18.175: Lecture 5

More integration and expectation

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Outline

Integration

Expectation

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  - f is non-negative (hint: reduce to previous case by taking $f \wedge N$ for $N \to \infty$).
  - f is any measurable function (hint: treat positive/negative parts separately, difference makes sense if both integrals finite).

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  - $|\int f d\mu| \leq \int |f| d\mu$.
- When $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{R}^d, \lambda)$, write $\int_E f(x) dx = \int 1_E f d\lambda$.

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- EX^k is called k**th moment of** X. Also, if $m = EX$ then $E(X - m)^2$ is called the **variance** of X.

Properties of expectation/integration

- **Jensen's inequality:** If μ is probability measure and $\phi : \mathbb{R} \to \mathbb{R}$ is convex then $\phi(\int f d\mu) \leq \int \phi(f) d\mu$. If X is random variable then $E\phi(X) \geq \phi(EX)$.

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- **Cauchy-Schwarz inequality:** Special case $p = q = 2$. Gives $\int |fg| d\mu \leq \|f\|_2 \|g\|_2$. Says that dot product of two vectors is at most product of vector lengths.

Bounded convergence theorem

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- **Main idea of proof:** for any ϵ, δ can take n large enough so $\int |f_n - f| d\mu < M\delta + \epsilon$.

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(Counterexample for opposite-direction inequality using thin and tall rectangles?)
- **Main idea of proof:** first reduce to case that the f_n are increasing by writing $g_n(x) = \inf_{m\geq n} f_m(x)$ and observing that $g_n(x) \uparrow g(x) = \liminf_{n\to\infty} f_n(x)$. Then truncate, used bounded convergence, take limits.

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- **Main idea of proof:** Fatou for functions $g + f_n \geq 0$ gives one side. Fatou for $g - f_n \geq 0$ gives other.

Computing expectations

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- Prove by checking for indicators, simple functions, non-negative functions, integrable functions.
- Examples: normal, exponential, Bernoulli, Poisson, geometric...