

18.175: Lecture 37

More Brownian motion

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Brownian motion properties and construction

Markov property, Blumenthal's 0-1 law

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- ▶ **Continuity:** With probability one, $t \rightarrow B_t$ is continuous.
- ▶ Hmm... does this mean we need to use a σ -algebra in which the event " B_t is continuous" is a measurable?
- ▶ Suppose Ω is set of all functions of t , and we use smallest σ -field that makes each B_t a measurable random variable... does that fail?

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- ▶ Another characterization: B is jointly Gaussian, $EB_s = 0$, $EB_s B_t = s \wedge t$, and $t \rightarrow B_t$ a.s. continuous.

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- ▶ We can use the Kolmogorov continuity theorem (next slide).
- ▶ Can prove Hölder continuity using similar estimates (see problem set).
- ▶ Can extend to higher dimensions: make each coordinate independent Brownian motion.

- ▶ **Kolmogorov continuity theorem:** Suppose $E|X_s - X_t|^\beta \leq K|t - s|^{1+\alpha}$ where $\alpha, \beta > 0$. If $\gamma < \alpha/\beta$ then with probability one there is a constant $C(\omega)$ so that $|X(q) - X(r)| \leq C|q - r|^\gamma$ for all $q, r \in \mathbb{Q}_2 \cap [0, 1]$.

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- ▶ **Proof idea:** First look at values at all multiples of 2^{-0} , then at all multiples of 2^{-1} , then multiples of 2^{-2} , etc.
- ▶ At each stage we can draw a nice piecewise linear approximation of the process. How much does the approximation change in supremum norm (or some other Hölder norm) on the i th step? Can we say it probably doesn't change very much? Can we say the sequence of approximations is a.s. Cauchy in the appropriate normed space?

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- ▶ **Argument from Durrett (Pemantle):** Write $G_n = \{|X(i/2^n) - X((i-1)/2^n)|\} \leq C|q - r|^\lambda$ for $0 < i \leq 2^n$.

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- ▶ Chebyshev implies $P(|Y| > a) \leq a^{-\beta} E|Y|^\beta$, so if $\lambda = \alpha - \beta\gamma > 0$ then

$$P(G_n^c) \leq 2^n \cdot 2^{n\beta\gamma} \cdot E|X(j2^{-n})|^\beta = K2^{-n\lambda}.$$

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Easy observations

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- ▶ Brownian motion is almost surely not differentiable.
- ▶ Brownian motion is almost surely not Lipschitz.
- ▶ Kolmogorov-Centsov theorem applies to higher dimensions (with adjusted exponents). One can construct a.s. continuous functions from \mathbb{R}^n to \mathbb{R} .

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More σ -algebra thoughts

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- ▶ Write $\mathcal{F}_s^+ = \bigcap_{t>s} \mathcal{F}_t^o$
- ▶ Note right continuity: $\bigcap_{t>s} \mathcal{F}_t^+ = \mathcal{F}_s^+$.
- ▶ \mathcal{F}_s^+ allows an “infinitesimal peek at future”

- ▶ If $s \geq 0$ and Y is bounded and \mathcal{C} -measurable, then for all $x \in \mathbb{R}^d$, we have

$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_{B_s} Y,$$

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- ▶ **Proof idea:** First establish this for some simple functions Y (depending on finitely many time values) and then use measure theory (monotone class theorem) to extend to general case.

- ▶ **Theorem:** If Z is bounded, measurable then for $s \geq 0$ have

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- ▶ If $A \in \mathcal{F}_0^+$, then $P(A) \in \{0, 1\}$ (if P is probability law for Brownian motion started at fixed value x at time 0).

Blumenthal's 0-1 law

- ▶ If $A \in \mathcal{F}_0^+$, then $P(A) \in \{0, 1\}$ (if P is probability law for Brownian motion started at fixed value x at time 0).
- ▶ There's nothing you can learn from infinitesimal neighborhood of future.