18.175: Lecture 28

Even more on martingales

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Outline

Recollections

More martingale theorems

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More martingale theorems

Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of** X **given** $\mathcal{F}$ is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.

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- **Theorem:** Up to redefinition on a measure zero set, the random variable $E(X|\mathcal{F})$ exists and is unique.
- This follows from Radon-Nikodym theorem.
- **Theorem:** $E(X|\mathcal{F}_i)$ is a martingale if $\mathcal{F}_i$ is an increasing sequence of σ-algebras and $E(|X|) < \infty$.

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- A sequence X_n is **adapted** to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ for all n. If X_n is an adapted sequence (with $E|X_n| < \infty$) then it is called a **martingale** if
$$E(X_{n+1}|\mathcal{F}_n) = X_n$$
for all n. It's a **supermartingale** (resp., **submartingale**) if same thing holds with $=$ replaced by $\leq$ (resp., $\geq$).

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- **Proof:** Just a special case of statement about $(H \cdot X)$ if stopping time is bounded.
- **Martingale convergence:** A non-negative martingale almost surely has a limit.
- **Idea of proof:** Count upcrossings (times martingale crosses a fixed interval) and devise gambling strategy that makes lots of money if the number of these is not a.s. finite. Basically, you buy every time price gets below the interval, sell each time it gets above.

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- How many primary candidates does one expect to ever exceed 20 percent on Intrade primary nomination market? (Asked by Aldous.)
- Compute probability of having a martingale price reach a before b if martingale prices vary continuously.
- Polya's urn: r red and g green balls. Repeatedly sample randomly and add extra ball of sampled color. Ratio of red to green is martingale, hence a.s. converges to limit.

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- **Proof idea:** Have $(EX_n^+)^p \le (E|X_n|)^p \le E|X_n|^p$ for martingale convergence theorem $X_n \to X$ a.s. Use L^p maximal inequality to get L^p convergence.

Orthogonality of martingale increments

- **Theorem:** Let X_n be a martingale with $EX_n^2 < \infty$ for all n. If $m \leq n$ and $Y \in \mathcal{F}_m$ with $EY^2 < \infty$, then $E\big((X_n - X_m)Y\big) = 0$.

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- **Conditional variance theorem:** If X_n is a martingale with $EX_n^2 < \infty$ for all n then $E\big((X_n - X_m)^2|\mathcal{F}_m\big) = E(X_n^2|\mathcal{F}_m) - X_m^2$.

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- Suppose we have a martingale X_n with $EX_n^2 < \infty$ for all n.
- We know X_n^2 is a submartingale. By Doob's decomposition, an write $X_n^2 = M_n + A_n$ where M_n is a martingale, and

$$A_n = \sum_{m=1}^{n} E(X_m^2|\mathcal{F}_{m-1}) - X_{m-1}^2 = \sum_{m=1}^{n} E\big((X_m - X_{m-1})^2|\mathcal{F}_{m-1}\big).$$

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- A_n in some sense measures total accumulated variance by time n.
- **Theorem:** $E(\sup_m |X_m|^2) \leq 4EA_\infty$
- **Proof idea:** L^2 maximal equality gives $E(\sup_{0 \leq m \leq n} |X_m|^2) \leq 4EX_n^2 = 4EA_n$. Use monotone convergence.

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- **Theorem:** $\lim_{n\to\infty} X_n$ exists and is finite a.s. on $\{A_\infty < \infty\}$.
- **Proof idea:** Try fixing a and truncating at time $N = \inf\{n : A_{n+1} > a^2\}$, use L^2 convergence theorem.

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- **Theorem:** If $X_n \to X$ in probability then the following are equivalent:
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  - $X_n \to X$ in L^1
  - $E|X_n| \to E|X| < \infty$

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- **Theorem:** $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. and in L^1.
- **Proof idea:** Use upcrosing inequality to show expected number of upcrossings of any interval is finite. Since $X_n = E(X_0|\mathcal{F}_n)$ the X_n are uniformly integrable, and we can deduce convergence in L^1.

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- **Theorem:** For any stopping time $N \leq \infty$, we have $EX_0 \leq EX_N \leq EX_\infty$ where $X_\infty = \lim X_n$.
- **Fairly general form of optional stopping theorem:** If $L \leq M$ are stopping times and $Y_{M\wedge n}$ is a uniformly integrable submartingale, then $EY_L \leq EY_M$ and $Y_L \leq E(Y_M|\mathcal{F}_L)$.