18.175: Lecture 2

Extension theorems: a tool for constructing measures

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Outline

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Distributions on $\mathbb{R}$

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- Price to this decision: for the rest of our lives, whenever we talk about a measure on any space (a Euclidean space, a space of differentiable functions, a space of fractal curves embedded in a plane, etc.), we have to worry about what the σ-algebra might be.

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- Answer: use extension theorems.

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- Measure μ is **probability measure** if $\mu(\Omega) = 1$.
- The **Borel σ-algebra** $\mathcal{B}$ on a topological space is the smallest σ-algebra containing all open sets.

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- If we're given such a function F, then we know how to compute the measure of any set of the form $(a, b]$.
- We would like to *extend* the measure defined for these subsets to a measure defined for the whole σ algebra generated by these subsets.
- Seems clear how to define measure of countable union of disjoint intervals of the form $(a, b]$ (just using countable additivity). But are we confident we can extend the definition to *all* Borel measurable sets in a consistent way?

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- One lemma: If $\mathcal{S}$ is a semialgebra, then the set $\overline{S}$ of finite disjoint unions of sets in $\mathcal{S}$ is an algebra, called the **algebra generated by** $\mathcal{S}$.

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- THEOREM: If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system that contains $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$, where $\sigma(\mathcal{A})$ denotes smallest σ-algebra containing $\mathcal{A}$.

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- Detailed proof is somewhat involved, but let's take a look at it.
- We can use this extension theorem prove existence of a unique translation invariant measure (Lebesgue measure) on the Borel sets of $\mathbb{R}^d$ that assigns unit mass to a unit cube. (Borel σ-algebra $\mathcal{R}^d$ is the smallest one containing all open sets of $\mathbb{R}^d$. Given any space with a topology, we can define a σ-algebra this way.)

Extension theorem for semialgebras

- Say $\mathcal{S}$ is semialgebra and μ is defined on $\mathcal{S}$ with $\mu(\emptyset = 0)$, such that μ is finitely additive and countably subadditive. [This means that if $S \in \mathcal{S}$ is a finite disjoint union of sets $S_i \in \mathcal{S}$ then $\mu(S) = \sum_i \mu(S_i)$. If it is a countable disjoint union of $S_i \in \mathcal{S}$ then $\mu(S) \leq \sum_i \mu(S_i)$.] Then μ has a unique extension $\bar{\mu}$ that is a measure on the algebra $\overline{\mathcal{S}}$ generated by $\mathcal{S}$. If $\bar{\mu}$ is sigma-finite, then there is an extension that is a measure on $\sigma(S)$.