18.175: Lecture 36

Brownian motion

Scott Sheffield

MIT

Outline

Brownian motion properties and construction

Markov property, Blumenthal's 0-1 law

Outline

Brownian motion properties and construction

Markov property, Blumenthal's 0-1 law

Basic properties

- Brownian motion is real-valued process B_t, $t \geq 0$.

Basic properties

- Brownian motion is real-valued process $B_t,\ t \geq 0$.
- **Independent increments:** If $t_0 < t_1 < t_2 \dots$ then $B(t_0), B(t_1 - t_0), B(t_2 - t_1), \dots$ are independent.

Basic properties

- Brownian motion is real-valued process $B_t,\ t \geq 0$.
- **Independent increments:** If $t_0 < t_1 < t_2 \ldots$ then $B(t_0), B(t_1 - t_0), B(t_2 - t_1), \ldots$ are independent.
- **Gaussian increments:** If $s, t \geq 0$ then $B(s+t) - B(s)$ is normal with variance t.

Basic properties

- Brownian motion is real-valued process B_t, $t \geq 0$.
- **Independent increments:** If $t_0 < t_1 < t_2 \ldots$ then $B(t_0), B(t_1 - t_0), B(t_2 - t_1), \ldots$ are independent.
- **Gaussian increments:** If $s, t \geq 0$ then $B(s + t) - B(s)$ is normal with variance t.
- **Continuity:** With probability one, $t \to B_t$ is continuous.

Basic properties

- Brownian motion is real-valued process B_t, $t \geq 0$.
- **Independent increments:** If $t_0 < t_1 < t_2 \ldots$ then $B(t_0), B(t_1 - t_0), B(t_2 - t_1), \ldots$ are independent.
- **Gaussian increments:** If $s, t \geq 0$ then $B(s+t) - B(s)$ is normal with variance t.
- **Continuity:** With probability one, $t \to B_t$ is continuous.
- Hmm... does this mean we need to use a σ-algebra in which the event "B_t is continuous" is a measurable?

Basic properties

- Brownian motion is real-valued process B_t, $t \geq 0$.
- **Independent increments:** If $t_0 < t_1 < t_2 \ldots$ then $B(t_0), B(t_1 - t_0), B(t_2 - t_1), \ldots$ are independent.
- **Gaussian increments:** If $s, t \geq 0$ then $B(s+t) - B(s)$ is normal with variance t.
- **Continuity:** With probability one, $t \to B_t$ is continuous.
- Hmm... does this mean we need to use a σ-algebra in which the event "B_t is continuous" is a measurable?
- Suppose Ω is set of all functions of t, and we use smallest σ-field that makes each B_t a measurable random variable... does that fail?

Basic properties

- Translation invariance: is $B_{t_0+t} - B_{t_0}$ a Brownian motion?

Basic properties

- Translation invariance: is $B_{t_0+t} - B_{t_0}$ a Brownian motion?
- Brownian scaling: fix c, then B_{ct} agrees in law with $c^{1/2}B_t$.

Basic properties

- Translation invariance: is $B_{t_0+t} - B_{t_0}$ a Brownian motion?
- Brownian scaling: fix c, then B_{ct} agrees in law with $c^{1/2}B_t$.
- Another characterization: B is jointly Gaussian, $EB_s = 0$, $EB_sB_t = s \wedge t$, and $t \to B_t$ a.s. continuous.

Defining Brownian motion

- Can define joint law of B_t values for any finite collection of values.

Defining Brownian motion

- Can define joint law of B_t values for any finite collection of values.
- Can observe consistency and extend to countable set by Kolmogorov. This gives us measure in σ-field $\mathcal{F}_0$ generated by cylinder sets.

Defining Brownian motion

- Can define joint law of B_t values for any finite collection of values.
- Can observe consistency and extend to countable set by Kolmogorov. This gives us measure in σ-field $\mathcal{F}_0$ generated by cylinder sets.
- But not enough to get a.s. continuity.

Defining Brownian motion

- Can define joint law of B_t values for any finite collection of values.
- Can observe consistency and extend to countable set by Kolmogorov. This gives us measure in σ-field $\mathcal{F}_0$ generated by cylinder sets.
- But not enough to get a.s. continuity.
- Can define Brownian motion jointly on diadic rationals pretty easily. And claim that this a.s. extends to continuous path in unique way.

Defining Brownian motion

- Can define joint law of B_t values for any finite collection of values.
- Can observe consistency and extend to countable set by Kolmogorov. This gives us measure in σ-field $\mathcal{F}_0$ generated by cylinder sets.
- But not enough to get a.s. continuity.
- Can define Brownian motion jointly on diadic rationals pretty easily. And claim that this a.s. extends to continuous path in unique way.
- Check out Kolmogorov continuity theorem.

Defining Brownian motion

- Can define joint law of B_t values for any finite collection of values.
- Can observe consistency and extend to countable set by Kolmogorov. This gives us measure in σ-field $\mathcal{F}_0$ generated by cylinder sets.
- But not enough to get a.s. continuity.
- Can define Brownian motion jointly on diadic rationals pretty easily. And claim that this a.s. extends to continuous path in unique way.
- Check out Kolmogorov continuity theorem.
- Can prove Hölder continuity using similar estimates (see problem set).

Defining Brownian motion

- Can define joint law of B_t values for any finite collection of values.
- Can observe consistency and extend to countable set by Kolmogorov. This gives us measure in σ-field $\mathcal{F}_0$ generated by cylinder sets.
- But not enough to get a.s. continuity.
- Can define Brownian motion jointly on diadic rationals pretty easily. And claim that this a.s. extends to continuous path in unique way.
- Check out Kolmogorov continuity theorem.
- Can prove Hölder continuity using similar estimates (see problem set).
- Can extend to higher dimensions: make each coordinate independent Brownian motion.

Outline

Brownian motion properties and construction

Markov property, Blumenthal's 0-1 law

Outline

Brownian motion properties and construction

Markov property, Blumenthal's 0-1 law

More σ-algebra thoughts

- Write $\mathcal{F}_s^o = \sigma(B_r : r \leq s)$.

More σ-algebra thoughts

- Write $\mathcal{F}_s^o = \sigma(B_r : r \leq s)$.
- Write $\mathcal{F}_s^+ = \cap_{t>s} \mathcal{F}_t^o$

More σ-algebra thoughts

- Write $\mathcal{F}_s^o = \sigma(B_r : r \leq s)$.
- Write $\mathcal{F}_s^+ = \cap_{t>s} \mathcal{F}_t^o$
- Note right continuity: $\cap_{t>s} \mathcal{F}_t^+ = \mathcal{F}_s^+$.

More σ-algebra thoughts

- Write $\mathcal{F}_s^o = \sigma(B_r : r \le s)$.
- Write $\mathcal{F}_s^+ = \cap_{t>s}\mathcal{F}_t^o$
- Note right continuity: $\cap_{t>s}\mathcal{F}_t^+ = \mathcal{F}_s^+$.
- $\mathcal{F}_s^+$ allows an "infinitesimal peek at future"

Markov property

- If $s \geq 0$ and Y is bounded and $\mathcal{C}$-measurable, then for all $x \in \mathbb{R}^d$, we have
$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_{B_s} Y,$$
where the RHS is function $\phi(x) = E_x Y$ evaluated at $x = B_s$.

Blumenthal's 0-1 law

- If $A \in \mathcal{F}_0^+$, then $P(A) \in \{0, 1\}$ (if P is probability law for Brownian motion started at fixed value x at time 0).

Blumenthal's 0-1 law

- If $A \in \mathcal{F}_0^+$, then $P(A) \in \{0, 1\}$ (if P is probability law for Brownian motion started at fixed value x at time 0).
- There's nothing you can learn from infinitesimal neighborhood of future.