Characterizing measures on $\mathbb{R}^d$

Random variables
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Recall definitions

- **Probability space** is triple \((\Omega, \mathcal{F}, P)\) where \(\Omega\) is sample space, \(\mathcal{F}\) is set of events (the \(\sigma\)-algebra) and \(P : \mathcal{F} \to [0, 1]\) is the probability function.
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- **$\sigma$-algebra** is collection of subsets closed under complementation and countable unions. Call $(\Omega, \mathcal{F})$ a measure space.

- A **measure** $\mu$ is a function $\mu : \mathcal{F} \to \mathbb{R}$ satisfying $\mu(\emptyset) = 0$ for all $A \in \mathcal{F}$ and countable additivity:
  $$\mu \left( \bigcup_{i} A_i \right) = \sum_{i} \mu(A_i)$$
  for disjoint $A_i$.

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Recall $\sigma$-algebra story

- Want, a priori, to define measure of \textit{any} subsets of $[0, 1)$. 

Borel $\sigma$-algebra is generated by open sets. Sometimes consider "completion" formed by tossing in measure zero sets.

Carathéodory Extension Theorem tells us that if we want to construct a measure on a $\sigma$-algebra, it is enough to construct the measure on an algebra that generates it.
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- Want, a priori, to define measure of *any* subsets of $[0, 1)$.
- Find that if we allow the axiom of choice and require measures to be countably additive (as we do) then we run into trouble. No valid translation invariant way to assign a finite measure to all subsets of $[0, 1)$.
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Recall construction of measures on $\mathbb{R}$

- Write $F(a) = P((-\infty, a])$.

Theorem: for each right continuous, non-decreasing function $F$, tending to 0 at $-\infty$ and to 1 at $\infty$, there is a unique measure defined on the Borel sets of $\mathbb{R}$ with $P((a, b]) = F(b) - F(a)$.

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Characterizing probability measures on $\mathbb{R}^d$

- Want to have $F(x) = \mu(-\infty, x_1] \times (\infty, x_2] \times \ldots \times (-\infty, x_n]$. 

- Given such an $F$, can compute $\mu$ of any finite rectangle of form $\prod (a_i, b_i]$ by taking differences of $F$ applied to vertices.

- Theorem: Given $F$, there is a unique measure whose values on finite rectangles are determined this way (provided that $F$ is non-decreasing, right continuous, and assigns a non-negative value to each rectangle).

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Outline

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Defining random variables

- Random variable is a *measurable* function from \((\Omega, \mathcal{F})\) to \((\mathbb{R}, \mathcal{B})\). That is, a function \(X : \Omega \rightarrow \mathbb{R}\) such that the preimage of every set in \(\mathcal{B}\) is in \(\mathcal{F}\). Say \(X\) is \(\mathcal{F}\)-measurable.
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- Question: to prove \(X\) is measurable, is it enough to show that the pre-image of every open set is in \(\mathcal{F}\)?

- Example of random variable: indicator function of a set. Or sum of finitely many indicator functions of sets.

- Let \(F(x) = F_X(x) = P(X \leq x)\) be distribution function for \(X\). Write \(f = f_X = F_X'\) for density function of \(X\).
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- **Theorem:** If \(X^{-1}(A) \in \mathcal{F}\) for all \(A \in \mathcal{A}\) and \(\mathcal{A}\) generates \(\mathcal{S}\), then \(X\) is a measurable map from \((\Omega, \mathcal{F})\) to \((\mathcal{S}, \mathcal{S})\).
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Random variable is a *measurable* function from \((Ω, F)\) to \((\mathbb{R}, \mathcal{B})\). That is, a function \(X : Ω → \mathbb{R}\) such that the preimage of every set in \(\mathcal{B}\) is in \(F\). Say \(X\) is \(F\)-measurable.

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What functions can be distributions of random variables?
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- What functions can be distributions of random variables?

- Non-decreasing, right-continuous, with $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$. 

Examples of possible random variable laws

- Other examples of distribution functions: uniform on $[0, 1]$, exponential with rate $\lambda$, standard normal, Cantor set measure.
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- Higher dimensional density functions analogously defined.
Other properties

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- If $X_1, \ldots, X_n$ are random variables in $\mathbb{R}$, defined on the same measure space, then $(X_1, \ldots, X_n)$ is a random variable in $\mathbb{R}^n$.
- Sums and products of finitely many random variables are random variables. If $X_i$ is a countable sequence of random variables, then $\inf_n X_n$ is a random variable. Same for $\lim \inf$, $\sup$, $\lim \sup$.
- Given an infinite sequence of random variables, consider the event that they converge to a limit. Is this a measurable event?
- Yes. If it has measure one, we say the sequence converges almost surely.
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