18.175: Lecture 3

Random variables and distributions

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Outline

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Characterizing measures on $\mathbb{R}^d$

Random variables

Recall definitions

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- The **Borel** σ-**algebra** $\mathcal{B}$ on a topological space is the smallest σ-algebra containing all open sets.

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- Borel σ-algebra is generated by open sets. Sometimes consider "completion" formed by tossing in measure zero sets.
- Caratheéodory Extension Theorem tells us that if we want to construct a measure on a σ-algebra, it is enough to construct the measure on an algebra that generates it.

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- Also proved using Caratheéodory Extension Theorem.

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- What functions can be distributions of random variables?
- Non-decreasing, right-continuous, with $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

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- Higher dimensional density functions analogously defined.

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- Given infinite sequence of random variables, consider the event that they converge to a limit. Is this a measurable event?
- Yes. If it has measure one, we say sequence converges almost surely.