The multiplicative Barratt-Priddy-Quillen theorem and beyond

Saul Glasman

MIT

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\( E_\infty \) ring spaces

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- \( K : \text{SymMonCat} \to \text{Spectra} \) is the algebraic \( K \)-theory of symmetric monoidal categories;
- \( \mathbb{S} \) is the sphere spectrum.
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Remark

It’s enough to prove that $N(\Sigma)$ is the free $E_\infty$ space on one generator.
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If this is a general pattern, then there must be a “free ring category on one object” whose $K$-theory is the free $E_\infty$ ring spectrum on one generator

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My purpose in this talk is to give an easy model for this free ring category, and to prove that its $K$-theory is indeed $\mathbb{S}[x]$. 
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With this definition, we can give a precise statement of our main result:

**Theorem (G.)**

\[ K(P) \cong S[x] \]

as \( E_\infty \) ring spectra.
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- This gives us a new perspective from which to try to understand $\mathbb{S}[x]$, and thus power operations in an arbitrary homology theory.
The proof will proceed by showing that the nerve of the maximal subgroupoid $\text{iso}(\mathcal{P})$ of $\mathcal{P}$ is the free $E_\infty$ ring space on one generator, so we’d better fix our model for $E_\infty$ ring spaces.
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Our approach is very close to May’s approach with $\mathcal{F} \wr \mathcal{F}$ spaces, although we’ll denote it $\mathcal{F}^\wedge$ instead.
The theory of $\mathcal{F}^\wedge$-spaces is a multiplicative analog of the theory of $\Gamma$-spaces.
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Recall that a (special) $\Gamma$-space is a “weak functor” $X : \mathcal{F} \to \text{Top}$ such that the $n$ inert maps $\langle n \rangle \to \langle 1 \rangle$ give a homotopy product decomposition $X(\langle n \rangle) \cong X(\langle 1 \rangle)^{\times n}$.

(A morphism in $\mathcal{F}$ is inert if it’s an isomorphism mod throwing some points away to the basepoint).
Definition

$\mathcal{F}^\wedge$ is the category where

- an object, denoted $[T, (U_t)_{t \in T^o}]$, is a finite pointed set $T$ together with a finite pointed set $U_t$ for each $t \in T^o$;
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- a morphism from $[T, (U_t)_{t \in T^o}]$ to $[V, (W_v)_{v \in V^o}]$ is a map $f : T \to V$ and, for each $v \in V^o$, a map

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\]

We have the object \( e = [ \langle 1 \rangle, (\langle 1 \rangle) ] \), and a natural bijection

\[
  \text{Inert}( [ T, (U_t)_{t \in T^o} ], e ) \cong \left( \coprod_{t \in T^o} U_t^o \right).
\]
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We think of $M(e)$ as the underlying space of $M$. 

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It’ll be important to unpack what I mean by “weak functor”, so let’s do that.
Proposition (Lurie)

Let $I$ be a small category. There’s an equivalence of homotopy theories

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\phi : \text{Fun}(N(I), \mathcal{T}op) \sim \{\text{Left fibrations } Y \to N(I)\},
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This is extremely useful for us, because unlike weak functors, left fibrations can frequently be constructed at the level of 1-categories.
The first step is to model the $E_\infty$ ring structure on $N(\text{iso}(\mathcal{P}))$. 

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The first step is to model the $E_\infty$ ring structure on $N(\text{iso}(\mathcal{P}))$.

We do this by giving an explicit left fibration

$$\Sigma[x] \to \mathcal{F}^\wedge$$

such that the fiber over $\Sigma[x]_e$ over $e$ is $\text{iso}(\mathcal{P})$. 
Definition

An object of $\Sigma[x]$ is an object $[T, (U_t)]$ of $\mathcal{F}^\wedge$ together with

- for each $t \in T^o$, a morphism $\phi_U : U^+_t \to U_t$ in $\mathcal{F}$;
- a morphism $\psi_U : U^+ \to \bigvee_t U^+_t$ in $\mathcal{F}$.

A morphism from $[T, (U_t), (U^+_t), U^+]$ to $[V, (W_v), (W^+_v), W^+]$ consists of

- A morphism $[f, (f_v)] : [T, (U_t)] \to [V, (W_v)]$ in $\mathcal{F}^\wedge$;
- for each $v \in V$ and $w \in W_v$, an isomorphism

$$f_w^+ : \left( \bigvee_{u_1 \wedge u_2 \wedge \cdots \wedge u_k \in f_v^{-1}(w)} \bigwedge_{i=1}^k \phi_U^{-1}(u_i) \right) \xrightarrow{\sim} \phi_W^{-1}(w);$$

- and for each $w^+ \in W^+_v$, an isomorphism

$$f_w^+ : \psi_V^{-1}(w^+) \xrightarrow{\sim} \bigvee_{i=1}^k \psi_U^{-1}(u_i^+),$$

where the $u_i^+$ are uniquely chosen so that $f_w^+(u_1^+ \wedge u_2^+ \wedge \cdots \wedge u_k^+) = w^+$. 
If \( N(\Sigma[x]) \) is to be thought of as the free \( E_\infty \) ring space on one generator, it ought to corepresent the underlying space functor on the category of \( E_\infty \) ring spaces.
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Let \( i : \bullet_e \to \Sigma[x] \) be the inclusion of the object \( \langle 1 \rangle \in \mathcal{P} \cong \Sigma[x]_e \).
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Let $i: \bullet_e \to \Sigma[x]$ be the inclusion of the object $\langle 1 \rangle \in \mathcal{P} \cong \Sigma[x]_e$.

Our game is to show that $i$ induces a bijection

$$\text{ho}(\text{sSet}_{/\mathcal{F}^\wedge})[\Sigma[x], M] \sim \text{ho}(\text{sSet}_{/\mathcal{F}^\wedge})[\bullet_e, M].$$
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The hard part is extending a map $\bullet_e \rightarrow M$ over $\Sigma[x]$. 

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1. Use $i$ to define a new simplicial set $Z$ over $\mathcal{F}^\wedge$. A simplex of $Z$ is a simplex of $M$ with some decorations making it look like a simplex of $\Sigma[x]$. 
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3. The decorations are rigid enough that $Z \to \Sigma[x]$ is a trivial fibration.
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Let $\mathcal{Sh}$ be the shuffle operad (multicategory): its objects are finite ordered sets and to give a morphism

$$(S_1, S_2, \cdots, S_n) \to T$$

is to express $T$ as a shuffle of $S_1, \cdots, S_n$. 
Proposition

- $\text{Fun}(\mathcal{S}h, \mathcal{F})$ is a symmetric bimonoidal category under Day convolution and objectwise coproduct.
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- $\text{Fun}(\text{Sh}, \mathcal{F})$ is a symmetric bimonoidal category under Day convolution and objectwise coproduct.

- $K(\text{Fun}(\text{Sh}, \mathcal{F}))$ is equivalent, as an $E_\infty$ ring spectrum, to the free connective divided power spectrum on one generator.
Thanks for listening!