The multiplicative Barratt-Priddy-Quillen theorem and beyond

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Outline

1. Background

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- K : **SymMonCat** $\to$ **Spectra** is the algebraic K-theory of symmetric monoidal categories;
- $\mathbb{S}$ is the sphere spectrum.

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Remark

It's enough to prove that $N(\Sigma)$ is the free E_∞ space on one generator.

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If this is a general pattern, then there must be a "free ring category on one object" whose K-theory is the free E_∞ ring spectrum on one generator

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My purpose in this talk is to give an easy model for this free ring category, and to prove that its K-theory is indeed $\mathbb{S}[x]$.

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With this definition, we can give a precise statement of our main result:

Theorem (G.)

$$K(\mathcal{P}) \cong \mathbb{S}[x]$$

as E_∞ ring spectra.

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- This gives us a new perspective from which to try to understand $\mathbb{S}[x]$, and thus power operations in an arbitrary homology theory.

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Our approach is very close to May's approach with $\mathcal{F} \wr \mathcal{F}$ spaces, although we'll denote it $\mathcal{F}^\wedge$ instead.

The theory of $\mathcal{F}^\wedge$-spaces is a multiplicative analog of the theory of Γ-spaces.

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Recall that a (special) Γ-space is a "weak functor" $X : \mathcal{F} \to \mathbf{Top}$ such that the n inert maps $\langle n \rangle \to \langle 1 \rangle$ give a homotopy product decomposition $X(\langle n \rangle) \cong X(\langle 1 \rangle)^{\times n}$.

(A morphism in $\mathcal{F}$ is inert if it's an isomorphism mod throwing some points away to the basepoint).

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We have the object $e = [\langle 1 \rangle, (\langle 1 \rangle)]$, and a natural bijection

$$\mathrm{Inert}([T, (U_t)_{t \in T^o}], e) \cong \left(\coprod_{t \in T^o} U_t^o \right).$$

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It'll be important to unpack what I mean by "weak functor", so let's do that.

Proposition (Lurie)

Let I be a small category. There's an equivalence of homotopy theories

$$\phi : \mathrm{Fun}(N(I), \mathcal{T}op) \xrightarrow{\sim} \{\text{Left fibrations } Y \to N(I)\},$$

where N is the nerve functor and $\mathcal{T}op$ is the ∞-category of spaces, such that

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This is extremely useful for us, because unlike weak functors, left fibrations can frequently be constructed at the level of 1-categories.

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We do this by giving an explicit left fibration

$$\Sigma[x] \to \mathcal{F}^\wedge$$

such that the fiber over $\Sigma[x]_e$ over e is $\mathrm{iso}(\mathcal{P})$.

Definition

An object of $\Sigma[x]$ is an object $[T, (U_t)]$ of $\mathcal{F}^\wedge$ together with

- for each $t \in T^o$, a morphism $\phi_{U_t} : U_t^+ \to U_t$ in $\mathcal{F}$;
- a morphism $\psi_U : U^+ \to \bigvee_t U_t^+$ in $\mathcal{F}$.

A morphism from $[T, (U_t), (U_t^+), U^+]$ to $[V, (W_v), (W_v^+), W^+]$ consists of

- A morphism $[f, (f_v)] : [T, (U_t)] \to [V, (W_v)]$ in $\mathcal{F}^\wedge$;
- for each $v \in V$ and $w \in W_v$, an isomorphism

$$f_w^+ : \left(\bigvee_{u_1 \wedge u_2 \wedge \cdots \wedge u_k \in f_v^{-1}(w)} \bigwedge_{i=1}^{k} \phi_{U_t}^{-1}(u_i) \right) \xrightarrow{\sim} \phi_{W_v}^{-1}(w);$$

- and for each $w^+ \in W_v^+$, an isomorphism

$$f_{w^+}^+ : \psi_V^{-1}(w^+) \xrightarrow{\sim} \bigvee_{i=1}^{k} \psi_U^{-1}(u_i^+)$$

where the u_i^+ are uniquely chosen so that $f_w^+(u_1^+ \wedge u_2^+ \wedge \cdots \wedge u_k^+) = w^+$.

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Let $i : \bullet_e \to \Sigma[x]$ be the inclusion of the object $\langle 1 \rangle \in \mathcal{P} \cong \Sigma[x]_e$.

Our game is to show that i induces a bijection

$$\mathrm{ho}(\mathbf{sSet}_{/\mathcal{F}^\wedge})[\Sigma[x], M] \xrightarrow{\sim} \mathrm{ho}(\mathbf{sSet}_{/\mathcal{F}^\wedge})[\bullet_e, M].$$

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The hard part is extending a map $\bullet_e \to M$ over $\Sigma[x]$.

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2. We have maps $\Sigma[x] \leftarrow Z \to M$, and the inclusion of $\bullet_e$ factors through Z.
3. The decorations are rigid enough that $Z \to \Sigma[x]$ is a trivial fibration. $\square$

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Let Sh be the shuffle operad (multicategory): its objects are finite ordered sets and to give a morphism

$$(S_1, S_2, \cdots, S_n) \to T$$

is to express T as a shuffle of $S_1, \cdots, S_n$.

Proposition

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- $K(\mathrm{Fun}(Sh, \mathcal{F}))$ is equivalent, as an E_∞ ring spectrum, to the free connective divided power spectrum on one generator.

Thanks for listening!