18.175: Lecture 4

Integration

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Outline

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Expectation

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Recall definitions

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- The **Borel** σ-**algebra** $\mathcal{B}$ on a topological space is the smallest σ-algebra containing all open sets.

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- Note: to prove X is measurable, it is enough to show that the pre-image of every open set is in $\mathcal{F}$.
- Can talk about σ-algebra generated by random variable(s): smallest σ-algebra that makes a random variable (or a collection of random variables) measurable.

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  - f is non-negative (hint: reduce to previous case by taking $f \wedge N$ for $N \to \infty$).
  - f is any measurable function (hint: treat positive/negative parts separately, difference makes sense if both integrals finite).

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- When $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{R}^d, \lambda)$, write $\int_E f(x) dx = \int 1_E f d\lambda$.

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- EX^k is called k**th moment of** X. Also, if $m = EX$ then $E(X - m)^2$ is called the **variance** of X.