18.175: Lecture 1

Probability spaces and σ-algebras

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Outline

Probability spaces and σ-algebras

Distributions on $\mathbb{R}$

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Probability space notation

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- Measure μ is **probability measure** if $\mu(\Omega) = 1$.

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- **continuity from above:** measures of sets A_i in decreasing sequence converge to measure of intersection $\cap_i A_i$

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- Thus $[0,1) = \cup \tau_r(A)$ as r ranges over rationals in $[0,1)$.
- If $P(A) = 0$, then $P(S) = \sum_r P(\tau_r(A)) = 0$. If $P(A) > 0$ then $P(S) = \sum_r P(\tau_r(A)) = \infty$. Contradicts $P(S) = 1$ axiom.

Three ways to get around this

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- 3. **Keep the axiom of choice and countable additivity but don't define probabilities of all sets:** Restrict attention to some σ-algebra of measurable sets.
- Most mainstream probability and analysis takes the third approach. But good to be aware of alternatives (e.g., **axiom of determinacy** which implies that all sets are Lebesgue measurable).

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- Why does this notion make sense? If $\mathcal{F}_i$ are σ-fields (for i in possibly uncountable index set I) does this imply that $\cap_{i \in I} \mathcal{F}_i$ is a σ-field?

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- **Theorem:** for each right continuous, non-decreasing function F, tending to 0 at $-\infty$ and to 1 at ∞, there is a unique measure defined on the Borel sets of $\mathbb{R}$ with $P((a,b]) = F(b) - F(a)$.