18.175: Lecture 27

More on martingales

Scott Sheffield

MIT

Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories

Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories

Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of** X **given** $\mathcal{F}$ is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.

Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of X given $\mathcal{F}$** is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.
- We require that Y is $\mathcal{F}$ measurable and that for all A in $\mathcal{F}$, we have $\int_A XdP = \int_A YdP$.

Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of X given $\mathcal{F}$** is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.
- We require that Y is $\mathcal{F}$ measurable and that for all A in $\mathcal{F}$, we have $\int_A X dP = \int_A Y dP$.
- Any Y satisfying these properties is called a **version** of $E(X|\mathcal{F})$.

Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of X given $\mathcal{F}$** is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.
- We require that Y is $\mathcal{F}$ measurable and that for all A in $\mathcal{F}$, we have $\int_A XdP = \int_A YdP$.
- Any Y satisfying these properties is called a **version** of $E(X|\mathcal{F})$.
- **Theorem:** Up to redefinition on a measure zero set, the random variable $E(X|\mathcal{F})$ exists and is unique.

Recall: conditional expectation

- Say we're given a probability space $(\Omega, \mathcal{F}_0, P)$ and a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a random variable X measurable w.r.t. $\mathcal{F}_0$, with $E|X| < \infty$. The **conditional expectation of X given $\mathcal{F}$** is a new random variable, which we can denote by $Y = E(X|\mathcal{F})$.
- We require that Y is $\mathcal{F}$ measurable and that for all A in $\mathcal{F}$, we have $\int_A X dP = \int_A Y dP$.
- Any Y satisfying these properties is called a **version** of $E(X|\mathcal{F})$.
- **Theorem:** Up to redefinition on a measure zero set, the random variable $E(X|\mathcal{F})$ exists and is unique.
- This follows from Radon-Nikodym theorem.

Conditional expectation observations

- Linearity: $E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.

Conditional expectation observations

- Linearity: $E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.
- If $X \leq Y$ then $E(E|\mathcal{F}) \leq E(Y|\mathcal{F})$.

Conditional expectation observations

- Linearity: $E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.
- If $X \leq Y$ then $E(E|\mathcal{F}) \leq E(Y|\mathcal{F})$.
- If $X_n \geq 0$ and $X_n \uparrow X$ with $EX < \infty$, then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$ (by dominated convergence).

Conditional expectation observations

- Linearity: $E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.
- If $X \leq Y$ then $E(E|\mathcal{F}) \leq E(Y|\mathcal{F})$.
- If $X_n \geq 0$ and $X_n \uparrow X$ with $EX < \infty$, then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$ (by dominated convergence).
- If $\mathcal{F}_1 \subset \mathcal{F}_2$ then

Conditional expectation observations

- Linearity: $E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.
- If $X \le Y$ then $E(E|\mathcal{F}) \le E(Y|\mathcal{F})$.
- If $X_n \ge 0$ and $X_n \uparrow X$ with $EX < \infty$, then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$ (by dominated convergence).
- If $\mathcal{F}_1 \subset \mathcal{F}_2$ then
  - $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$.

Conditional expectation observations

- Linearity: $E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.
- If $X \leq Y$ then $E(E|\mathcal{F}) \leq E(Y|\mathcal{F})$.
- If $X_n \geq 0$ and $X_n \uparrow X$ with $EX < \infty$, then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$ (by dominated convergence).
- If $\mathcal{F}_1 \subset \mathcal{F}_2$ then
  - $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$.
  - $E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.

Conditional expectation observations

- Linearity: $E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.
- If $X \leq Y$ then $E(E|\mathcal{F}) \leq E(Y|\mathcal{F})$.
- If $X_n \geq 0$ and $X_n \uparrow X$ with $EX < \infty$, then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$ (by dominated convergence).
- If $\mathcal{F}_1 \subset \mathcal{F}_2$ then
  - $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$.
  - $E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.
- Second is kind of interesting: says, after I learn $\mathcal{F}_1$, my best guess of what my best guess for X will be after learning $\mathcal{F}_2$ is simply my current best guess for X.

Conditional expectation observations

- Linearity: $E(aX+Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$.
- If $X \le Y$ then $E(E|\mathcal{F}) \le E(Y|\mathcal{F})$.
- If $X_n \ge 0$ and $X_n \uparrow X$ with $EX < \infty$, then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$ (by dominated convergence).
- If $\mathcal{F}_1 \subset \mathcal{F}_2$ then
  - $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$.
  - $E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.
- Second is kind of interesting: says, after I learn $\mathcal{F}_1$, my best guess of what my best guess for X will be after learning $\mathcal{F}_2$ is simply my current best guess for X.
- Deduce that $E(X|\mathcal{F}_i)$ is a martingale if $\mathcal{F}_i$ is an increasing sequence of σ-algebras and $E(|X|) < \infty$.

Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories

Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories

Martingales

- Let $\mathcal{F}_n$ be increasing sequence of σ-fields (called a **filtration**).

Martingales

- Let $\mathcal{F}_n$ be increasing sequence of σ-fields (called a **filtration**).
- A sequence X_n is **adapted** to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ for all n. If X_n is an adapted sequence (with $E|X_n| < \infty$) then it is called a **martingale** if
$$E(X_{n+1}|\mathcal{F}_n) = X_n$$
for all n. It's a **supermartingale** (resp., **submartingale**) if same thing holds with $=$ replaced by $\leq$ (resp., $\geq$).

Martingale observations

- **Claim:** If X_n is a supermartingale then for $n > m$ we have $E(X_n|\mathcal{F}_m) \le X_m$.

Martingale observations

- **Claim:** If X_n is a supermartingale then for $n > m$ we have $E(X_n|\mathcal{F}_m) \leq X_m$.
- **Proof idea:** Follows if $n = m + 1$ by definition; take $n = m + k$ and use induction on k.

Martingale observations

- **Claim:** If X_n is a supermartingale then for $n > m$ we have $E(X_n|\mathcal{F}_m) \le X_m$.
- **Proof idea:** Follows if $n = m + 1$ by definition; take $n = m + k$ and use induction on k.
- Similar result holds for submartingales. Also, if X_n is a martingale and $n > m$ then $E(X_n|\mathcal{F}_m) = X_m$.

Martingale observations

- **Claim:** If X_n is a supermartingale then for $n > m$ we have $E(X_n|\mathcal{F}_m) \le X_m$.
- **Proof idea:** Follows if $n = m + 1$ by definition; take $n = m + k$ and use induction on k.
- Similar result holds for submartingales. Also, if X_n is a martingale and $n > m$ then $E(X_n|\mathcal{F}_m) = X_m$.
- **Claim:** if X_n is a martingale w.r.t. $\mathcal{F}_n$ and ϕ is convex with $E|\phi(X_n)| < \infty$ then $\phi(X_n)$ is a submartingale.

Martingale observations

- **Claim:** If X_n is a supermartingale then for $n > m$ we have $E(X_n|\mathcal{F}_m) \leq X_m$.
- **Proof idea:** Follows if $n = m + 1$ by definition; take $n = m + k$ and use induction on k.
- Similar result holds for submartingales. Also, if X_n is a martingale and $n > m$ then $E(X_n|\mathcal{F}_m) = X_m$.
- **Claim:** if X_n is a martingale w.r.t. $\mathcal{F}_n$ and ϕ is convex with $E|\phi(X_n)| < \infty$ then $\phi(X_n)$ is a submartingale.
- **Proof idea:** Immediate from Jensen's inequality and martingale definition.

Martingale observations

- **Claim:** If X_n is a supermartingale then for $n > m$ we have $E(X_n|\mathcal{F}_m) \le X_m$.
- **Proof idea:** Follows if $n = m + 1$ by definition; take $n = m + k$ and use induction on k.
- Similar result holds for submartingales. Also, if X_n is a martingale and $n > m$ then $E(X_n|\mathcal{F}_m) = X_m$.
- **Claim:** if X_n is a martingale w.r.t. $\mathcal{F}_n$ and ϕ is convex with $E|\phi(X_n)| < \infty$ then $\phi(X_n)$ is a submartingale.
- **Proof idea:** Immediate from Jensen's inequality and martingale definition.
- Example: take $\phi(x) = \max\{x, 0\}$.

Predictable sequence

- Call H_n **predictable** if each $H + n$ is $\mathcal{F}_{n-1}$ measurable.

Predictable sequence

- Call H_n **predictable** if each $H+n$ is $\mathcal{F}_{n-1}$ measurable.
- Maybe H_n represents amount of shares of asset investor has at nth stage.

Predictable sequence

- Call H_n **predictable** if each $H + n$ is $\mathcal{F}_{n-1}$ measurable.
- Maybe H_n represents amount of shares of asset investor has at nth stage.
- Write $(H \cdot X)_n = \sum_{m=1}^n H_m(X_m - X_{m-1})$.

Predictable sequence

- Call H_n **predictable** if each $H+n$ is $\mathcal{F}_{n-1}$ measurable.
- Maybe H_n represents amount of shares of asset investor has at nth stage.
- Write $(H\cdot X)_n = \sum_{m=1}^n H_m(X_m - X_{m-1})$.
- **Observe:** If X_n is a supermartingale and the $H_n \geq 0$ are bounded, then $(H\cdot X)_n$ is a supermartingale.

Predictable sequence

- Call H_n **predictable** if each $H+n$ is $\mathcal{F}_{n-1}$ measurable.
- Maybe H_n represents amount of shares of asset investor has at nth stage.
- Write $(H \cdot X)_n = \sum_{m=1}^n H_m(X_m - X_{m-1})$.
- **Observe:** If X_n is a supermartingale and the $H_n \geq 0$ are bounded, then $(H \cdot X)_n$ is a supermartingale.
- Example: take $H_n = 1_{N \geq n}$ for stopping time N.

Two big results

- **Optional stopping theorem:** Can't make money in expectation by timing sale of asset whose price is non-negative martingale.

Two big results

- **Optional stopping theorem:** Can't make money in expectation by timing sale of asset whose price is non-negative martingale.
- **Proof:** Just a special case of statement about $(H \cdot X)$ if stopping time is bounded.

Two big results

- **Optional stopping theorem:** Can't make money in expectation by timing sale of asset whose price is non-negative martingale.
- **Proof:** Just a special case of statement about $(H \cdot X)$ if stopping time is bounded.
- **Martingale convergence:** A non-negative martingale almost surely has a limit.

Two big results

- **Optional stopping theorem:** Can't make money in expectation by timing sale of asset whose price is non-negative martingale.
- **Proof:** Just a special case of statement about $(H \cdot X)$ if stopping time is bounded.
- **Martingale convergence:** A non-negative martingale almost surely has a limit.
- **Idea of proof:** Count upcrossings (times martingale crosses a fixed interval) and devise gambling strategy that makes lots of money if the number of these is not a.s. finite. Basically, you buy every time price gets below the interval, sell each time it gets above.

Two big results

- **Optional stopping theorem:** Can't make money in expectation by timing sale of asset whose price is non-negative martingale.
- **Proof:** Just a special case of statement about $(H \cdot X)$ if stopping time is bounded.
- **Martingale convergence:** A non-negative martingale almost surely has a limit.
- **Idea of proof:** Count upcrossings (times martingale crosses a fixed interval) and devise gambling strategy that makes lots of money if the number of these is not a.s. finite. Basically, you buy every time price gets below the interval, sell each time it gets above.
- **Stronger convergence statement:** If X_n is a submartingale with $\sup EX_n^+ < \infty$ then as $n \to \infty$, X_+n converges a.s. to a limit X with $E|X| < \infty$.

Other statements

- If X_n is a supermartingale then as $n \to \infty$, $X_n \to X$ a.s. and $EX \leq EX_0$.

Other statements

- If X_n is a supermartingale then as $n \to \infty$, $X_n \to X$ a.s. and $EX \le EX_0$.
- **Proof:** $Y_n = -X_n \le 0$ is a submartingale with $EY^+ = 0$. Since $EX_0 \ge EX_n$, inequality follows from Fatou's lemma.

Other statements

- If X_n is a supermartingale then as $n \to \infty$, $X_n \to X$ a.s. and $EX \le EX_0$.
- **Proof:** $Y_n = -X_n \le 0$ is a submartingale with $EY^+ = 0$. Since $EX_0 \ge EX_n$, inequality follows from Fatou's lemma.
- **Doob's decomposition:** Any submartingale X_n can be written in a unique way as $X_n = M_n + A_n$ where M_n is a martingale and A_n is a predictable increasing sequence with $A_0 = 0$.

Other statements

- If X_n is a supermartingale then as $n \to \infty$, $X_n \to X$ a.s. and $EX \le EX_0$.
- **Proof:** $Y_n = -X_n \le 0$ is a submartingale with $EY^+ = 0$. Since $EX_0 \ge EX_n$, inequality follows from Fatou's lemma.
- **Doob's decomposition:** Any submartingale X_n can be written in a unique way as $X_n = M_n + A_n$ where M_n is a martingale and A_n is a predictable increasing sequence with $A_0 = 0$.
- **Proof idea:** Just let M_n be sum of "surprises" (i.e., the values $X_n - E(X_n|\mathcal{F}_{n-1})$).

Other statements

- If X_n is a supermartingale then as $n \to \infty$, $X_n \to X$ a.s. and $EX \leq EX_0$.
- **Proof:** $Y_n = -X_n \leq 0$ is a submartingale with $EY^+ = 0$. Since $EX_0 \geq EX_n$, inequality follows from Fatou's lemma.
- **Doob's decomposition:** Any submartingale X_n can be written in a unique way as $X_n = M_n + A_n$ where M_n is a martingale and A_n is a predictable increasing sequence with $A_0 = 0$.
- **Proof idea:** Just let M_n be sum of "surprises" (i.e., the values $X_n - E(X_n|\mathcal{F}_{n-1})$).
- A martingale with bounded increments a.s. either converges to limit or oscillates between $\pm\infty$. That is, a.s. either $\lim X_n < \infty$ exists or $\limsup X_n = +\infty$ and $\liminf X_n = -\infty$.

Problems

- How many primary candidates does one expect to ever exceed 20 percent on Intrade? (Asked by Aldous.)

Problems

- How many primary candidates does one expect to ever exceed 20 percent on Intrade? (Asked by Aldous.)
- Compute probability of having a martingale price reach a before b if martingale prices vary continuously.

Problems

- How many primary candidates does one expect to ever exceed 20 percent on Intrade? (Asked by Aldous.)
- Compute probability of having a martingale price reach a before b if martingale prices vary continuously.
- Polya's urn: r red and g green balls. Repeatedly sample randomly and add extra ball of sampled color. Ratio of red to green is martingale, hence a.s. converges to limit.

Wald

- **Wald's equation:** Let X_i be i.i.d. with $E|X_i| < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = EX_1 EN$.

Wald

- **Wald's equation:** Let X_i be i.i.d. with $E|X_i| < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = EX_1 EN$.
- **Wald's second equation:** Let X_i be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$.

Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}\}$. What is $\mathbb{E}S_N$?

Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}$. What is $\mathbb{E}S_N$?
- What is $\mathbb{E}N$?

Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories

Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories

Reflection principle

- How many walks from $(0, x)$ to (n, y) that don't cross the horizontal axis?

Reflection principle

- How many walks from $(0, x)$ to (n, y) that don't cross the horizontal axis?
- Try counting walks that *do* cross by giving bijection to walks from $(0, -x)$ to (n, y).

Ballot Theorem

- Suppose that in election candidate A gets α votes and B gets $\beta < \alpha$ votes. What's probability that A is ahead throughout the counting?

Ballot Theorem

- Suppose that in election candidate A gets α votes and B gets $\beta < \alpha$ votes. What's probability that A is ahead throughout the counting?
- Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.

Arcsin theorem

- Theorem for last hitting time.

Arcsin theorem

- Theorem for last hitting time.
- Theorem for amount of positive positive time.