18.175: Lecture 9

Borel-Cantelli and strong law

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Outline

Laws of large numbers: Borel-Cantelli applications

Strong law of large numbers

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Strong law of large numbers

Borel-Cantelli lemmas

- **First Borel-Cantelli lemma:** If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n \ \text{i.o.}) = 0$.

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- **First Borel-Cantelli lemma:** If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n \text{ i.o.}) = 0$.
- **Second Borel-Cantelli lemma:** If A_n are independent, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$.

Convergence in probability subsequential a.s. convergence

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- **Main idea of proof:** Consider event E_n that X_n and X differ by ϵ. Do the E_n occur i.o.? Use Borel-Cantelli.

Pairwise independence example

- **Theorem:** Suppose $A_1, A_2, \ldots$ are pairwise independent and $\sum P(A_n) = \infty$, and write $S_n = \sum_{i=1}^n 1_{A_i}$. Then the ratio S_n/ES_n tends a.s. to 1.

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- **Main idea of proof:** First, pairwise independence implies that variances add. Conclude (by checking term by term) that $\mathrm{Var} S_n \leq ES_n$. Then Chebyshev implies

$$P(|S_n - ES_n| > \delta ES_n) \leq \mathrm{Var}(S_n)/(\delta ES_n)^2 \to 0,$$

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- Second, take a smart subsequence. Let $n_k = \inf\{n : ES_n \ge k^2\}$. Use Borel Cantelli to get a.s. convergence along this subsequence. Check that convergence along this subsequence deterministically implies the non-subsequential convergence.

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Laws of large numbers: Borel-Cantelli applications

Strong law of large numbers

General strong law of large numbers

- **Theorem (strong law):** If $X_1, X_2, \ldots$ are i.i.d. real-valued random variables with expectation m and $A_n := n^{-1} \sum_{i=1}^n X_i$ are the *empirical means* then $\lim_{n \to \infty} A_n = m$ almost surely.

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- Expand $(X_1 + \ldots + X_n)^4$. Five kinds of terms: $X_iX_jX_kX_l$ and $X_iX_jX_k^2$ and $X_iX_j^3$ and $X_i^2X_j^2$ and X_i^4.

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- The first three terms all have expectation zero. There are $\binom{n}{2}$ of the fourth type and n of the last type, each equal to at most K. So $E[A_n^4] \leq n^{-4}\left(6\binom{n}{2} + n\right)K$.

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- The first three terms all have expectation zero. There are $\binom{n}{2}$ of the fourth type and n of the last type, each equal to at most K. So $E[A_n^4] \leq n^{-4}\Big(6\binom{n}{2} + n\Big)K$.
- Thus $E[\sum_{n=1}^\infty A_n^4] = \sum_{n=1}^\infty E[A_n^4] < \infty$. So $\sum_{n=1}^\infty A_n^4 < \infty$ (and hence $A_n \to 0$) with probability 1.

General proof of strong law

- Suppose X_k are i.i.d. with finite mean. Let $Y_k = X_k 1_{|X_k| \leq k}$. Write $T_n = Y_1 + \ldots + Y_n$. **Claim:** $X_k = Y_k$ all but finitely often a.s. so suffices to show $T_n/n \to \mu$. (Borel Cantelli, expectation of positive r.v. is area between cdf and line $y = 1$)

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- **Claim:** $\sum_{k=1}^\infty \mathrm{Var}(Y_k)/k^2 \le 4E|X_1| < \infty$. How to prove it?
- **Observe:** $\mathrm{Var}(Y_k) \le E(Y_k^2) = \int_0^\infty 2yP(|Y_k| > y)dy \le \int_0^k 2yP(|X_1| > y)dy$. Use Fubini (interchange sum/integral, since everything positive)

$$\sum_{k=1}^\infty E(Y_k^2)/k^2 \le \sum_{k=1}^\infty k^{-2} \int_0^\infty 1_{(y<k)} 2yP(|X_1| > y)dy =$$

$$\int_0^\infty \Big(\sum_{k=1}^\infty k^{-2} 1_{(y<k)}\Big) 2yP(|X_1| > y)dy.$$

Since $E|X_1| = \int_0^\infty P(|X_1| > y)dy$, complete proof of claim by showing that if $y \ge 0$ then $2y \sum_{k>y} k^{-2} \le 4$.

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- Consider subsequence $k(n) = [\alpha^n]$ for arbitrary $\alpha > 1$. Using Chebyshev, if $\epsilon > 0$ then

$$\sum_{n=1}^{\infty} P\big(|T_{k(n)} - ET_{k(n)}| > \epsilon k(n)\big) \leq \epsilon^{-1} \sum_{n=1}^{\infty} \mathrm{Var}(T_{k(n)})/k(n)^2$$

$$= \epsilon^{-2} \sum_{n=1}^{\infty} k(n)^{-2} \sum_{m=1}^{k(n)} \mathrm{Var}(Y_m) = \epsilon^{-2} \sum_{m=1}^{\infty} \mathrm{Var}(Y_m) \sum_{n:k(n)\geq m} k(n)^{-2}.$$

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$\sum_{n:\alpha^n\ge m} [\alpha^n]^{-2} \le 4 \sum_{n:\alpha^n \ge m} \alpha^{-2n} \le 4(1-\alpha^{-2})^{-1} m^{-2}.$

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- **Combine computations (observe RHS below is finite):**

$$\sum_{n=1}^{\infty} P(|T_{k(n)} - ET_k(n)| > \epsilon k(n)) \leq 4(1-\alpha^{-2})^{-1}\epsilon^{-2} \sum_{m=1}^{\infty} E(Y_m^2) m^{-2}.$$

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- Since ϵ is arbitrary, get $(T_{k(n)} - ET_{k(n)})/k(n) \to 0$ a.s.

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- Can extend to the case that X_1 is a.s. positive within infinite mean.
- Generally, can consider X_1^+ and X_1^-, and it is enough if one of them has a finite mean.