift-diffusion processes, as on the Wikipedia page) that

$$df(X_t) = f'(X_t) dX_t + \frac{\sigma_t^2}{2} f''(X_t) dt,$$

where f' and f'' are complex derivatives.

C. Let

$$dg_t(z) = \frac{2}{(g_t(z) - W_t)} dt$$

be the usual Loewner flow and write $f_t(z) = g_t(z) - W_t$. Fix $x \in \mathbb{H}$ and compute the following (with all the Ito's formula details).

1. $df_t(z)$
2. $d \log f_t(z)$
3. $df_t'(z)$
4. $d \log f_t'(z)$

Do the same in the case of reverse flow, where $dg_t = \frac{-2}{(g_t(z)-W_t)}dt$. In each case, find a linear combination $h_t(z)$ of $\log f_t(z)$ and $\log f_t'(z)$ which is a local martingale. Suppose we write $a_t = g_t(-1)$ and fix $\alpha \in \mathbb{R}$. Can you define a process W_t that makes $\log(g_t(z) - a_t) + h_t(z)$ a local martingale for each fixed $z \in \mathbb{H}$?

D. Let α be a complex number in the upper half plane, and consider the continuous path $f(s) = \alpha s$. Compute explicitly the half-plane capacity $f([0, s])$ and the conformal map $\Phi : \mathbb{H} \setminus f([0, s])$ to $\mathbb{H}$, normalized so that $\lim_{z\to\infty} \Phi(z) - z = 0$. Verify Loewner's theorem in this setting.

E. Read the (short) Wikipedia article on Bessel processes and verify the claims about the path being recurrent, transient, neighborhood recurrent depending on dimension. You can consult Revuz-Yor (or another source) if needed.

F. Read the St. Flour lecture notes by Wendelin Werner and write a one-paragraph summary of each of your two favorite sections.