The moduli stack of elliptic curves

Functor of points!

\[ Y : \mathcal{G} \rightarrow [\text{Aff}^{\text{op}}, \text{Set}] \]

Elements of \( X(R) \) are called \( R \)-points, because...

Great for constructing morphisms. Less good for constructing objects: no general sufficient conditions, but many necessary ones, for example various sheaf conditions.

To specify a map, enough to specify it on an open Zariski cover. To encode this, let (say) \( U, V \) be a Zariski open cover of \( X \), and let \( F : \text{Sch}^{\text{op}} \rightarrow \text{Set} \).

The Zariski descent for this cover means

\[ T \emptyset = U \cup V \quad T \emptyset \rightarrow X \]

\[ E \times X E \subseteq E \text{ (reflexive coeq / } \perp \text{-truncated simplicial diagram)} \]

\[ F(X) \cong \text{lim} \left( F(E) \subseteq F(E \times X E) \right) \]

We'll come back to this later.

An elliptic curve \( E/S \) is a genus smooth, proper alg curve of genus 1 (i.e. \( \Omega_{E/S} \) is trivial) with a distinguished \( S \)-point. \( S \) will usually be a DVR or field.
This is a group scheme:

- Can embed \( E \) in \( \mathbb{P}^2 \), as a cubic. Any line \( L \subset \mathbb{P}^2 \) intersects \( E \) (with mult) at three points \( P_1, P_2, P_3 \). Say \( P_1 + P_2 + P_3 = 0 \).

- Over \( \mathbb{C} \), \( E \) is (analytically) a complex torus. Usually torus addition.

- \( E \) parametrises line bundles over itself:
  \[
  E(\mathbb{R}) \equiv \{ \text{line bundles on } E \times \mathbb{R} \}/\sim.
  \]
  If \( R = k \) a field (for example) \( \sim \) is just isomorphism.
  The group law on \( E \) corresponds to \( \otimes \).

Seek a space \( \text{Mell} \) parametrising ell curves. First work over \( \mathbb{C} \). An ell curve is

so that \( \text{Mell}(\mathbb{R}) \)

\[
\{ \text{families of ell curves} \}/\sim.
\]

An \( EC \) is given by a lattice \( \mathbb{Z} \langle 1, \tau_1, \tau_2 \rangle \), where we may take \( \tau \in \mathbb{H} \), the upper half plane. Two \( \tau_1, \tau_2 \) give the same lattices \( \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) for some \( b \)

\[
(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in \text{SL}_2(\mathbb{Z}) \iff \tau_2 = \frac{a \tau_1 + b}{c \tau_1 + d}
\]

So we might try

\[
\text{Mell} := \frac{\mathbb{H}}{\text{SL}_2(\mathbb{Z})}.
\]

Assuming this is algebraic, \( \text{Mell}(\mathbb{C}) = \{ EC/\text{C} \}/\text{iso} \).
So this is a good first approximation, but can such an approach ever work? No, due to isotrivial families.

Let $S = \text{Spec } k$, char $k \neq 2$. Then fix an elliptic curve $E$. Let

$$X = G_m \times E \quad \frac{X}{\mathbb{Z}/2} \text{ by } (u \mapsto -u) \text{ on } G_m$$

$$(e \mapsto -e) \text{ on } E$$

$X/\mathbb{Z}/2$ is another $E$-bundle over $G_m$ in that all fibres are isomorphic to $E$. So the induced map $G_m \to \text{Mell}$ is constant. But $X/\mathbb{Z}/2 \neq G_m \times E$. How to fix this?

If $\text{Mell}$ is to be "just a scheme", there's really no way. We must look in the mirror and define an object by the functor it represents, as the functor $\text{Mell}$ represents should be groupoid-valued, to account for isomorphisms.

Sheaf of groupoids – homotopy descent.

Etale maps, etale descent.

So that's a stack – a groupoid-valued functor w/ sheaf condition (mention)

Aff$^{\text{op}} \to \text{Grd}$

You might think these things would be hard to work with, but one thing is clear: how to define limits.
for many nice stacks \( \mathcal{M} \)

In fact in many cases there is a morphism \( X \to \mathcal{M} \)

for \( X \) a scheme such that for any other \( \mathcal{M} \) from

a scheme \( Y \to \mathcal{M} \), \( X \times \mathcal{M} Y \) is a scheme, e.g.

moduli space of ell curves with level \( S \) structure,

\( \mathcal{M}_{ell}[S] \). No automorphism preserves this, so

and compact 1-d complex manifold, so alg curve.

\[
\begin{array}{c}
G_m \times E \\
\downarrow \text{r} \mapsto r^2 \\
G_m \\
text{exists.} \quad \mathcal{M}_{ell}
\end{array}
\]

Noncontractible loop.

A really down-to-earth presentation of \( \mathcal{M}_{ell} \): any

elliptic curve can be written as

(suppose for simplicity \( \text{char } k \neq 2 \))

affine eq \( y^2 = x^3 + ax + b \) (projective completion

has 1 extra pl, which

we take to be the

identity)

\( \Delta = -16(4a^3 + 27b^2) \neq 0 \)

\((y^2 = x^3 + ax + b) \cong (y^2 = x^3 + a'x + b') \)

\((-2) \{ u \in k \quad a' = u^4a, b' = u^6b \}

So the moduli stack of \( \mathcal{M}_{ell} \) can be given by

\( \text{Spec } R \to \{ \text{objects } (a,b) \in R^2 \ \Delta(a,b) \neq 0 \}

\{ \text{morphisms } (a,b) \to (u^4a, u^6b) \ \forall u \in k \}

\)

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\{ \text{morphisms } (a,b) \to (u^4a, u^6b) \ \forall u \in k \}

\)
This presentation is very convenient for calculations.

Note: If \( j = \frac{-1728(4A)^3}{D} \) then \( j \) is an \( \mathbb{F}_p \)-invariant.

In fact, if \( \mathbb{F}_p \)-alg closed, there is (unique up to \( \mathbb{F}_p \)-iso) \( E/\mathbb{F}_p \) with \( j(E) = r \). "up to stackiness", over \( \operatorname{ACF} \)

\[
j: \mathcal{M}_{ell} \twoheadrightarrow \mathcal{A}_0^1.
\]

We have the universal ells curve

\[
\begin{array}{ccc}
3 & \longrightarrow & 3 \\
\downarrow & & \downarrow \\
\mathcal{M}_{ell} & \twoheadrightarrow & \overline{\mathcal{M}}_{ell}
\end{array}
\]

Can compactify by adding nodal cubics, which are some curves with \( \Delta = 0 \). All iso over \( \operatorname{ACF} \), so \( \cong \mathbb{P}_1/\{0, \infty\} \).

\[
j: \overline{\mathcal{M}}_{ell} \twoheadrightarrow \mathbb{P}^1. \text{ Extra point called the cusp.}
\]

Change of tack: what's a 1-form on \( \mathcal{M}_{ell} \)? It must be an \( \mathfrak{f} \) if we work over \( \mathbb{C} \).

If we think of \( \mathcal{M}_{ell} \) as \( H/\operatorname{SL}_2(\mathbb{Z}) \), then this must be a \( \operatorname{SL}_2(\mathbb{Z}) \)-equivariant 1-form on \( H \), i.e.

\[
\omega \in \Omega^1(H) \quad \text{such that } \quad g^* \omega = \omega \quad \text{all } g \in \operatorname{SL}_2(\mathbb{Z})
\]

(Do calc) If \( \omega = f(z)dz \), then

\[
f\left( \frac{az+b}{cz+d} \right) = \frac{1}{(cz+d)^2} f(z), \text{ all } z \in H
\]

Such a \( f \) is a modular form of weight 2 if it extends to the cusp.
(This is iff $f$ has a limit as $z \to$ vertical $\infty$).

So a modular form of weight 2 is a section of $\Omega^1(\mathbb{H}_0)$.

Note: $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. 

So what does it look like?