On Representations of Rational Cherednik Algebras

by

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Abstract

This thesis introduces and studies two constructions related to the representation theory of rational Cherednik algebras: the refined filtration by supports for the category O and the Dunkl weight function.

The refined filtration by supports provides an analogue for rational Cherednik algebras of the Harish-Chandra series appearing in the representation theory of finite groups of Lie type. In particular, irreducible representations in the rational Cherednik category O with particular generalized support conditions correspond to irreducible representations of associated generalized Hecke algebras. An explicit presentation for these generalized Hecke algebras is given in the Coxeter case, classifying the irreducible finite-dimensional representations in many new cases.

The Dunkl weight function K is a holomorphic family of tempered distributions on the real reflection representation of a finite Coxeter group W with values in linear endomorphisms of an irreducible representation of W. The distribution K gives rise to an integral formula for the Gaussian inner product on a Verma module in the rational Cherednik category O. At real parameter values, the restriction of K to the regular locus in the real reflection representation can be interpreted as an analytic function taking values in Hermitian forms, invariant under the braid group, on the image of a Verma module under the Knizhnik-Zamolodchikov (KZ) functor. This provides a bridge between the study of invariant Hermitian forms on representations of rational Cherednik algebras and of Hecke algebras, allowing for a proof that the KZ functor preserves signatures in an appropriate sense.

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Some acknowledgements relating to the mathematical content of this thesis are in order. The content of Chapter 3 of this thesis is closely based on my joint paper [52] with my co-advisor Ivan Losev. Many ideas appearing in that chapter, including the original idea for the project, are due to Ivan. I would also like to thank him for countless useful conversations throughout the completion of that project, surely without which the project would never have been successful. That project further benefited from Ivan's conversations with Raphaël Rouquier, from my conversations with Pavel Etingof and José Simental, and from comments from Emily Norton on a preliminary draft of [52].

Chapter 4 of this thesis is closely based on the preprint [67], which arose from a project developed by Pavel Etingof and which benefited from his conversations with Larry Guth, Richard Melrose, Leonid Polterovich, and Vivek Shende. I would like to express my deep gratitude to Pavel for suggesting that project, for many useful conversations, for comments on a preliminary draft of [67], and for his ideas and insights that permeate [67] and Chapter 4 of this thesis. I would also like to thank Semyon Dyatlov for providing essentially all of the content of Sections 4.4.1 and 4.4.2, notably including the crucial Lemma 4.4.2.2 and its proof, and for explaining to me the techniques from semiclassical analysis used in those sections.

Last, but certainly not least, I would like to thank my parents, Dana Shelley and Phil Abrahamson, and my fiancée, Olivia Hentz, for their constant love and support. I would be lost without you.

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Chapter 1

Introduction

This thesis presents two new constructions in the representation theory of rational Cherednik algebras. The first, the refined filtration by supports on the category $\mathcal{O}_c(W, \mathfrak{h})$ appearing in the paper [52], is discussed in Chapter 3 and permits the classification of the finite-dimensional irreducible representations of the rational Cherednik algebra $H_c(W, \mathfrak{h})$ in many new cases. The second, the Dunkl weight function introduced in the preprint [67], is discussed in Chapter 4 and provides an integral formula for Cherednik's Gaussian inner product, giving a bridge between the study of invariant Hermitian forms on representations of rational Cherednik algebras and Hecke algebras.

Let W be a finite complex reflection group with reflection representation $\mathfrak{h}$, and let $c : S \to \mathbb{C}$ be a W-invariant complex-valued function on the set of complex reflections $S \subset W$. Associated to this data one has the rational Cherednik algebra $H_c(W, \mathfrak{h})$, introduced by Etingof and Ginzburg [29]. The algebras $H_c(W, \mathfrak{h})$, parameterized by such functions $c : S \to \mathbb{C}$, form a family of infinite-dimensional noncommutative associative algebras providing a flat deformation of the algebra $H_0(W, \mathfrak{h}) = \mathbb{C}W \ltimes D(\mathfrak{h})$, the semidirect product of W with the algebra of polynomial differential operators on $\mathfrak{h}$.

Since their introduction these algebras and their representation theory have been intensely studied. Of particular interest is the category $\mathcal{O}_c(W, \mathfrak{h})$ of representations of $H_c(W, \mathfrak{h})$, introduced by Ginzburg, Guay, Opdam, and Rouquier [38], deforming the category of finite-dimensional representations of W and analogous to the classical Bernstein-Gelfand-Gelfand category $\mathcal{O}$ attached to a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$. In addition to being of interest from the purely representation-theoretic point of view taken in this thesis, the category $\mathcal{O}_c(W, \mathfrak{h})$ has proved to have connections with various other topics, including Hilbert schemes [41, 42], torus knot invariants [30, 43], quantum integrable systems [25, 29, 34], and categorification [65, 66].

The category $\mathcal{O}_c(W, \mathfrak{h})$ has many structures and properties in common with classical categories $\mathcal{O}$. It is a highest weight category with standard objects labeled by the set $\mathrm{Irr}(W)$ of irreducible representations of the group W; to each $\lambda \in \mathrm{Irr}(W)$ there is an associated standard module $\Delta_c(\lambda)$ with lowest weight λ, analogous to the Verma modules appearing in the representation theory of semisimple Lie algebras. Each standard module $\Delta_c(\lambda)$ has a unique simple quotient $L_c(\lambda)$, and the correspondence $\lambda \mapsto L_c(\lambda)$ provides a bijection between $\mathrm{Irr}(W)$ and the isomorphism classes of irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$. All of the finite-dimensional representations of $H_c(W, \mathfrak{h})$ belong to $\mathcal{O}_c(W, \mathfrak{h})$. Furthermore, each representation M in $\mathcal{O}_c(W, \mathfrak{h})$ is naturally graded and therefore has an associated character, defined to be the generating function recording the characters of the finite-dimensional representations of W appearing in each graded component. However, unlike for semisimple Lie algebras, explicit character formulas for irreducible representations and a classification of the finite-dimensional representations remain unknown in many cases.

In the interest of keeping this thesis as self-contained as possible, and to fix notations, Chapter 2 will provide relevant background and definitions. In particular, Chapter 2 will recall definitions and basic results involving the rational Cherednik algebra $H_c(W, \mathfrak{h})$; its category $\mathcal{O}_c(W, \mathfrak{h})$ of representations and related constructions; finite Hecke algebras $\mathsf{H}_q(W)$ and the Knizhnik-Zamolodchikov (KZ) functor; the Bezrukavnikov-Etingof parabolic induction and restriction functors; and the partial KZ functors.

Chapter 3 introduces the refined filtration by supports for the category $\mathcal{O}_c(W, \mathfrak{h})$. This is a filtration of $\mathcal{O}_c(W, \mathfrak{h})$ by Serre subcategories, refining the filtration by supports, with filtration subquotients labeled by W-orbits of pairs (W', L) of a parabolic subgroup W' and an irreducible finite-dimensional representation L of $H_c(W', \mathfrak{h}_{W'})$, where $\mathfrak{h}_{W'}$ denotes the reflection representation of W'. The definition of this filtration is in direct analogy with the notion of Harish-Chandra series in the context of representations of finite groups of Lie type, with the irreducible representations appearing in the subquotient labeled by (W', L) viewed as belonging to the same Harish-Chandra series. The subquotient category labeled by (W', L) is equivalent to the category of finite-dimensional representations of the opposite endomorphism algebra $\mathrm{End}_{H_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L)^{opp}$, and the main technical result of Chapter 3 is the following concrete description of these finite-dimensional algebras, appearing in Theorems 3.3.2.14 and 3.3.2.11:

Theorem. *Let W be a finite Coxeter group with simple reflections S, let $c : S \to \mathbb{C}$ be a class function, let W' be a standard parabolic subgroup generated by the simple reflections S', and let L be an irreducible finite-dimensional representation of the rational Cherednik algebra $H_c(W', \mathfrak{h}_{W'})$. Let $N_{W'}$ denote the canonical complement to W' in its normalizer $N_W(W')$, let $S_{W'} \subset N_{W'}$ denote the set of reflections in $N_{W'}$ with respect to its representation in the fixed space $\mathfrak{h}^{W'}$, and let $N_{W'}^{ref}$ denote the reflection subgroup of $N_{W'}$ generated by $S_{W'}$. Let $N_{W'}^{comp}$ be a complement for $N_{W'}^{ref}$ in $N_{W'}$, given as the stabilizer of a choice of fundamental Weyl chamber for the action of $N_{W'}^{ref}$ on $\mathfrak{h}^{W'}$. Then there is a class function $q_{W',L} : S_{W'} \to \mathbb{C}^\times$ and an isomorphism of algebras*

$$End_{H_c(W,\mathfrak{h})}(Ind_{W'}^W L)^{opp} \cong N_{W'}^{comp} \ltimes \mathsf{H}_{q_{W',L}}(N_{W'}^{ref})$$

where the semidirect product is defined by the action of $N_{W'}^{comp}$ on $\mathsf{H}_{q_{W',L}}(N_{W'}^{ref})$ by diagram automorphisms arising from the conjugation action of $N_{W'}^{comp}$ on $N_{W'}^{ref}$. Furthermore, the parameter $q_{W',L}$ is explicitly computable.

The above theorem makes it possible to count, for any finite Coxeter group W, the number of irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ of any given support. In particular, this allows for the determination of the number of isomorphism classes of finite-dimensional irreducible representations of $H_c(W, \mathfrak{h})$. This number is computed explicitly for all exceptional finite Coxeter groups W, leading to the classification of the finite-dimensional irreducible representations of $H_c(W, \mathfrak{h})$ in many new cases. Chapter 3 is closely based on the joint paper [52] with my co-advisor Ivan Losev.

Chapter 4 introduces the Dunkl weight function, generalizing previous results of Dunkl to an arbitrary finite Coxeter group W with an irreducible representation λ. The Dunkl weight function $K_{c,\lambda}$ is a family of $\mathrm{End}_{\mathbb{C}}(\lambda)$-valued tempered distributions, holomorphic in the complex multi-parameter c, on the real reflection representation of W. For real parameters c, the distribution $K_{c,\lambda}$ provides an integral formula for Cherednik's Gaussian inner product $\gamma_{c,\lambda}$ on the standard module $\Delta_c(\lambda)$:

Theorem. *For any finite Coxeter group W and irreducible representation λ of W, there is a unique family $K_{c,\lambda}$, holomorphic in c, of $\mathrm{End}_{\mathbb{C}}(\lambda)$-valued tempered distributions on the real reflection representation $\mathfrak{h}_{\mathbb{R}}$ of W such that the following integral representation of the Gaussian pairing $\gamma_{c,\lambda}$ holds:*

$$\gamma_{c,\lambda}(P, Q) = \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger K_{c,\lambda}(x) P(x) e^{-|x|^2/2} dx \qquad \textit{for all } P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda.$$

For all real parameters c, the restriction of the distribution $K_{c,\lambda}$ to the complement $\mathfrak{h}_{\mathbb{R},reg}$ of the reflection hyperplanes is given by integration against an analytic function that may be viewed as taking values in braid group-invariant Hermitian forms on $KZ(\Delta_c(\lambda))$. This provides a concrete connection between the study of invariant Hermitian forms on representations of rational Cherednik algebras and Hecke algebras, permitting a proof that the KZ functor preserves signatures, and hence unitarizability, in an appropriate sense. Chapter 4 is closely based on the preprint [67].

Chapter 2

Background and Definitions

In this chapter we will recall the definition of rational Cherednik algebras $H_c(W, \mathfrak{h})$ and some related objects and constructions that will be important later, including the category of representations $\mathcal{O}_c(W, \mathfrak{h})$, standard modules $\Delta_c(\lambda)$, the KZ functor, and the Bezrukavnikov-Etingof parabolic induction and restriction functors. Most of the material recalled in this section can be found in [31], [38], and [5].

2.1 Rational Cherednik Algebras

A *real reflection* is an invertible linear transformation $s \in GL(\mathfrak{h}_\mathbb{R})$ of a finite dimensional real vector space $\mathfrak{h}_\mathbb{R}$ such that $s^2 = \mathrm{Id}$ and $\dim\ker(s-1) = \dim \mathfrak{h}_\mathbb{R} - 1$. A *finite real reflection group* is a finite group W along with a real faithful representation $\mathfrak{h}_\mathbb{R}$ such that the set of elements $S \subset W$ acting in $\mathfrak{h}_\mathbb{R}$ as real reflections generates W. We will refer to the representation $\mathfrak{h}_\mathbb{R}$ as the real reflection representation of W. A finite group W may be a finite real reflection group in more than one way, and we will always consider the pair $(W, \mathfrak{h}_\mathbb{R})$ of W along with a given real reflection representation $\mathfrak{h}_\mathbb{R}$. The finite real reflection groups coincide with the finite Coxeter groups. The standard classification of finite real reflection groups and their basic properties can be found, for example, in [48].

Let $\mathfrak{h}$ be a finite-dimensional complex vector space. A *complex reflection* is an invertible linear operator $s \in GL(\mathfrak{h})$ of finite order such that $\mathrm{rank}(1-s) = 1$. Let

$W \subset GL(\mathfrak{h})$ be a *complex reflection group*, i.e. a finite subgroup of $GL(\mathfrak{h})$ generated by the subset $S \subset W$ of complex reflections lying in W. The complex reflection groups were classified by Shephard and Todd in 1954 [69]; each such group decomposes as a product of irreducible complex reflection groups, and every irreducible complex reflection group either appears in the infinite family of complex reflection groups $G(m, p, n)$ indexed by integers $m, p, n \geq 1$ with $p \mid m$ or is one of the 34 exceptional irreducible complex reflection groups. Important special cases of complex reflection groups include the finite real reflection groups, i.e. the finite Coxeter groups, which may be regarded as complex reflection groups via complexification $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ of the real reflection representation $\mathfrak{h}_{\mathbb{R}}$. For example, the symmetric groups S_n are the complex reflection groups $G(1, 1, n)$.

Fix a finite complex reflection group W with complex reflection representation $\mathfrak{h}$, and let $S \subset \mathbb{C}$ denote the set of complex reflections in W with respect to the representation $\mathfrak{h}$. Let $(\cdot, \cdot)$ denote the natural pairing $\mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$. Mimicking the roots and coroots appearing in Lie theory, for each complex reflection $s \in S$ choose eigenvectors $\alpha_s \in \mathfrak{h}^*$ and $\alpha_s^\vee \in \mathfrak{h}$ for s with eigenvalues $\lambda_s \neq 1$ and $\lambda_s^{-1} \neq 1$, respectively, partially normalized so that $(\alpha_s, \alpha_s^\vee) = 2$. Let $c : S \to \mathbb{C}$ be a W-invariant function with respect to the action of W on S by conjugation. We will refer to such c as a *parameter*, and $\mathfrak{p}$ will denote the $\mathbb{C}$-vector space of such parameters. Given a parameter $c \in \mathfrak{p}$ one may define a *rational Cherednik algebra*:

Definition 2.1.0.1. *The* rational Cherednik algebra $H_c(W, \mathfrak{h})$ *is the associative unital algebra with presentation*

$$\frac{\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)}{\langle [x, x'], [y, y'], [y, x] - (y, x) + \sum_{s \in S} c_s(\alpha_s, y)(x, \alpha_s^\vee)s : x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h} \rangle}.$$

In the definition above, $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ denotes the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$, and

$$\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$$

denotes the semidirect product algebra of $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ with $\mathbb{C}W$ with respect to the natural action of W on $T(\mathfrak{h} \oplus \mathfrak{h}^*)$: as a vector space we have

$$\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*) = \mathbb{C}W \otimes T(\mathfrak{h} \oplus \mathfrak{h}^*),$$

and the multiplication is given by $(w_1 \otimes t_1)(w_2 \otimes t_2) = w_1 w_2 \otimes w_2^{-1}(t_1)t_2$. For example, when $c = 0$ the algebra $H_0(W, \mathfrak{h})$ is naturally identified with the algebra $\mathbb{C}W \ltimes D(\mathfrak{h})$, where $D(\mathfrak{h})$ denotes the algebra of polynomial differential operators on $\mathfrak{h}$.

The rational Cherednik algebra $H_c(W, \mathfrak{h})$ may alternatively be defined via its faithful *polynomial* representation in the space $\mathbb{C}[\mathfrak{h}]$ in which an element $w \in W$ acts via the representation of W on $\mathfrak{h}$, an element $x \in \mathfrak{h}^*$ acts by multiplication, and an element $y \in \mathfrak{h}$ acts by the *Dunkl operator*

$$D_y := \partial_y - \sum_{s \in S} \frac{2c_s \alpha_s(y)}{(1-\lambda_s)\alpha_s}(1-s),$$

where ∂_y denotes the derivative with respect to y.

2.2 PBW Theorem, Category $\mathcal{O}_c$, and Supports

From the above presentation it is clear that $H_c(W, \mathfrak{h})$ admits a filtration with $\deg W = \deg \mathfrak{h} = 0$ and $\deg \mathfrak{h}^* = 1$, analogous to the usual filtration on $D(\mathfrak{h})$ by the order of differential operators. Etingof and Ginzburg [29, Theorem 1.3] showed that the associated graded algebra of $H_c(W, \mathfrak{h})$ with respect to this filtration is naturally identified with $\mathbb{C}W \ltimes S(\mathfrak{h}^* \oplus \mathfrak{h})$, and in particular the natural multiplication map $\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \to H_c(W, \mathfrak{h})$ is an isomorphism of $\mathbb{C}$-vector spaces:

Theorem 2.2.0.1 (PBW Theorem for Rational Cherednik Algebras)**.** *For any parameter c, the natural map*

$$\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \to H_c(W, \mathfrak{h})$$

given by multiplication is an isomorphism of vector spaces.

This isomorphism should be viewed as an analogue for rational Cherednik algebras of the triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$ of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a choice of Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$. The category $\mathcal{O}_c(W, \mathfrak{h})$ of representations of $H_c(W, \mathfrak{h})$ introduced in [38] is then defined in parallel with the classical Bernstein-Gelfand-Gelfand category $\mathcal{O}$:

Definition 2.2.0.2 (Category $\mathcal{O}$ for Rational Cherednik Algebras, [38])**.** *The category $\mathcal{O}_c(W, \mathfrak{h})$ is the full subcategory of the category of representations of $H_c(W, \mathfrak{h})$ consisting of those representations M that are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and on which $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently.*

An important class of representations in $\mathcal{O}_c(W, \mathfrak{h})$ are the *standard modules*, which are direct analogues of the classical Verma modules of semisimple Lie algebras:

Definition 2.2.0.3 (Standard Modules and Lowest Weight Representations)**.** *For any irreducible representation $\lambda \in Irr(W)$, the* standard module $\Delta_c(\lambda) \in \mathcal{O}_c(W, \mathfrak{h})$ with lowest weight λ *is*

$$\Delta_c(\lambda) := H_c(W, \mathfrak{h}) \otimes_{\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^*]} \lambda,$$

where $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ acts on λ by 0. A module $M \in \mathcal{O}_c(W, \mathfrak{h})$ will be called lowest weight with lowest weight λ *if M is isomorphic to a nonzero quotient of $\Delta_c(\lambda)$.*

Note that by the PBW theorem any standard module $\Delta_c(\lambda)$ is naturally isomorphic to $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ as a module over $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}] \subset H_c(W, \mathfrak{h})$. We will use this identification frequently.

Modules in the category $\mathcal{O}_c(W, \mathfrak{h})$ are naturally graded by their decomposition into generalized eigenvectors for the *grading element* $\mathbf{h} \in H_c(W, \mathfrak{h})$, a deformation of the Euler vector field:

Definition 2.2.0.4 (Grading Element)**.** *The* grading element $\mathbf{h} \in H_c(W, \mathfrak{h})$ *is the element*

$$\mathbf{h} := \sum_{i=1}^{\dim \mathfrak{h}} x_i y_i + \frac{\dim \mathfrak{h}}{2} - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} s,$$

where $x_1, ..., x_{\dim \mathfrak{h}}$ is any basis of $\mathfrak{h}^*$ and $y_1, ..., y_{\dim \mathfrak{h}}$ is the associated dual basis of $\mathfrak{h}$.

Clearly, the element $\mathbf{h}$ does not depend on the choice of basis $x_1, ..., x_{\dim \mathfrak{h}}$. When W is a finite real reflection group, as will be the case most relevant for this thesis, we have $\lambda_s = -1$ for all $s \in S$, so $\mathbf{h}$ takes the form $\mathbf{h} = \sum_{i=1}^{\dim \mathfrak{h}} x_i y_i + \frac{\dim \mathfrak{h}}{2} - \sum_{s \in S} c_s s$. From [31, Proposition 3.18, Theorem 3.28] we have:

Proposition 2.2.0.5. *The element $\mathbf{h} \in H_c(W, \mathfrak{h})$ satisfies the commutation relations*

$$[\mathbf{h}, x] = x, x \in \mathfrak{h}^* \qquad [\mathbf{h}, y] = -y, y \in \mathfrak{h}.$$

Furthermore, $\mathbf{h}$ acts locally finitely on any $M \in \mathcal{O}_c(W, \mathfrak{h})$.

It follows that any $M \in \mathcal{O}_c(W, \mathfrak{h})$ has a direct sum decomposition $M = \bigoplus_{z \in \mathbb{C}} M_z$ into generalized eigenspaces for the action of $\mathbf{h}$ and that with respect to this decomposition M is a graded module with elements $x \in \mathfrak{h}^*$ acting by degree 1 operators, elements $y \in \mathfrak{h}$ acting by degree -1 operators, and elements $w \in W$ acting by degree 0 operators.

This grading on $\Delta_c(\lambda)$ coincides with the usual grading on $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ shifted by $h_c(\lambda)$, where $h_c(\lambda) = (\dim \mathfrak{h})/2 - \chi_\lambda(\sum_{s \in S} \frac{2c_s}{1-\lambda_s})$ is the scalar by which the element $(\dim \mathfrak{h})/2 - \sum_{s \in S} \frac{2c_s}{1-\lambda_s} s \in Z(\mathbb{C}W)$ acts in the irreducible representation λ of W and χ_λ denotes the character of the representation λ. In particular, the $\mathbf{h}$-weights appearing nontrivially in the standard module are those in the set $\{h_c(\lambda) + n : n \in \mathbb{Z}^{\geq 0}\}$. Any proper submodule of $\Delta_c(\lambda)$ is graded and has all nontrivial $\mathbf{h}$-weights appearing in the set $\{h_c(\lambda) + n : n \in \mathbb{Z}^{>0}\}$, and it follows that there is a maximal proper submodule $N_c(\lambda)$ of $\Delta_c(\lambda)$. We denote the irreducible quotient $\Delta_c(\lambda)/N_c(\lambda)$ by $L_c(\lambda)$. By [31, Proposition 3.30] we have

Proposition 2.2.0.6. *Any irreducible representation $L \in \mathcal{O}_c(W, \mathfrak{h})$ is isomorphic to some $L_c(\lambda)$ for a unique $\lambda \in Irr(W)$.*

If $N_c(\lambda) \neq 0$, then there is a nonzero homomorphism $\Delta_c(\mu) \to N_c(\lambda)$ for some $\mu \in \mathrm{Irr}(W)$ such that $h_c(\mu) = h_c(\lambda) + k$ for some integer $k > 0$. The quantity

$h_c(\mu) - h_c(\lambda)$ is visibly a linear function of c, and $\{c \in \mathfrak{p} : h_c(\mu) = h_c(\lambda) + k\}$ is an affine hyperplane in $\mathfrak{p}$. There are countably many such hyperplanes for various $\lambda, \mu \in \mathrm{Irr}(W)$ and $k \in \mathbb{Z}^{>0}$, and when c lies on none of these hyperplanes it follows that $\Delta_c(\lambda) = L_c(\lambda)$ for all $\lambda \in \mathrm{Irr}(W)$. Furthermore, for such c we have by a similar argument that $\mathrm{Ext}^1_{\mathcal{O}_c(W,\mathfrak{h})}(\Delta_c(\lambda), \Delta_c(\mu)) = 0$ for all $\lambda, \mu \in \mathrm{Irr}(W)$. In summary, we have [31, Proposition 3.35]:

Proposition 2.2.0.7. *For Weil generic $c \in \mathfrak{p}$, the category $\mathcal{O}_c(W, \mathfrak{h})$ is semisimple with simple objects $L_c(\lambda)$ for $\lambda \in Irr(W)$.*

Finally, we will need the notion of the *support* of a module $M \in \mathcal{O}_c(W, \mathfrak{h})$:

Definition 2.2.0.8. *Any $M \in \mathcal{O}_c(W, \mathfrak{h})$ is by definition a finitely generated module over $\mathbb{C}[\mathfrak{h}]$ with additional structure. Let $Supp(M) \subset \mathfrak{h}$ denote the* support *of M. When $Supp(M) = \mathfrak{h}$ we will say that M has* full support.

For example, the standard module $\Delta_c(\lambda)$ is a finitely generated free module over $\mathbb{C}[\mathfrak{h}]$, and in particular $\mathrm{Supp}(\Delta_c(\lambda)) = \mathfrak{h}$.

The determination of the support varieties $\mathrm{Supp}(L_c(\lambda))$ of the irreducible representations $L_c(\lambda)$ is a fundamental question about $\mathcal{O}_c(W, \mathfrak{h})$. Using parabolic induction and restriction functors, Bezrukavnikov and Etingof [5] showed that the subvarieties of $\mathfrak{h}$ appearing as supports of irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ can only be the form $X(W') := W\mathfrak{h}^{W'}$ where W' is a parabolic subgroup of W, i.e. the stabilizer of a point $b \in \mathfrak{h}$. Note that $X(W') = X(W'')$ exactly when W' and W'' are conjugate in W, and in particular the possible supports of simple modules in $\mathcal{O}_c(W, \mathfrak{h})$ are indexed by conjugacy classes of parabolic subgroups of W. It is important to note that not necessarily all such varietes $X(W')$ appear as the support varieties of representations in $\mathcal{O}_c(W, \mathfrak{h})$, and in fact for generic values of the parameter c all irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ have full support, in which case only the variety $X(1) = \mathfrak{h}$ appears.

2.3 The Localization Lemma, Hecke Algebras, and the KZ Functor

Let $\mathfrak{h}_{reg} := \mathfrak{h} \setminus \bigcup_{s \in S} \ker(\alpha_s)$, the *regular locus* for the action of W on $\mathfrak{h}$, be the complement in $\mathfrak{h}$ of the arrangement of reflection hyperplanes for the action of W on $\mathfrak{h}$. Note that $\mathfrak{h}_{reg}$ is precisely the principal affine open subset of $\mathfrak{h}$ defined by the non-vanishing of the discriminant element $\delta := \prod_{s \in S} \alpha_s$. As $\delta^{|W|}$ is W-invariant and has locally nilpotent adjoint action on $H_c(W, \mathfrak{h})$, it follows that the noncommutative localization $H_c(W, \mathfrak{h})[\delta^{-1}]$ coincides with the localization of $H_c(W, \mathfrak{h})$ as a $\mathbb{C}[\mathfrak{h}]$-module, and this localized algebra is isomorphic to the algebra $\mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ in a natural way (see [38, Theorem 5.6]). Thus, a module M in $\mathcal{O}_c(W, \mathfrak{h})$ determines a module $M[\delta^{-1}]$ over $\mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ by localization, which may be regarded as a W-equivariant D-module on $\mathfrak{h}_{reg}$. The W-equivariant $D(\mathfrak{h}_{reg})$-modules occurring in this way are $\mathcal{O}(\mathfrak{h}_{reg})$-coherent, by definition of the category $\mathcal{O}_c(W, \mathfrak{h})$, and have regular singularities [38]. By the Riemann-Hilbert correspondence [16], descending to $\mathfrak{h}_{reg}/W$ and taking the monodromy representation at a base point $b \in \mathfrak{h}_{reg}$ defines an equivalence of categories

$$\mathbb{C}W \ltimes D(\mathfrak{h}_{reg})\text{-mod}_{\mathcal{O}(\mathfrak{h}_{reg})\text{-coh, r.s.}} \cong \pi_1(\mathfrak{h}_{reg}/W, b)\text{-mod}_{f.d.}$$

where the subscript $\mathcal{O}(\mathfrak{h}_{reg})$-coh, r.s. indicates that the $D(\mathfrak{h}_{reg})$-modules are $\mathcal{O}(\mathfrak{h}_{reg})$-coherent with regular singularities and the subscript $f.d.$ indicates that $\pi_1(\mathfrak{h}_{reg}/W, b)$-modules are finite-dimensional. This procedure of localization to $\mathfrak{h}_{reg}$ followed by descent to $\mathfrak{h}_{reg}$ and the Riemann-Hilbert correspondence thus defines an exact functor

$$\mathcal{O}_c(W, \mathfrak{h}) \to \pi_1(\mathfrak{h}_{reg}/W, b)\text{-mod}_{f.d.}.$$

The fundamental group $\pi_1(\mathfrak{h}_{reg}/W, b)$ is the *generalized braid group* attached to W and is denoted B_W. In [38], it was shown that the above functor in fact factors through the category $\mathsf{H}_q(W)\text{-mod}_{f.d.}$ of finite-dimensional representations over a certain quotient $\mathsf{H}_q(W)$, the *Hecke algebra*, of the group algebra $\mathbb{C}B_W$. The resulting exact functor

$$KZ : \mathcal{O}_c(W, \mathfrak{h}) \to \mathsf{H}_q(W)\text{-mod}_{f.d.}$$

is the KZ functor. The quotient $\mathsf{H}_q(W)$ is defined as follows [7]. Let $\mathcal{H} = \{\ker(s) : s \in S\}$ be the set of reflection hyperplanes for the action of W on $\mathfrak{h}$. For each reflection hyperplane $H \in \mathcal{H}$, let T_H be a representative of the conjugacy class in B_W determined by a small loop, oriented counterclockwise, in $\mathfrak{h}_{reg}/W$ about the hyperplane H. The group B_W is generated by the union of these conjugacy classes ([7, Theorem 2.17]). For each $H \in \mathcal{H}$, let l_H be the order of the cyclic subgroup of W stabilizing H pointwise, and let $q_{1,H}, ...q_{l_H-1,H} \in \mathbb{C}^\times$ be nonzero complex numbers that are W-invariant, i.e. $q_{j,H} = q_{j,H'}$ whenever $H = wH'$ for some $w \in W$. Let q denote the collection of these $q_{j,H}$, and define the Hecke algebra $\mathsf{H}_q(W)$ as the quotient of $\mathbb{C}B_W$ by the relations

$$(T_H - 1) \prod_{j=1}^{l_H-1} (T_H - e^{2\pi i j/l_H} q_{j,H}) = 0.$$

The choice of the parameter q such that the functor KZ is defined as above is given explicitly in [38] as a function of the parameter c for the algebra $H_c(W, \mathfrak{h})$. In the special case that W is a real reflection group, which is the primary case relevant to this thesis, the dependence of q on c is especially simple. In that case we have $l_H = 2$ for all $H \in \mathcal{H}$, and $q_{1,H} = e^{-2\pi i c_s}$ where $s \in S$ is the reflection such that $H = \ker(s)$ (note that the sign convention for c in this thesis differs from that appearing in [38]). In particular, in the Coxeter case the Hecke algebra $\mathsf{H}_q(W)$ appearing in the KZ functor is precisely the Iwahori-Hecke algebra attached to the Coxeter group W with generators $\{T_s : s \in S\}$ satisfying the relations in the braid group B_W and the quadratic relations

$$(T_s - 1)(T_s + e^{-2\pi i c_s}) = 0.$$

When it is relevant, we will explicitly emphasize the dependence of the functor KZ on the base point $b \in \mathfrak{h}_{reg}$ using the notation KZ_b. Clearly, the functor KZ_b does not depend on $b \in \mathfrak{h}_{reg}$ up to isomorphism. At the level of vector spaces, $KZ_b(M)$ is the fiber at b of the $\mathbb{C}[\mathfrak{h}]$-module $M \in \mathcal{O}_c(W, \mathfrak{h})$. As a consequence, $KZ_b(M) \neq 0$ if and only if M has full support.

In Chapter 4, we will be particularly concerned with the image of the standard modules $\Delta_c(\lambda)$ under KZ_b, and it is worth considering the $KZ_b(\Delta_c(\lambda))$ in some detail here. Recall that by the PBW theorem for rational Cherednik algebras we have $\Delta_c(\lambda) = \mathbb{C}[\mathfrak{h}] \otimes \lambda$ as a $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}]$-module, giving rise to an identification $\Delta_c(\lambda)[\delta^{-1}] = \mathbb{C}[\mathfrak{h}_{reg}] \otimes \lambda$ of $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}_{reg}]$-modules. For any $y \in \mathfrak{h} \subset H_c(W, \mathfrak{h})$ we have $y\lambda = 0$, by the definition of $\Delta_c(\lambda)$. At the same time, with respect to the natural identification of $H_c(W, \mathfrak{h})[\delta^{-1}]$ with $\mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ the element $y \in \mathfrak{h}$ corresponds to the Dunkl operator $D_y = \partial_y - \sum_{s \in S} c_s \frac{\alpha_s(y)}{\alpha_s}(1-s)$. In particular, it follows that the vector field $\partial_y \in \mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ acts on $\lambda \subset \Delta_c(\lambda)[\delta^{-1}]$ by

$$\partial_y v = \sum_{s \in S} c_s \frac{\alpha_s(y)}{\alpha_s}(1-s)$$

for all $v \in \lambda \subset \Delta_c(\lambda)[\delta^{-1}]$. It folows that the W-equivariant $D(\mathfrak{h}_{reg})$-module structure on $\Delta_c(\lambda)[\delta^{-1}]$ arises from the flat *KZ connection*

$$\nabla_{KZ} := d + \sum_{s \in S} c_s \frac{d\alpha_s}{\alpha_s}(1-s)$$

on the trivial vector bundle on $\mathfrak{h}_{reg}$ with fiber λ.

2.4 Bezrukavnikov-Etingof Parabolic Induction and Restriction Functors

Parabolic induction and restriction functors are central to the study of representations of Iwahori-Hecke algebras. The definition of these functors relies on the natural embedding $\mathsf{H}_q(W') \subset \mathsf{H}_q(W)$ of the Hecke algebra $\mathsf{H}_q(W')$ attached to a parabolic subgroup W' of a Coxeter group W in the Hecke algebra $\mathsf{H}_q(W)$ attached to W. Parabolic restriction is then simply the naive restriction, and parabolic induction is the tensor product $\mathsf{H}_q(W) \otimes_{\mathsf{H}_q(W')} \bullet$. As for Hecke algebras, parabolic induction and restriction functors are central to the study of representations of rational Cherednik algebras $H_c(W, \mathfrak{h})$. Let $\mathfrak{h}_{W'}$ denote the unique W'-stable complement to $\mathfrak{h}^{W'}$ in $\mathfrak{h}$. Then W' acts on $\mathfrak{h}_{W'}$, and the action is generated by reflections. Let $H_c(W', \mathfrak{h}_{W'})$ denote the associated rational Cherednik algebra, where by abuse of notation c here denotes the restriction of c to the reflections $S' = S \cap W'$ in W lying in W'. Then, unlike for Hecke algebras, the algebra $H_c(W', \mathfrak{h}_{W'})$ does not embed as a subalgebra of $H_c(W, \mathfrak{h})$ in a natural way, and the restriction and induction functors must be defined differently.

Bezrukavnikov and Etingof [5] circumvented this difficulty by using the geometric interpretation of rational Cherednik algebras. The price to pay for this approach is that rather than defining a single parabolic restriction functor

$$\mathrm{Res}_{W'}^W : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_c(W', \mathfrak{h}_{W'})$$

and a single parabolic induction functor

$$\mathrm{Ind}_{W'}^W : \mathcal{O}_c(W', \mathfrak{h}_{W'}) \to \mathcal{O}_c(W, \mathfrak{h}),$$

one instead defines a restriction functor Res_b and induction functor Ind_b defined for every point b in the locally closed subvariety $\mathfrak{h}_{reg}^{W'}$ of $\mathfrak{h}$ consisting of those points $b \in \mathfrak{h}$ with stabilizer $\mathrm{Stab}_W(b) = W'$. Bezrukavnikov and Etingof explain the dependence of Res_b on the choice of $b \in \mathfrak{h}_{reg}^{W'}$ in the following elegant manner. They construct an exact functor

$$\underline{\mathrm{Res}}_{W'}^W : \mathcal{O}_c(W, \mathfrak{h}) \to (\mathcal{O}_c(W', \mathfrak{h}_{W'}) \boxtimes \mathrm{Loc}(\mathfrak{h}_{reg}^{W'}))^{N_W(W')},$$

where $\mathrm{Loc}(\mathfrak{h}_{reg}^{W'})$ denotes the category of local systems on $\mathfrak{h}_{reg}^{W'}$ and the $N_W(W')$ in the superscript indicates equivariance for the normalizer $N_W(W')$ of W' in W, and they show that the fiber $(\underline{\mathrm{Res}}_{W'}^W)_b$ of the resulting local system at $b \in \mathfrak{h}_{reg}^{W'}$ gives precisely the parabolic restriction functor Res_b. In particular, for any points $b, b' \in \mathfrak{h}_{reg}^{W'}$, parallel transport along any path γ in $\mathfrak{h}_{reg}^{W'}$ connecting b and b' defines an isomorphism between the functors Res_b and $\mathrm{Res}_{b'}$. As explained in [5], in the case $W' = 1$, the functor $\underline{\mathrm{Res}}_{W'}^{W}$ can be identified with the KZ functor recalled above in Section 2.3, and therefore the functors $\underline{\mathrm{Res}}_{W'}^{W}$ can be regarded as relative versions of the KZ functor. We refer to the functors $\underline{\mathrm{Res}}_{W'}^{W}$ as the *partial KZ functors*, and these functors are central to the approach we take here.

We recall the construction of the functors $\mathrm{Res}_b, \mathrm{Ind}_b$, and $\underline{\mathrm{Res}}_{W'}^{W}$ below, following [5, Section 3].

2.4.1 Construction of Res_b and Ind_b

Let $b \in \mathfrak{h}_{reg}^{W'}$, i.e. a point in $\mathfrak{h}$ with stabilizer W' in W. The completion $\widehat{H}_c(W, \mathfrak{h})_b$ of the rational Cherednik algebra $H_c(W, \mathfrak{h})$ at the orbit $Wb \subset \mathfrak{h}$ is defined to be the associative algebra

$$\widehat{H}_c(W, \mathfrak{h})_b := \mathbb{C}[\mathfrak{h}]^{\wedge_{Wb}} \otimes_{\mathbb{C}[\mathfrak{h}]} H_c(W, \mathfrak{h}),$$

where $\mathbb{C}[\mathfrak{h}]^{\wedge_{Wb}}$ denotes the completion of $\mathbb{C}[\mathfrak{h}]$ at the orbit Wb. Restricting the parameter $c : S \to \mathbb{C}$ to the set of reflections $S' \subset W'$, one may form the rational Cherednik algebra $H_c(W', \mathfrak{h})$ and its completion $\widehat{H}_c(W', \mathfrak{h})_0$ at $0 \in \mathfrak{h}$. As explained in [5, Section 3.3], one may think of the algebra $\widehat{H}_c(W, \mathfrak{h})_b$ in a more geometric manner as the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}]^{\wedge_{Wb}})$ generated by $\mathbb{C}[\mathfrak{h}]^{\wedge_{Wb}}$, the group W, and the Dunkl operators D_y associated to points $y \in \mathfrak{h}$. This geometric interpretation leads naturally to an isomorphism ([5, Theorem 3.2])

$$\theta : \widehat{H}_c(W, \mathfrak{h})_b \to Z(W, W', \widehat{H}_c(W', \mathfrak{h})_0)$$

where the algebra $Z(W, W', \widehat{H}_c(W', \mathfrak{h})_0)$, the *centralizer algebra*, is the endomorphism algebra of the right $\widehat{H}_c(W, \mathfrak{h})_0$-module $Fun_{W'}(W, \widehat{H}_c(W', \mathfrak{h})_0)$ of functions $f : W \to \widehat{H}_c(W', \mathfrak{h})_0$ satisfying $f(w'w) = w'f(w)$ for all $w \in W, w' \in W'$. We may take the completion $\widehat{H}_c(W', \mathfrak{h})_0$ of $H_c(W', \mathfrak{h})$ at 0 rather than at b in the centralizer because the assignments $w' \mapsto w'$ for $w' \in W'$, $y \mapsto y$ for $y \in \mathfrak{h}$, and $x \mapsto x - b$ extends to an isomorphism $\widehat{H}_c(W',\mathfrak{h})_0 \cong \widehat{H}_c(W',\mathfrak{h})_b$. The isomorphism θ determines, by transfer of structure, an equivalence of categories

$$\theta_* : \widehat{H}_c(W,\mathfrak{h})_b\text{-mod} \to Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)\text{-mod}.$$

The centralizer algebra $Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)$ is non-canonically isomorphic to the matrix algebra of size $|W/W'|$ over $\widehat{H}_c(W',\mathfrak{h})_0$, and in particular the isomorphism θ shows that $\widehat{H}_c(W,\mathfrak{h})_b$ is Morita equivalent to $\widehat{H}_c(W',\mathfrak{h})_0$. A particularly natural choice of Morita equivalence is given by the idempotent $e_{W'} \in Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)$ defined by $e_{W'}(f)(w) = f(w)$ for $w \in W'$ and $e_{W'}(f)(w) = 0$ for $w \notin W'$. Note that $e_{W'}$ is the image under θ_* of the idempotent $1_b \in \mathbb{C}[\mathfrak{h}]^{\wedge_{Wb}} \subset \widehat{H}_c(W,\mathfrak{h})_b$ that is the indicator function of the point b in its W-orbit. As the two-sided ideal in the centralizer algebra generated by $e_{W'}$ is the entire centralizer algebra and as the associated spherical subalgebra $e_{W'}Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)e_{W'}$ is naturally identified with $\widehat{H}_c(W',\mathfrak{h})_0$, the functors

$$I : \widehat{H}_c(W',\mathfrak{h})_0\text{-mod} \to Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)\text{-mod}$$

$$M \mapsto I(M) := Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)e_{W'} \otimes_{e_{W'}Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)e_{W'}} M$$

and

$$I^{-1} : Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)\text{-mod} \to \widehat{H}_c(W',\mathfrak{h})_0\text{-mod}$$

$$N \mapsto I^{-1}(N) := e_{W'}N$$

are mutually quasi-inverse equivalences.

Let $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_b$ denote the full subcategory of $\widehat{H}_c(W,\mathfrak{h})_b$-modules which are finitely generated over $\mathbb{C}[\mathfrak{h}]^{\wedge_{Wb}}$. Let

$$\widehat{\ }_b : \mathcal{O}_c(W,\mathfrak{h}) \to \widehat{\mathcal{O}}_c(W,\mathfrak{h})_b$$

be the functor of completion at the orbit Wb and let

$$E^b : \widehat{\mathcal{O}_c}(W,\mathfrak{h})_b \to \mathcal{O}_c(W,\mathfrak{h})$$

be the functor that sends a module M to its subspace $E^b(M)$ of $\mathfrak{h}$-locally nilpotent vectors. By [5, Proposition 3.6, Proposition 3.8], these functors are exact and E^b is the right adjoint of $\widehat{\ }_b$. Similarly, we have the category $\widehat{\mathcal{O}}_c(W', \mathfrak{h})_0$, the completion functor $\widehat{\ }_0 : \mathcal{O}_c(W', \mathfrak{h}) \to \widehat{\mathcal{O}}_c(W', \mathfrak{h})_0$ and the functor $E^0 : \widehat{\mathcal{O}}_c(W', \mathfrak{h})_0 \to \mathcal{O}_c(W', \mathfrak{h})$ taking $\mathfrak{h}$-locally nilpotent vectors, which are in fact category equivalences [5, Theorem 2.3].

Let $\mathfrak{h}_{W'}$ denote the unique W'-stable complement to $\mathfrak{h}^{W'}$ in $\mathfrak{h}$. Clearly, an element $s \in W'$ acts on $\mathfrak{h}$ as a complex reflection if and only if it acts on $\mathfrak{h}_{W'}$ as a complex reflection, and W' acts on $\mathfrak{h}_{W'}$ faithfully. By restriction of the parameter $c : S \to \mathbb{C}$ to the set of reflections S' in W', we may form the rational Cherednik algebra $H_c(W', \mathfrak{h}_{W'})$ and its associated category $\mathcal{O}_c(W', \mathfrak{h}_{W'})$. Let

$$\zeta : \mathcal{O}_c(W', \mathfrak{h}) \to \mathcal{O}_c(W', \mathfrak{h}_{W'})$$

be the functor defined by

$$\zeta(M) = \{m \in M : ym = 0 \ \forall y \in \mathfrak{h}^{W'}\}.$$

As discussed in [5, Section 2.3], ζ is an equivalence of categories - this is essentially an instance of Kashiwara's lemma for $D(\mathfrak{h}^{W'})$-modules, in view of the natural tensor product decomposition $H_c(W', \mathfrak{h}) \cong H_c(W', \mathfrak{h}_{W'}) \otimes D(\mathfrak{h}^{W'})$.

Definition 2.4.1.1. *The parabolic restriction functor $Res_b : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_c(W', \mathfrak{h}_{W'})$ is the composition*

$$Res_b := \zeta \circ E^0 \circ I^{-1} \circ \theta_* \circ \widehat{\ }_b$$

and the parabolic induction functor $Ind_b : \mathcal{O}_c(W', \mathfrak{h}_{W'}) \to \mathcal{O}_c(W, \mathfrak{h})$ is the composition

$$E^b \circ \theta_*^{-1} \circ I \circ \widehat{\ }_0 \circ \zeta^{-1}.$$

Proposition 2.4.1.2. *The functors Ind_b and Res_b are exact and biadjoint and do not depend on the choice of $b \in \mathfrak{h}^{W'}_{reg}$ up to isomorphism.*

Proof. Exactness is [5, Proposition 3.9(i)], and that Ind_b is right adjoint to Res_b is [5, Theorem 3.10]. That Ind_b is also left adjoint to Res_b was established by Shan [65, Section 2.4] under some assumptions and later in full generality by Losev [54]. The independence up to isomorphism on the choice of b was established in [5, Section 3.7] using the partial KZ functor $\underline{\mathrm{Res}}_{W'}^{W}$ to be recalled in the following section. $\square$

2.5 Partial KZ Functors

In this section we give in some detail a construction of the *partial KZ* functor $\underline{\mathrm{Res}}_{W'}^{W}$ introduced in [5, Section 3.7]. Let $\widehat{H}_c(W,\mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}}$ denote the completion of $H_c(W,\mathfrak{h})$ at the W-stable locally closed subvariety $W\mathfrak{h}_{reg}^{W'} \subset \mathfrak{h}$. As for the case of completion at the W-orbit of a point, the completion $\widehat{H}_c(W,\mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}}$ can be realized as the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}]^{\wedge_{W\mathfrak{h}_{reg}^{W'}}})$ generated by the functions $\mathbb{C}[\mathfrak{h}]^{\wedge_{W\mathfrak{h}_{reg}^{W'}}}$ on the formal neighborhood of $W\mathfrak{h}_{reg}^{W'}$ in $\mathfrak{h}$, the group W, and the Dunkl operators D_y for $y \in \mathfrak{h}$ associated to the action of W on $\mathfrak{h}$. (Recall that the algebra of functions $\mathbb{C}[\mathfrak{h}]^{\wedge_{W\mathfrak{h}_{reg}^{W'}}}$ is obtained by first localizing to a principal open subset in which $W\mathfrak{h}_{reg}^{W'}$ is a closed subvariety, followed by completion at the ideal defining $W\mathfrak{h}_{reg}^{W'}$ in this principal open subset.) Note that the variety $W\mathfrak{h}_{reg}^{W'}$ is the disjoint union of the W-translates of $\mathfrak{h}_{reg}^{W'}$, and the set-wise stabilizer of $\mathfrak{h}_{reg}^{W'}$ in W is precisely $N_W(W')$. Let $1_{\mathfrak{h}_{reg}^{W'}} \in \mathbb{C}[\mathfrak{h}]^{\wedge_{W\mathfrak{h}_{reg}^{W'}}}$ denote the indicator function of $\mathfrak{h}_{reg}^{W'}$ as a function on the formal neighborhood in $\mathfrak{h}$ of its W-orbit. Similar to the isomorphism

$$1_b\widehat{H}_c(W,\mathfrak{h})_b 1_b \cong \widehat{H}_c(W',\mathfrak{h})_b$$

from [5] discussed in the previous section, we have a natural isomorphism of algebras

$$1_{\mathfrak{h}_{reg}^{W'}}\widehat{H}_c(W,\mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}}1_{\mathfrak{h}_{reg}^{W'}} \cong \widehat{H}_c(N_W(W'),\mathfrak{h})_{\mathfrak{h}_{reg}^{W'}},$$

where the algebra $\widehat{H}_c(N_W(W'),\mathfrak{h})_{\mathfrak{h}_{reg}^{W'}}$ denotes the completion of the algebra

$$H_c(N_W(W'),\mathfrak{h}) = \mathbb{C}N_W(W') \ltimes_{W'} H_c(W',\mathfrak{h})$$

at the $N_W(W')$-stable locally closed subvariety $\mathfrak{h}_{reg}^{W'} \subset \mathfrak{h}$ (for comparison with the case of étale pullbacks rather than completions, see [45, Theorem 3.2]). The formal neighborhood of $\mathfrak{h}_{reg}^{W'}$ inside $\mathfrak{h}$ is canonically identified with its formal neighborhood inside the principal open subset $\mathfrak{h}_{reg}^{W'} \times \mathfrak{h}_{W'} \subset \mathfrak{h}$, which is precisely $(\mathfrak{h}_{reg}^{W'} \times \mathfrak{h}_{W'})^{\wedge_{\mathfrak{h}_{reg}^{W'}}} = \mathfrak{h}_{reg}^{W'} \widehat{\times} \mathfrak{h}_{W'}^{\wedge_0}$ and has ring of formal functions $\mathbb{C}[\mathfrak{h}^{W'}]\widehat{\otimes}\mathbb{C}[[\mathfrak{h}_{W'}]]$. From this it follows immediately that there is a natural isomorphism

$$\widehat{H}_c(N_W(W'), \mathfrak{h})_{\mathfrak{h}_{reg}^{W'}} \cong N_W(W') \ltimes_{W'} (\widehat{H}_c(W', \mathfrak{h}_{W'})_0 \widehat{\otimes} D(\mathfrak{h}_{reg}^{W'})).$$

Let

$$\psi : 1_{\mathfrak{h}_{reg}^{W'}} \widehat{H}_c(W, \mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}} 1_{\mathfrak{h}_{reg}^{W'}} \to N_W(W') \ltimes_{W'} (\widehat{H}_c(W', \mathfrak{h}_{W'})_0 \widehat{\otimes} D(\mathfrak{h}_{reg}^{W'}))$$

be the isomorphism obtained as the composition of the two natural isomorphisms discussed above, and let ψ_* denote the induced equivalence of module categories.

As in Definition 2.2.0.4, let $\mathbf{h}_{W'} \in H_c(W', \mathfrak{h}_{W'})$ denote the grading element

$$\mathbf{h}_{W'} := \sum_{i=1}^{n} x_i y_i + \frac{n}{2} - \sum_{s \in S'} \frac{2c_s}{1-\lambda_s} s$$

where $x_1, ..., x_n \in \mathfrak{h}_{W'}^*$ is any basis of $\mathfrak{h}_{W'}^*$ and $y_1, ..., y_n$ is the dual basis of $\mathfrak{h}_{W'}$. Recall that $\mathbf{h}_{W'}$ satisfies the commutation relations $[\mathbf{h}_{W'}, x] = x$ for all $x \in \mathfrak{h}_{W'}^*$, $[\mathbf{h}_{W'}, y] = -y$ for all $y \in \mathfrak{h}_{W'}$, and $[\mathbf{h}_{W'}, w] = 0$ for all $w \in W$. Note that in our setting, $\mathbf{h}_{W'}$ is also fixed by the action of $N_W(W')$ on $H_c(W', \mathfrak{h}_{W'})$ because the parameter $c : S' \to \mathbb{C}$ is obtained by restriction of the W-invariant parameter $c : S \to \mathbb{C}$.

Let $\mathcal{O}(N_W(W') \ltimes_{W'} (H_c(W', \mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'})))$ denote the category of modules over the algebra

$$N_W(W') \ltimes_{W'} (H_c(W', \mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'}))$$

that are finitely generated over $\mathbb{C}[\mathfrak{h}_{reg}^{W'}] \otimes \mathbb{C}[\mathfrak{h}_{W'}]$ and on which $\mathfrak{h}_{W'}$ acts locally nilpotently, and similarly let $\widehat{\mathcal{O}}(N_W(W') \ltimes_{W'} (\widehat{H}_c(W', \mathfrak{h}_{W'})_0 \widehat{\otimes} D(\mathfrak{h}_{reg}^{W'})))$ denote the category of $N_W(W') \ltimes_{W'} (\widehat{H}_c(W', \mathfrak{h}_{W'})_0 \widehat{\otimes} D(\mathfrak{h}_{reg}^{W'}))$-modules that are finitely generated over $\mathbb{C}[\mathfrak{h}_{reg}^{W'}]\widehat{\otimes}\mathbb{C}[[\mathfrak{h}_{W'}]]$. By the proof of [5, Proposition 2.4], $\mathbf{h}_{W'}$ acts locally finitely on any $H_c(W', \mathfrak{h}_{W'})$-module that has locally nilpotent action of $\mathfrak{h}_{W'}$, and in particular $\mathbf{h}_{W'}$ acts locally finitely on any $M \in \mathcal{O}(N_W(W') \ltimes_{W'} (H_c(W', \mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'})))$. It follows in particular that such M are naturally graded as

$$M = \bigoplus_{a \in \mathbb{C}} M_a$$

where M_a is the generalized a-eigenspace of $\mathbf{h}_{W'}$ in M. As the actions of $\mathbf{h}_{W'}$ and $N_W(W') \ltimes D(\mathfrak{h}_{reg}^{W'})$ on M commute, it follows that the above decomposition of M is a decomposition as $N_W(W') \times D(\mathfrak{h}_{reg}^{W'})$-modules. As $\mathfrak{h}_{W'} M_a \subset M_{a+1}$ for all $a \in \mathbb{C}$ and as M is finitely generated over $\mathbb{C}[\mathfrak{h}_{W'}] \otimes \mathbb{C}[\mathfrak{h}_{reg}^{W'}]$, it follows that each M_a is finitely generated over $\mathbb{C}[\mathfrak{h}_{reg}^{W'}]$ and that the set of $a \in \mathbb{C}$ such that $M_a \neq 0$ is a subset of a finite union of sets of the form $z + \mathbb{Z}^{\geq 0}$. In particular, the generalized eigenspaces M_a are $N_W(W')$-equivariant $\mathcal{O}(\mathfrak{h}_{reg}^{W'})$-coherent $D(\mathfrak{h}_{reg}^{W'})$-modules. An argument entirely analogous to the proof of Theorem 2.3 in [5] then shows that the functor

$$E : \widehat{\mathcal{O}}(N_W(W') \ltimes_{W'} (\widehat{H}_c(W', \mathfrak{h}_{W'})_0 \widehat{\otimes} D(\mathfrak{h}_{reg}^{W'})))$$

$$\to \mathcal{O}(N_W(W') \ltimes_{W'} (H_c(W', \mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'})))$$

sending a module M to its subspace of $\mathbf{h}_{W'}$-locally finite vectors is quasi-inverse to the functor

$$\widehat{}_{\mathfrak{h}_{reg}^{W'}} : \mathcal{O}(N_W(W') \ltimes_{W'} (H_c(W', \mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'})))$$

$$\to \widehat{\mathcal{O}}(N_W(W') \ltimes_{W'} (\widehat{H}_c(W', \mathfrak{h}_{W'})_0 \widehat{\otimes} D(\mathfrak{h}_{reg}^{W'})))$$

of completion at $\mathfrak{h}_{reg}^{W'}$.

Let $\widehat{\mathcal{O}}(\widehat{H}_c(W, \mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}})$ denote the category of $\widehat{H}_c(W, \mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}}$-modules finitely generated over the ring of functions $\mathbb{C}[\mathfrak{h}]^{\wedge_{W\mathfrak{h}_{reg}^{W'}}}$ on the formal neighborhood of $W\mathfrak{h}_{reg}^{W'}$ in $\mathfrak{h}$. Completion at $W\mathfrak{h}^{W'}$ defines an exact functor

$$\widehat{}_{W\mathfrak{h}_{reg}^{W'}} : \mathcal{O}_c(W, \mathfrak{h}) \to \widehat{\mathcal{O}}(\widehat{H}_c(W, \mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}}).$$

Note also that the discussion above and the observation that $1_{\mathfrak{h}_{reg}^{W'}}$ generates the unit ideal in $\widehat{H}_c(W,\mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}}$ shows that the composition

$$\psi_* \circ 1_{\mathfrak{h}_{reg}^{W'}} : \widehat{\mathcal{O}}(\widehat{H}_c(W,\mathfrak{h})_{W\mathfrak{h}_{reg}^{W'}}) \to \widehat{\mathcal{O}}(N_W(W') \ltimes_{W'} (\widehat{H}_c(W',\mathfrak{h}_{W'})_0 \widehat{\otimes} D(\mathfrak{h}_{reg}^{W'})))$$

is an equivalence of categories. Next, consider the composite functor

$$E \circ \psi_* \circ 1_{\mathfrak{h}_{reg}^{W'}} \circ \widehat{\ }_{W\mathfrak{h}_{reg}^{W'}} : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}(N_W(W') \ltimes_{W'} (H_c(W',\mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'})))$$

which as we've seen is a completion functor followed by equivalences of categories.

Lemma 2.5.0.1. *The image of the composite functor $E \circ \psi_* \circ 1_{\mathfrak{h}_{reg}^{W'}} \circ \widehat{\ }_{W\mathfrak{h}_{reg}^{W'}}$ lies in the full subcategory*

$$\mathcal{O}(N_W(W') \ltimes_{W'} (H_c(W',\mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'})))_{r.s.}$$

of

$$\mathcal{O}(N_W(W') \ltimes_{W'} (H_c(W',\mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'})))$$

consisting of those modules M whose generalized $\mathbf{h}_{W'}$ eigenspaces M_a have regular singularities as $\mathcal{O}(\mathfrak{h}_{reg}^{W'})$-coherent $D(\mathfrak{h}_{reg}^{W'})$-modules.

Proof. Let N be a module in $\mathcal{O}_c(W,\mathfrak{h})$ and let M be its image under the composite functor $E \circ \psi_* \circ 1_{\mathfrak{h}_{reg}^{W'}} \circ \widehat{\ }_{W\mathfrak{h}_{reg}^{W'}}$. As the full subcategory of $\mathcal{O}(\mathfrak{h}_{reg}^{W'})$-coherent $D(\mathfrak{h}_{reg}^{W'})$-modules consisting of those D-modules with regular singularities is a Serre subcategory, to show that the $D(\mathfrak{h}_{reg}^{W'})$-module M_a has regular singularities it suffices to show that every irreducible $D(\mathfrak{h}_{reg}^{W'})$-module appearing as a composition factor in M has regular singularities. Note that $\mathfrak{h}_{W'}^*$ acts on M by $D(\mathfrak{h}_{reg}^{W'})$-module homomorphisms of degree 1 with respect to the $\mathbf{h}_{W'}$-grading. It follows that $(\mathfrak{h}_{W'}^*)^k M$ is a $D(\mathfrak{h}_{reg}^{W'})$-submodule of M for all integers $k \geq 0$, and, as M is finitely generated over $\mathbb{C}[\mathfrak{h}_{W'}] \otimes \mathbb{C}[\mathfrak{h}_{reg}^{W'}]$, that any generalized $\mathbf{h}_{W'}$-eigenspace M_a of M embeds in some quotient $M/(\mathfrak{h}_{W'}^*)^k M$ for sufficiently large k. Therefore, it suffices to show that the module $M/(\mathfrak{h}_{W'}^*)^k M$ has regular singularities for all integers $k > 0$. But $M/\mathfrak{h}_{W'}^* M$ is precisely the $D(\mathfrak{h}_{reg}^{W'})$-module obtained by pulling N back to $\mathfrak{h}_{reg}^{W'}$ as a coherent sheaf as in [73, Proposition 1.2], which has regular singularities by [73, Proposition 1.3]. That $M/(\mathfrak{h}_{W'}^*)^k M$ has regular singularities then follows from the exact sequence

$$0 \to \frac{(\mathfrak{h}_{W'}^*)^{k-1}M}{(\mathfrak{h}_{W'}^*)^k M} \to \frac{M}{(\mathfrak{h}_{W'}^*)^k M} \to \frac{M}{(\mathfrak{h}_{W'}^*)^{k-1} M} \to 0$$

and induction on k. □

By the Riemann-Hilbert correspondence [16], passing to local systems of flat sections on $\mathfrak{h}_{reg}^{W'}$ in each generalized eigenspace then defines an equivalence of categories

$$RH : \mathcal{O}(N_W(W') \ltimes_{W'} (H_c(W', \mathfrak{h}_{W'}) \otimes D(\mathfrak{h}_{reg}^{W'})))_{r.s.} \to (\mathcal{O}_c(W', \mathfrak{h}_{W'}) \boxtimes \mathrm{Loc}(\mathfrak{h}_{reg}^{W'}))^{N_W(W')}$$

where $\mathcal{O}_c(W', \mathfrak{h}_{W'}) \boxtimes \mathrm{Loc}(\mathfrak{h}_{reg}^{W'})$ is the category of local systems on $\mathfrak{h}_{reg}^{W'}$ of $H_c(W', \mathfrak{h}_{W'})$-modules in $\mathcal{O}_c(W', \mathfrak{h}_{W'})$ and where $(\mathcal{O}_c(W', \mathfrak{h}_{W'}) \boxtimes \mathrm{Loc}(\mathfrak{h}_{reg}^{W'}))^{N_W(W')}$ is the associated category of $N_W(W')$-equivariant objects.

Definition 2.5.0.2. *Let* $\underline{Res}_{W'}^W$ *be the* partial KZ functor

$$\underline{Res}_{W'}^W : \mathcal{O}_c(W, \mathfrak{h}) \to (\mathcal{O}_c(W', \mathfrak{h}_{W'}) \boxtimes Loc(\mathfrak{h}_{reg}^{W'}))^{N_W(W')}$$

defined by the composition

$$\underline{Res}_{W'}^W := RH \circ E \circ \psi_* \circ 1_{\mathfrak{h}_{reg}^{W'}} \circ \widehat{\ }_{W\mathfrak{h}_{reg}^{W'}}.$$

Let $b \in \mathfrak{h}_{reg}^{W'}$ and let $\mathrm{Fiber}_b : (\mathcal{O}_c(W', \mathfrak{h}_{W'}) \boxtimes \mathrm{Loc}(\mathfrak{h}_{reg}^{W'}))^{N_W(W')} \to \mathcal{O}_c(W', \mathfrak{h}_{W'})$ denote the exact functor of taking the fiber of the local system at b. From [5, Section 3.7], the functor $\mathrm{Fiber}_b \circ \underline{\mathrm{Res}}_{W'}^W$ is naturally identified with the parabolic restriction functor Res_b. In particular, the monodromy in $\mathfrak{h}_{reg}^{W'}$ provides isomorphisms of functors $\mathrm{Res}_b \cong \mathrm{Res}_{b'}$ for any points $b, b' \in \mathfrak{h}_{reg}^{W'}$. Note, however, that such an isomorphism is not canonical and depends on a choice of path between b and b'. This monodromy action on Res_b will be crucial in what follows.

When there is no loss of clarity, we will suppress the choice of $b \in \mathfrak{h}^{W'}$ and write

$\mathrm{Res}_{W'}^{W}$ for the parabolic restriction functor Res_b, and similarly we will write $\mathrm{Ind}_{W'}^{W}$ for the parabolic induction functor Ind_b. The underlined $\underline{\mathrm{Res}}_{W'}^{W}$ will always denote the associated partial KZ functor with fibers $\mathrm{Res}_{W'}^{W}$.

2.6 Gordon-Martino Transitivity

Let $W'' \subset W' \subset W$ be a chain of parabolic subgroups of W. In [65, Corollary 2.5], Shan proved that the parabolic restriction functors are transitive in the sense that there is an isomorphism of functors

$$\mathrm{Res}_{W''}^{W} \cong \mathrm{Res}_{W''}^{W'} \circ \mathrm{Res}_{W'}^{W}.$$

Gordon and Martino [40] explained the sense in which this transitivity is compatible with the local systems appearing in the partial KZ functors. We recall their result below.

Consider the functor

$$\underline{\mathrm{Res}}_{W''}^{W'} \boxtimes \mathrm{Id} \circ \downarrow_{N_W(W',W'')}^{N_W(W')} \circ \underline{\mathrm{Res}}_{W'}^{W} : \mathcal{O}_c(W, \mathfrak{h})$$

$$\to \left(\mathcal{O}_c(W'', \mathfrak{h}_{W''}) \boxtimes \mathrm{Loc}((\mathfrak{h}_{W'})_{reg}^{W''}) \boxtimes \mathrm{Loc}(\mathfrak{h}_{reg}^{W'})\right)^{N_W(W',W'')}.$$

Here $N_W(W', W'')$ is the intersection of the normalizers $N_W(W')$ and $N_W(W'')$ and $\downarrow$ denotes the restriction of equivariant structure to a subgroup. Similarly to the notation $\mathfrak{h}_{reg}^{W'}$, the space $(\mathfrak{h}_{W'})_{reg}^{W''}$ is the locally closed locus of points in $\mathfrak{h}_{W'}$ with stabilizer W'' in W'. The goal is to relate this functor to the partial KZ functor $\underline{\mathrm{Res}}_{W''}^{W}$. However, the latter functor produces local systems on the space $\mathfrak{h}_{reg}^{W''}$, not on $(\mathfrak{h}_{W'})_{reg}^{W''} \times \mathfrak{h}_{reg}^{W'}$ as the functor considered above does. In general, viewed as subvarieties of $\mathfrak{h}$, these spaces do not coincide, and there is no obvious map between them. However, $(\mathfrak{h}_{W'})_{reg}^{W''} \times \mathfrak{h}_{reg}^{W'}$, viewed as a complex manifold, does contain a $N_W(W', W'')$-stable deformation retract that includes naturally in $\mathfrak{h}_{reg}^{W''}$, giving rise to a pullback functor

$$\iota^*_{W',W''} : \mathrm{Loc}(\mathfrak{h}^{W''}_{reg})^{N_W(W',W'')} \to \mathrm{Loc}((\mathfrak{h}_{W'})^{W''}_{reg} \times \mathfrak{h}^{W'}_{reg})^{N_W(W',W'')}.$$

This functor can be constructed as follows.

Choose a W-invariant norm $||\cdot||$ on $\mathfrak{h}$. Let $\epsilon > 0$ be a positive number. Define the subspaces

$$\mathfrak{h}^{W'}_{\epsilon-reg} := \{x \in \mathfrak{h}^{W'} : ||wx - x|| > \epsilon \text{ for all } w \in W \backslash W'\} \subset \mathfrak{h}^{W'}_{reg}$$

and

$$(\mathfrak{h}_{W'})^{W'',\epsilon}_{reg} := \{x \in (\mathfrak{h}_{W'})^{W''}_{reg} : ||x|| < \epsilon/2\} \subset (\mathfrak{h}_{W'})^{W''}_{reg}.$$

Note that these spaces are $N_W(W',W'')$-stable, that the subspace $(\mathfrak{h}_{W'})^{W'',\epsilon}_{reg} \times \mathfrak{h}^{W'}_{\epsilon-reg}$ is a deformation retract of $(\mathfrak{h}_{W'})^{W''}_{reg} \times \mathfrak{h}^{W'}_{reg}$ via a $N_W(W',W'')$-equivariant deformation retraction, and there is a natural inclusion $(\mathfrak{h}_{W'})^{W'',\epsilon}_{reg} \times \mathfrak{h}^{W'}_{\epsilon-reg} \subset \mathfrak{h}^{W''}_{reg}$. Pullback along the inclusion $(\mathfrak{h}_{W'})^{W'',\epsilon}_{reg} \times \mathfrak{h}^{W'}_{\epsilon-reg} \subset \mathfrak{h}^{W''}_{reg}$ defines a functor

$$\mathrm{Loc}(\mathfrak{h}^{W''}_{reg})^{N_W(W',W'')} \to \mathrm{Loc}((\mathfrak{h}_{W'})^{W'',\epsilon}_{reg} \times \mathfrak{h}^{W'}_{\epsilon-reg})^{N_W(W',W'')}$$

and the deformation retraction defines an equivalence of categories

$$\mathrm{Loc}((\mathfrak{h}_{W'})^{W'',\epsilon}_{reg} \times \mathfrak{h}^{W'}_{\epsilon-reg})^{N_W(W',W'')} \cong \mathrm{Loc}((\mathfrak{h}_{W'})^{W''}_{reg} \times \mathfrak{h}^{W'}_{reg})^{N_W(W',W'')}.$$

We define $\iota^*_{W',W''}$ to be the composition of these functors. Note that $\iota^*_{W',W''}$ does not depend on the choice of norm $||\cdot||$ or $\epsilon > 0$. We comment that the definition of $\iota^*_{W',W''}$ given in [40] was stated in terms of fundamental groups, but the version we give here is equivalent.

We can now state the transitivity result of Gordon-Martino:

Theorem 2.6.0.1. (Gordan-Martino, [40, Theorem 3.11]) *There is a natural isomor-*

phism

$$\iota^*_{W',W''} \circ \downarrow^{N_W(W'')}_{N_W(W',W'')} \circ \underline{Res}^W_{W''} \cong (\underline{Res}^{W'}_{W''} \boxtimes Id) \circ \downarrow^{N_W(W')}_{N_W(W',W'')} \circ \underline{Res}^W_{W'}$$

of functors

$$\mathcal{O}_c(W,\mathfrak{h}) \to \Big(\mathcal{O}_c(W'',\mathfrak{h}_{W''}) \boxtimes Loc((\mathfrak{h}_{W'})^{W''}_{reg}) \boxtimes Loc(\mathfrak{h}^{W'}_{reg})\Big)^{N_W(W',W'')}.$$

2.7 Mackey Formula for Rational Cherednik Algebras for Coxeter Groups

Shan and Vasserot established in [66, Lemma 2.5] that the natural analogue of the usual Mackey formula for the composition of induction and restriction functors for representations of finite groups holds for the Bezrukavnikov-Etingof parabolic induction and restriction functors at the level of Grothendieck groups. It will be convenient for us to know that, at least in the case of rational Cherednik algebras attached to finite Coxeter groups, the Mackey formula holds at the level of the parabolic induction and restriction functors themselves. This is established below, by lifting the Mackey formula for the associated Hecke algebras via the KZ functor.

Let (W,S) be a finite Coxeter system with simple reflections S, real reflection representation $\mathfrak{h}_{\mathbb{R}}$, and complexified reflection representation $\mathfrak{h}$. Let $c : S \to \mathbb{C}$ be a class function on the simple reflections, and let $q : S \to \mathbb{C}^\times$ be the associated class function $q(s) = e^{-2\pi i c(s)}$. Recall that the *Hecke algebra* $\mathsf{H}_q(W)$ attached to W and q is the associative $\mathbb{C}$-algebra generated by symbols $\{T_s : s \in S\}$ subject to *braid relations*

$$T_{s_1}T_{s_2}T_{s_1}\cdots = T_{s_2}T_{s_1}T_{s_2}\cdots$$

for $s_1 \neq s_2 \in S$, where there are m_{ij} terms on each side where m_{ij} is the order of the product s_1s_2, and the *quadratic relations*

$$(T_s - 1)(T_s + q_s) = 0$$

where we write q_s for $q(s)$. Recall that the *length function* $l : W \to \mathbb{Z}^{\geq 0}$ assigns to $w \in W$ the minimal length $l(w)$ of an expression $s_{i_1} \cdots s_{i_{l(w)}} = w$ of w as a product of simple reflections, and that such a factorization of w is called a *reduced expression*. The product $T_{s_{i_1}} \cdots T_{s_{i_{l(w)}}}$ in $\mathsf{H}_q(W)$ does not depend on the choice of reduced expression $s_{i_1} \cdots s_{i_{l(w)}}$ for W, and the resulting elements T_w form a $\mathbb{C}$-basis for $\mathsf{H}_q(W)$. For details and proofs on such basic structural results on Hecke algebras mentioned in this section, see [37] (note that their convention is to use quadratic relations $(T+1)(T-q) = 0$ rather than $(T-1)(T+q) = 0$).

For a subset $J \subset S$ of the simple reflections in W, let $W_J \subset W$ be the parabolic subgroup generated by $J \subset S$. Then (W_J, J) is a Coxeter subsystem of (W, S), and the (complexified) reflection representation of (W_J, J) is identified with the action of W_J on the unique W_J-stable complement $\mathfrak{h}_{W_J}$ of $\mathfrak{h}^{W_J}$ in $\mathfrak{h}$. The parameter $q : S \to \mathbb{C}^\times$ restricts to give a W_J-invariant function $J \to \mathbb{C}^\times$, which by abuse of notation we also denote by q. We therefore may form the Hecke algebra $\mathsf{H}_q(W_J)$. Reduced expressions of $w \in W_J$ as an element of W_J are precisely the reduced expressions of w as an element of W. In particular the length function l_{W_J} for (W_J, J) coincides with the restriction to W_J of the length function for (W, J), and in this way the assignment $T_w \mapsto T_w$ for $w \in W$ extends uniquely to a unital embedding of the algebra $\mathsf{H}_q(W_J)$ as a subalgebra of $\mathsf{H}_q(W)$. Via this embedding we can define the parabolic restriction functor

$$^H\mathrm{Res}^W_{W_J} : \mathsf{H}_q(W)\text{-mod} \to \mathsf{H}_q(W_J)\text{-mod}$$

and the parabolic induction functor

$$^H\mathrm{Ind}^W_{W_J} := \mathsf{H}_q(W) \otimes_{\mathsf{H}_q(W_J)} \bullet : \mathsf{H}_q(W_J)\text{-mod} \to \mathsf{H}_q(W)\text{-mod}.$$

Let $J, K \subset S$ be two subsets of simple reflections, with associated parabolic subgroups W_J and W_K. Each W_K-W_J double-coset in W contains a unique element of minimal length. Let $X_{KJ} \subset W$ denote this subset of distinguished double-coset representatives. For $d \in X_{KJ}$, conjugation $w \mapsto dwd^{-1}$ defines a length-preserving isomorphism $W_{J \cap K^d} \to W_{^dJ \cap K}$, where $K^d := d^{-1}Kd$ and $^dJ := dJd^{-1}$. In particular, the assignment $T_w \mapsto T_{dwd^{-1}}$ extends uniquely to an algebra isomorphism $\mathsf{H}_q(W_{J\cap K^d}) \to \mathsf{H}_q(W_{^dJ\cap K})$. Note that this isomorphism is realized inside $\mathsf{H}_q(W)$ by conjugation by T_d. Let

$$^H\mathrm{tw}_d : \mathsf{H}_q(W_{J\cap K^d})\text{-mod} \to \mathsf{H}_q(W_{^dJ\cap K})\text{-mod}$$

denote the equivalence of categories obtained by transfer of structure via this isomorphism.

We can now state the Mackey formula for Hecke algebras. Note that this formula holds for any numerical parameter $q : S \to \mathbb{C}^\times$.

Proposition 2.7.0.1. (Mackey Formula for Hecke Algebras, [37, Proposition 9.1.8]) *Let $J, K \subset S$ and let $X_{KJ} \subset W$ be the set of distinguished (minimal length) W_K-W_J double-coset representatives in W. There is an isomorphism*

$$^H Res^W_{W_K} \circ {}^H Ind^W_{W_J} \cong \bigoplus_{d \in X_{KJ}} {}^H Ind^W_{W_{^dJ\cap K}} \circ {}^H tw_d \circ {}^H Res^{W_J}_{J\cap K^d}$$

of functors $\mathsf{H}_q(W_J)$-mod $\to \mathsf{H}_q(W_K)$-mod.

By restriction of the parameter c to subsets $J \subset S$, one can form the associated rational Cherednik algebra $H_c(W_J, \mathfrak{h}_{W_J})$. Let $J, K \subset S$, and let $d \in X_{KJ}$. Note that the action of d on $\mathfrak{h}$ induces an isomorphism $\mathfrak{h}_{W_{J\cap K^d}} \to \mathfrak{h}_{W_{^dJ\cap K}}$, $y \mapsto dy$, and hence also an isomorphism $\mathfrak{h}^*_{W_{J\cap K^d}} \to \mathfrak{h}^*_{W_{^dJ\cap K}}$, $f \mapsto d(f) = f(d^{-1}\bullet)$. As in the comments preceding [66, Lemma 2.5], these isomorphisms along with the isomorphism $W_{J\cap K^d} \to W_{^dJ\cap K}$ discussed above induce an isomorphism $H_c(W_{J\cap K^d}, \mathfrak{h}_{W_{J\cap K^d}}) \cong H_c(W_{^dJ\cap K}, \mathfrak{h}_{W_{^dJ\cap K}})$ respecting triangular decompositions. Transfer of structure by this isomorphism therefore induces an equivalence of categories

$$\mathrm{tw}_d : \mathcal{O}_c(W_{J\cap K^d}, \mathfrak{h}_{W_{J\cap K^d}}) \cong \mathcal{O}_c(W_{^dJ\cap K}, \mathfrak{h}_{W_{^dJ\cap K}}).$$

Theorem 2.7.0.2. (Mackey Formula for Rational Cherednik Algebras for Coxeter Groups) *Let $J, K \subset S$ and let $X_{KJ} \subset W$ be the set of distinguished W_K-W_J double-*

coset representatives in W. There is an isomorphism

$$Res^W_{W_K} \circ Ind^W_{W_J} \cong \bigoplus_{d \in X_{KJ}} Ind^W_{W_{d_{J\cap K}}} \circ tw_d \circ Res^{W_J}_{J\cap K^d}$$

of functors $\mathcal{O}_c(W_J, \mathfrak{h}_{W_J}) \to \mathcal{O}_c(W_K, \mathfrak{h}_{W_K})$.

Proof. For a subset $A \subset S$, let $KZ(W_A, \mathfrak{h}_{W_A})$ denote the KZ functor $\mathcal{O}_c(W_A, \mathfrak{h}_{W_A}) \to \mathsf{H}_q(W_A)$-mod. It follows from [65, Lemma 2.4] that to show the existence of the desired isomorphism of functors, we need only check that the functors $KZ(W_K, \mathfrak{h}^{W_K}) \circ \mathrm{Res}^W_{W_K} \circ \mathrm{Ind}^W_{W_J}$ and $KZ(W_K, \mathfrak{h}_{W_K}) \circ \bigoplus_{d \in X_{KJ}} \mathrm{Ind}^W_{W_{d_{J\cap K}}} \circ \mathrm{tw}_d \circ \mathrm{Res}^{W_J}_{J\cap K^d}$ are isomorphic. Shan also proved ([65, Theorem 2.1]) that KZ commutes with parabolic restriction functors in the sense that there is an isomorphism of functors

$$KZ(W_A, \mathfrak{h}_{W_A}) \circ \mathrm{Res}^{W_B}_{W_A} \cong {}^H\mathrm{Res}^{W_B}_{W_A} \circ KZ(W_B, \mathfrak{h}_{W_B})$$

for any $A \subset B \subset S$. It is clear that KZ commutes with tw_d in the sense that there is an isomorphism of functors $KZ(W_{{}^dJ\cap K}, \mathfrak{h}_{W^{d}J\cap K}) \circ \mathrm{tw}_d \cong {}^H\mathrm{tw}_d \circ KZ(W_{J\cap K^d}, \mathfrak{h}_{W_{J\cap K^d}})$. Passing the KZ functor to the right using these isomorphisms, the desired statement then follows from the Mackey formula for Hecke algebras. $\square$

Chapter 3

The Refined Filtration By Supports

3.1 Introduction

A key technical tool for studying $H_c(W, \mathfrak{h})$ and the category $\mathcal{O}_c(W, \mathfrak{h})$ is the Knizhnik-Zamolodchikov (KZ) functor also introduced in [38]. The KZ functor

$$KZ : \mathcal{O}_c(W, \mathfrak{h}) \to \mathsf{H}_q(W)\text{-mod}_{f.d.}$$

is an exact functor, defined via monodromy, from $\mathcal{O}_c(W, \mathfrak{h})$ to the category

$$\mathsf{H}_q(W)\text{-mod}_{f.d.}$$

of finite-dimensional modules over the Hecke algebra $\mathsf{H}_q(W)$ associated to the reflection group W. The parameter q of the Hecke algebra $\mathsf{H}_q(W)$ depends on the parameter c of the rational Cherednik algebra $H_c(W, \mathfrak{h})$ in an exponential manner. KZ induces an equivalence of categories $\mathcal{O}_c(W, \mathfrak{h})/\mathcal{O}_c(W, \mathfrak{h})^{tor} \cong \mathsf{H}_q(W)\text{-mod}_{f.d}$ [38, 53], where $\mathcal{O}_c(W, \mathfrak{h})^{tor}$ denotes the Serre subcategory of modules in $\mathcal{O}_c(W, \mathfrak{h})$ supported on the union of the reflection hyperplanes $\cup_{s \in S} \ker(s)$. In this way, KZ establishes a bijection between the irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ of full support in $\mathfrak{h}$ and the finite-dimensional irreducible representations of $\mathsf{H}_q(W)$. Following previous work of Etingof-Rains [32], Marin-Pfeiffer [57], Losev [53], Chavli [9], and others towards proving the Broué-Malle-Rouquier conjecture [7], Etingof [27] recently showed that the Hecke algebra $\mathsf{H}_q(W)$ is always finite-dimensional with dimension $\#W$, even in the case of complex reflection groups.

In this chapter, in the case that W is a Coxeter group with complexified reflection representation $\mathfrak{h}$, we extend this correspondence between irreducible representations L in $\mathcal{O}_c(W, \mathfrak{h})$ and irreducible representations of finite-type Hecke algebras to include all cases in which the support of L is not equal to $\{0\} \subset \mathfrak{h}$, i.e. all cases in which L is not finite-dimensional over $\mathbb{C}$. Our approach is inspired by the Harish-Chandra series appearing in the representation theory of finite groups of Lie type. In place of the parabolic induction and restriction functors defined for finite groups of Lie type, in the setting of rational Cherednik algebras one has analogous parabolic induction and restriction functors introduced by Bezrukavnikov and Etingof [5]. In particular, suppose $(W', S') \subset (W, S)$ is a parabolic Coxeter subsystem of (W, S) with complexified reflection representation $\mathfrak{h}_{W'}$. Restricting the parameter c to S' we may form the rational Cherednik algebra $H_c(W', \mathfrak{h}_{W'})$ and the associated category of representations $\mathcal{O}_c(W', \mathfrak{h}_{W'})$. The algebra $H_c(W', \mathfrak{h}_{W'})$ does not naturally embed as a subalgebra of the algebra $H_c(W, \mathfrak{h})$ as is the case for Hecke algebras, so there is no naive definition of parabolic induction and restriction functors as there is for Hecke algebras. Rather, the Bezrukavinikov-Etingof parabolic induction functor $\mathrm{Ind}_{W'}^W : \mathcal{O}_c(W', \mathfrak{h}_{W'}) \to \mathcal{O}_c(W, \mathfrak{h})$ and restriction functor $\mathrm{Res}_{W'}^W : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_c(W', \mathfrak{h}_{W'})$ are more technical and are defined using the geometric interpretation of rational Cherednik algebras and a certain isomorphism involving completions of $H_c(W, \mathfrak{h})$ and $H_c(W', \mathfrak{h}_{W'})$. The functors $\mathrm{Res}_{W'}^W$ and $\mathrm{Ind}_{W'}^W$ are exact [5] and biadjoint [54, 65].

As for representations of finite groups of Lie type, one defines the cuspidal representations of a rational Cherednik algebra $H_c(W, \mathfrak{h})$ to be those irreducible representations L in $\mathcal{O}_c(W, \mathfrak{h})$ such that $\mathrm{Res}_{W'}^W L = 0$ for all proper parabolic subgroups $W' \subset W$. For rational Cherednik algebras, the cuspidal representations are precisely the finite-dimensional irreducible representations. The class of cuspidal representations is the smallest class of representations in the categories $\mathcal{O}_c(W, \mathfrak{h})$ such that any irreducible representation in a category $\mathcal{O}_c(W, \mathfrak{h})$ appears as a subobject (or, as a quotient) of a representation induced from a representation in that class. From this perspective, the study of irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ is largely reduced to the study of cuspidal representations L in categories $\mathcal{O}_c(W', \mathfrak{h}_{W'})$ for $W' \subset W$ a parabolic subgroup and of the structure and decomposition of the induced representations $\mathrm{Ind}_{W'}^W L$.

The most basic finite-dimensional irreducible representation of a rational Cherednik algebra is the trivial representation $\mathbb{C}$ of the trivial rational Cherednik algebra $H_c(1, \{0\}) = \mathbb{C}$. The induced representation $\mathrm{Ind}_1^W \mathbb{C}$, denoted P_{KZ}, is a remarkable projective object in $\mathcal{O}_c(W, \mathfrak{h})$. In particular, there is an algebra isomorphism $\mathrm{End}_{H_c(W,\mathfrak{h})}(P_{KZ})^{opp} \cong \mathsf{H}_q(W)$ with respect to which P_{KZ} represents the KZ functor. The equivalence of categories $\mathcal{O}_c(W, \mathfrak{h})/\mathcal{O}_c(W, \mathfrak{h})^{tor} \cong \mathsf{H}_q(W)\text{-mod}_{f.d.}$ induced by KZ therefore establishes a correspondence between the irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ of full support in $\mathfrak{h}$ and the irreducible representations of $\mathrm{End}_{H_c(W,\mathfrak{h})}(\mathrm{Ind}_1^W \mathbb{C})$.

In this chapter, we study the endomorphism algebras $\mathrm{End}_{H_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L)$ of induced cuspidal representations L in the general case that L is not necessarily the trivial representation $\mathbb{C}$ in $\mathcal{O}_c(1, \{0\})$ and provide results analogous to and generalizing those summarized above for the case $L = \mathbb{C}$. In the parallel setting of representations of finite groups of Lie type, endomorphism algebras of induced cuspidal representations have been studied in great detail and are closely related to Hecke algebras of finite type. For example, in the most basic case one considers the parabolic induction $\mathrm{Ind}_B^G \mathbb{C}$ of the trivial representation $\mathbb{C}$ of the trivial group 1 to the finite general linear group $G := GL_n(\mathbb{F}_q)$, where q is a prime power and B is the standard Borel subgroup of upper triangular matrices. In that case, the endomorphism algebra is precisely the Hecke algebra $\mathsf{H}_q(S_n)$ where q is the order of the finite field, exactly analogous to the case of $P_{KZ} = \mathrm{Ind}_1^W \mathbb{C}$ for rational Cherednik algebras. In the general case, Howlett and Lehrer [46, Theorem 4.14] showed that, in characteristic 0, the endomorphism algebra of a parabolically induced cuspidal representation of a finite group of Lie type can be described as a semidirect product of a finite type Hecke algebra by a finite group acting by a diagram automorphism, twisted by a certain 2-cocycle (see, for instance, [8, Theorem 10.8.5]). We obtain an exactly analogous result for the endomorphism algebras of induced representations $\mathrm{Ind}_{W'}^W L$ of finite-dimensional irreducible representations L of $H_c(W', \mathfrak{h}_{W'})$. Howlett and Lehrer conjectured [46, Conjecture 6.3] that the 2-cocycle appearing in the description of the endomorphism algebra was trivial, as was later proved (see Lusztig [56, Theorem 8.6] and Geck [35]). Appealing to the classification of irreducible finite Coxeter groups and to previous results about finite-dimensional irreducible representations of rational Cherednik algebras, we find that the 2-cocycles appearing in our setting are trivial as well. The connections between the categories $\mathcal{O}_c(W, \mathfrak{h})$ and the modular representation theory of finite groups of Lie type is more than an analogy; Norton [61] explains various facts and conjectures relating these two subjects, as well as ways in which these connections break down.

The methods used in the setting of finite groups of Lie type ultimately rely on the comparatively simple and explicit definition of parabolic induction, which allows for a basis of intertwining operators to be written down in closed form. Instead, in our setting of the more complicated Bezrukavnikov-Etingof parabolic induction functors for rational Cherednik algebras, we consider for each finite-dimensional irreducible representation L in $\mathcal{O}_c(W', \mathfrak{h}_{W'})$ the functor KZ_L represented by $\mathrm{Ind}_{W'}^W L$, analogous to the KZ functor. Recall that KZ induces an equivalence of categories $\mathcal{O}_c(W, \mathfrak{h})/\mathcal{O}_c(W, \mathfrak{h})^{tor} \cong \mathsf{H}_q(W)\text{-mod}_{f.d.}$; the functor KZ_L induces an analogous equivalence. Specifically, let $\mathcal{O}_c(W, \mathfrak{h})_{(W',L)}$ be the Serre subcategory of $\mathcal{O}_c(W, \mathfrak{h})$ generated by those irreducible representations M supported on $W\mathfrak{h}^{W'}$ as coherent sheaves on $\mathfrak{h}$ and with L appearing as a composition factor of $\mathrm{Res}_{W'}^W M$, and let $\mathcal{O}_c(W, \mathfrak{h})_{(W',L)}^{tor}$ be the kernel of $\mathrm{Res}_{W'}^W$ in $\mathcal{O}_c(W, \mathfrak{h})_{(W',L)}$. Then KZ_L induces an equivalence of categories

$$\mathcal{O}_c(W, \mathfrak{h})_{(W',L)}/\mathcal{O}_c(W, \mathfrak{h})_{(W',L)}^{tor} \cong \mathrm{End}_{H_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L)^{opp}\text{-mod}_{f.d}.$$

As with the KZ functor, we provide a geometric interpretation of the functor KZ_L in terms of the monodromy of an equivariant local system on $\mathfrak{h}_{reg}^{W'}$, the open locus of points inside the fixed space $\mathfrak{h}^{W'}$ with stabilizer precisely equal to W'. Using a transitivity result for local systems of parabolic restriction functors due to Gordon and

Martino [40], we reduce the problem of computing the eigenvalues of the monodromy around the missing hyperplanes, i.e. the parameters of the underlying "Hecke algebra" $\mathrm{End}_{H_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L)^{opp}$, to the case in which W' is a corank-1 parabolic subgroup of W. In that case, we use a different application of the Gordon-Martino transitivity result and the fact that the usual KZ functor is fully faithful on projective objects to further reduce the problem of computing eigenvalues of monodromy to concrete, although often involved, computations inside the Hecke algebra $\mathsf{H}_q(W)$ itself.

The main technical result of this chapter is the following:

Theorem. *Let W be a finite Coxeter group with simple reflections S, let $c : S \to \mathbb{C}$ be a class function, let W' be a standard parabolic subgroup generated by the simple reflections S', and let L be an irreducible finite-dimensional representation of the rational Cherednik algebra $H_c(W', \mathfrak{h}_{W'})$. Let $N_{W'}$ denote the canonical complement to W' in its normalizer $N_W(W')$, let $S_{W'} \subset N_{W'}$ denote the set of reflections in $N_{W'}$ with respect to its representation in the fixed space $\mathfrak{h}^{W'}$, and let $N_{W'}^{ref}$ denote the reflection subgroup of $N_{W'}$ generated by $S_{W'}$. Let $N_{W'}^{comp}$ be a complement for $N_{W'}^{ref}$ in $N_{W'}$, given as the stabilizer of a choice of fundamental Weyl chamber for the action of $N_{W'}^{ref}$ on $\mathfrak{h}^{W'}$. Then there is a class function $q_{W',L} : S_{W'} \to \mathbb{C}^\times$ and an isomorphism of algebras*

$$End_{H_c(W,\mathfrak{h})}(Ind_{W'}^W L)^{opp} \cong N_{W'}^{comp} \ltimes \mathsf{H}_{q_{W',L}}(N_{W'}^{ref})$$

where the semidirect product is defined by the action of $N_{W'}^{comp}$ on $\mathsf{H}_{q_{W',L}}(N_{W'}^{ref})$ by diagram automorphisms arising from the conjugation action of $N_{W'}^{comp}$ on $N_{W'}^{ref}$. Furthermore, the parameter $q_{W',L}$ is explicitly computable.

The content of the theorem above appears later in Theorem 3.3.2.14, for the isomorphism $\mathrm{End}_{H_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L)^{opp} \cong N_{W'}^{comp} \ltimes \mathsf{H}_{q_{W',L}}(N_{W'}^{ref})$, and Theorem 3.3.2.11, for the computation of $q_{W',L}$.

As a corollary of the explicit descriptions of the algebras

$$\mathrm{End}_{H_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L)^{opp}$$

we are able to count the number of irreducible representations in $\mathcal{O}_c(W,\mathfrak{h})$ with any given support variety in $\mathfrak{h}$. In particular, as an application we calculate the number of finite-dimensional irreducible representations of $H_c(W,\mathfrak{h})$ for all exceptional Coxeter groups W and all parameters c.

This chapter is organized as follows. In Section 3.2, we introduce the filtration on $\mathcal{O}_c(W,\mathfrak{h})$ that we study in this chapter. We construct the functor KZ_L using the monodromy of the Bezrukavnikov-Etingof parabolic restriction functors, providing a description of the algebra $\mathrm{End}_{H_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L)^{opp}$ as a quotient of a twisted group algebra of a fundamental group. We use the Gordon-Martino transitivity result to show that the generator of monodromy around a deleted hyperplane satisfies a polynomial relation of degree equal to the order of the stabilizer of that hyperplane in the inertia group of L (see Definition 3.2.1.5). In Section 3.3, we specialize to the case in which W is a Coxeter group. In this case, we provide a presentation of the endomorphism algebra as a generalized Hecke algebra directly analogous to the presentation by Howlett and Lehrer [46, Theorem 4.14]. Using the KZ functor, we provide a method for computing the parameters of these generalized Hecke algebras. We apply our methods systematically, using the classification of finite Coxeter groups and previously known results about finite-dimensional representations, to count the number of irreducible representations in $\mathcal{O}_c(W,\mathfrak{h})$ of given support for all exceptional Coxeter groups W and parameters c. We describe the generalized Hecke algebras appearing in types E and H explicitly.

The results we obtain for finite Coxeter groups confirm, unify, and extend many previously known results in both classical and exceptional types. In type A, we recover Wilcox's description [73, Theorem 1.2] of the subquotients of the filtration by supports of the categories $\mathcal{O}_c(S_n,\mathfrak{h})$. In type B, we show that the subquotients of our refined filtration are equivalent to categories of finite-dimensional modules over tensor products of Iwahori-Hecke algebras of type B, and we obtain similar descriptions in type D. These facts follows from results of Shan-Vasserot [66], although their results do not generalize to the exceptional types.

In the exceptional types, however, our methods provide many new results. Among the exceptional Coxeter types, complete knowledge of character formulas for all irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ for all parameters c is only known for the dihedral types, treated by Chmutova [13], and type H_3, treated by Balagovic-Puranik [2]. For parameters $c = 1/d$, where $d > 2$ is an integer dividing a fundamental degree of the Coxeter group W, Norton [61] computed the decomposition matrices for $\mathcal{O}_c(W, \mathfrak{h})$, thereby classifying the finite-dimensional irreducible representations and determining character formulas and supports for all irreducible representations, when W is a Coxeter group of type E_6, E_7, H_4, or F_4. After [52] was finished, I learned that Norton, in a sequel to [61] then to be announced, had produced complete decomposition matrices in type E_8 when the denominator d of c is not $2, 3, 4$ or 6, decomposition matrices for several blocks of $\mathcal{O}_c(E_8, \mathfrak{h})$ when the denominator d equals 4 or 6, as well as decomposition matrices in some unequal parameter cases in type F_4. However, to our knowledge, the classification of the finite-dimensional representations of rational Cherednik algebras remained unknown in types E_6, E_7, H_4 and F_4 at half-integer parameters, in type F_4 in most unequal parameter cases, and in type E_8 when the denominator d of c is 2,3,4, or 6.

Recently, Griffeth-Gusenbauer-Juteau-Lanini [45] produced a necessary condition for an irreducible representation $L_c(\lambda)$ in $\mathcal{O}_c(W, \mathfrak{h})$ to be finite-dimensional that applies to all complex reflection groups W and parameters c. Our results show that this condition is also sufficient in many of the previously unknown cases mentioned above, providing complete classifications of the irreducible finite-dimensional representations of $H_c(W, \mathfrak{h})$ in these cases. In combination with previous results of Balagovic-Puranik [2], Norton [61], in this way we complete the classification of finite-dimensional irreducible representations of $H_c(W, \mathfrak{h})$ in the case that W is an exceptional Coxeter group and $c = 1/d$, where d is a positive integer possibly equal to 2, except in the case in which both $W = E_8$ and also $d = 3$ simultaneously. In the case $W = E_8$ and $c = 1/3$, we show that there are 8 non-isomorphic finite-dimensional irreducible representations of $H_{1/3}(E_8, \mathfrak{h})$, and results of [45] rule out all but 9 of the 112 irreducible representations of the Coxeter group E_8 as the potential lowest weights of these representations. In particular, determining which one of these 9 irreducible representations is infinite-dimensional will complete the classification in type E entirely. This analysis in types E, H, and F is carried out Sections 3.3.7, 3.3.8, and 3.3.9, respectively.

In Sections 3.3.7 and 3.3.8, we describe the generalized Hecke algebras arising from exceptional Coxeter groups W of types E_6, E_7, E_8, H_3 and H_4 at all parameters c of the form $c = 1/d$, where $d > 1$ is an integer dividing a fundamental degree of W. In Section 3.3.9, we count the number of irreducible representations in $\mathcal{O}_{c_1,c_2}(F_4, \mathfrak{h})$ of given support in $\mathfrak{h}$ for all parameter values c_1, c_2, including the unequal parameter case. Along with previous results of Chmutova [13] for dihedral Coxeter groups, this completes the counting of irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ of given support for all exceptional Coxeter groups W and all, possibly unequal, parameters c.

These results for parameters $c = 1/d$ can be extended to parameters $c = r/d$ with $r > 0$ a positive integer relatively prime to d; the case $r < 0$ is then obtained by tensoring with the sign character. The reduction from the $c = r/d$ case to the $c = 1/d$ case can be achieved by results of Rouquier [63], Opdam [62], and Gordon-Griffeth [39] when $d \geq 3$. Specifically, a result of Rouquier, as it appears in [39, Theorem 2.7] after appropriate modifications, implies that for any integer $d \geq 3$ and integer $r > 1$ relatively prime to d there is a bijection φ on the set $\mathrm{Irr}(W)$ of the irreducible complex representations of the Coxeter group W such that for all $\lambda \in \mathrm{Irr}(W)$ the irreducible representations $L_{1/d}(\lambda)$ and $L_{r/d}(\varphi(\lambda))$ have the same support in $\mathfrak{h}$. In [62] these bijections are calculated, and in [39, Section 2.16] a method for computing the bijections φ is given which applies in greater generality. This allows the reduction of the classification of irreducible representations of given support in $\mathcal{O}_{r/d}(W, \mathfrak{h})$ to the classification of irreducible representations of the same given support in $\mathcal{O}_{1/d}(W, \mathfrak{h})$. In particular, as the finite-dimensional representations are those with support $\{0\} \subset \mathfrak{h}$, this provides the same reduction for classifying finite-dimensional irreducible representations. In type H_3, this analysis has been carried out in this way by Balagovic-Puranik [2, Section 5.4], and this approach generalizes to the other Coxeter types. When $d = 2$, our results show that the necessary condition for finite-dimensionality appearing in [45] is in fact also sufficient for the exceptional types E, H, and F, completing the classification of finite-dimensional irreducible representations for all half-integer parameters in these cases without the use of such bijections on the labels of the irreducibles.

3.2 Endomorphism Algebras Via Monodromy

Throughout this section, let W be a complex reflection group with reflection representation $\mathfrak{h}$, let $S \subset W$ be the set of reflections in W, and let $c : S \to \mathbb{C}$ be a W-invariant function. Let $H_c(W, \mathfrak{h})$ be the associated rational Cherednik algebra. In light of the isomorphism $H_c(W, \mathfrak{h}) \cong H_c(W, \mathfrak{h}_W) \otimes D(\mathfrak{h}^W)$, we will always assume that the fixed space $\mathfrak{h}^W$ is the trivial subspace. In order to understand and count irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ with a given support in $\mathfrak{h}$, it will be convenient to consider certain subcategories and subquotient categories of $\mathcal{O}_c(W, \mathfrak{h})$.

3.2.1 Harish-Chandra Series for Rational Cherednik Algebras

In the following definition, let $W' \subset W$ be a parabolic subgroup and let L be a finite-dimensional irreducible representation in $\mathcal{O}_c(W', \mathfrak{h}_{W'})$.

Definition 3.2.1.1. *Let $W' \subset W$ be a parabolic subgroup and let L be a finite-dimensional irreducible representation in $\mathcal{O}_c(W', \mathfrak{h}_{W'})$. Let $\mathcal{O}_{c,W'}(W, \mathfrak{h})$ be the full subcategory of $\mathcal{O}_c(W, \mathfrak{h})$ consisting of modules supported on $W\mathfrak{h}^{W'}$, and let $\mathcal{O}_{c,W',L}(W, \mathfrak{h})$ be the full subcategory of $\mathcal{O}_{c,W'}(W, \mathfrak{h})$ consisting of modules M such that $Res_{W'}^W M \in \mathcal{O}_c(W', \mathfrak{h}_{W'})$ is semisimple with all irreducible constituents in orbit of L for the action of $N_W(W')$ on $Irr(H_c(W', \mathfrak{h}_{W'}))$. Let $\overline{\mathcal{O}}_{c,W'}(W, \mathfrak{h})$ and $\overline{\mathcal{O}}_{c,W',L}(W, \mathfrak{h})$ be the respective quotient categories by the kernel of the exact functor $Res_{W'}^W$.*

When the ambient reflection representation $(W, \mathfrak{h})$ is clear, we will write $\mathcal{O}_{c,W'}$ for $\mathcal{O}_{c,W'}(W, \mathfrak{h})$, $\mathcal{O}_{c,W',L}$ for $\mathcal{O}_{c,W',L}(W, \mathfrak{h})$, and similarly for their respective quotients $\overline{\mathcal{O}}_{c,W'}$ and $\overline{\mathcal{O}}_{c,W',L}$.

Proposition 3.2.1.2. *The following hold:*

(1) We have

$$\mathit{Ext}^1_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L', L'') = 0$$

for all L', L'' in the $N_W(W')$-orbit of L.

(2) The categories $\mathcal{O}_{c,W'}$ and $\mathcal{O}_{c,W',L}$ are Serre subcategories of $\mathcal{O}_c(W, \mathfrak{h})$.

(3) Every simple object $S \in \mathcal{O}_c(W, \mathfrak{h})$ lies in such a category $\mathcal{O}_{c,W',L}$ for some parabolic subgroup $W' \subset W$ and finite dimensional irreducible L, and the pair (W', L) is uniquely determined by S up to the natural W-action on such pairs.

(4) If (W', L) labels the simple module S in this way, then $\mathit{Supp}(S) = W\mathfrak{h}^{W'}$.

Proof. For the Ext^1 statement, notice that the restriction of the parameter c to the set of reflection $S' \subset W'$ is not only W'-equivariant but also $N_W(W')$-equivariant. As the Euler grading element $\mathrm{eu}_{W'}$ is fixed by the $N_W(W')$-action, it follows that the lowest $\mathrm{eu}_{W'}$-weight spaces of L and L' have the same weight, and therefore there can be no nontrivial extensions between L and L' by usual highest weight theory (see, e.g. [38, Section 2]). More precisely, let L have lowest weight $\lambda \in \mathrm{Irr}(W')$ and L' have lowest weight $\lambda' = n.\lambda$ for some $n \in N_W(W')$, and consider a module $M \in \mathcal{O}_c(W', \mathfrak{h}_{W'})$ such that $[M] = [L_c(\lambda)] + [L_c(\lambda')]$ in $K_0(\mathcal{O}_c(W', \mathfrak{h}_{W'}))$. It suffices to show that $M \cong L_c(\lambda) \oplus L_c(\lambda')$. As the lowest weight spaces λ of $L_c(\lambda)$ and λ' of $L_c(\lambda')$ occur in the same graded degree with respect to the grading by generalized eigenspaces of $\mathrm{eu}_{W'}$, the lowest weight space of M is isomorphic to $\lambda \oplus \lambda'$ as a representation of W'. It follows that there is an $H_c(W', \mathfrak{h}_{W'})$-module homomorphism $\Delta_c(\lambda) \oplus \Delta_c(\lambda') \to M$ that is an isomorphism in the lowest weight space, and as $[M] = [L_c(\lambda)] + [L_c(\lambda')]$ this map must annihilate the unique maximal proper submodules of each $\Delta_c(\lambda)$ and $\Delta_c(\lambda')$, as their simple constituents have strictly higher lowest weight spaces, and hence factor through an isomorphism $L_c(\lambda) \oplus L_c(\lambda') \to M$, as needed. That $\mathcal{O}_{c,W'}$ and $\mathcal{O}_{c,W',L}$ are Serre subcategories then follows from the exactness of $\mathrm{Res}^W_{W'}$.

Given a simple module $S \in \mathcal{O}_c(W, \mathfrak{h})$, its support is of the form $W\mathfrak{h}^{W'}$ for a parabolic subgroup $W' \subset W$ uniquely determined up to conjugation, and $\mathrm{Res}^W_{W'}S$ is finite dimensional and nonzero for a parabolic subgroup $W' \subset W$ if and only if $\mathrm{Supp}(S) = W\mathfrak{h}^{W'}$, see [5, Proposition 3.2]. So, there exists a finite dimensional simple module $L \in \mathcal{O}_c(W', \mathfrak{h}_{W'})$ and a nonzero homomorphism $L \to \mathrm{Res}^W_{W'} S$. By adjunction, there is therefore a nonzero homomorphism $\mathrm{Ind}^W_{W'} L \to S$, which is a surjection because S is simple. So it suffices to check $\mathrm{Ind}^W_{W'} L \in \mathcal{O}_{c,W',L}$, i.e. that the irreducible constituents of $\mathrm{Res}^W_{W'}\mathrm{Ind}^W_{W'} L$ lie in the $N_W(W')$-orbit of L. This follows from the Mackey formula for rational Cherednik algebras at the level of Grothendieck groups ([66, Lemma 2.5]) and the fact that L is annihilated by any nontrivial parabolic restriction functor. $\square$

Remark 3.2.1.3. *Proposition 3.2.1.2 can also be obtained from [51, Theorem 3.4.6].*

The labeling of simple modules in $\mathcal{O}_c(W, \mathfrak{h})$ by pairs (W', L) as in Proposition 3.2.1.2 is an analogue in the setting of rational Cherednik algebras of the partitioning of irreducible representations of finite groups of Lie type into Harish-Chandra series.

Proposition 3.2.1.4. *The image of $Ind^W_{W'} L$ in the quotient category $\overline{\mathcal{O}}_{c,W',L}$ is a projective generator. Furthermore, the natural map*

$$End_{\mathcal{O}_{c,W',L}}(Ind^W_{W'} L) \to End_{\overline{\mathcal{O}}_{c,W',L}}(Ind^W_{W'} L)$$

is an isomorphism, and in particular there is an equivalence of categories

$$\overline{\mathcal{O}}_{c,W',L} \cong End_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(Ind^W_{W'} L)^{opp}\text{-mod}_{f.d.}.$$

Proof. The statement comparing the endomorphism algebra of $\mathrm{Ind}^W_{W'} L$ in $\mathcal{O}_{c,W',L}$ and in $\overline{\mathcal{O}}_{c,W',L}$ follows from the observation that $\mathrm{Ind}^W_{W'} L$ has no nonzero submodules or quotients annihilated by $\mathrm{Res}^W_{W'}$ and from the definition of morphisms in the quotient category. Otherwise, there would exist a simple module $S \in \mathcal{O}_{c,W',L}$ such that $\mathrm{Res}^W_{W'} S = 0$ but such that either $\mathrm{Hom}_{\mathcal{O}_{c,W',L}}(S, \mathrm{Ind}^W_{W'} L)$ or $\mathrm{Hom}_{\mathcal{O}_{c,W',L}}(\mathrm{Ind}^W_{W'} L, S)$ is nonzero. As $\mathrm{Ind}^W_{W'}$ and $\mathrm{Res}^W_{W'}$ are biadjoint, this is equivalent to one of the hom spaces $\mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(\mathrm{Res}^W_{W'} S, L)$ or $\mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L, \mathrm{Res}^W_{W'} S)$ being nonzero, which contradicts $\mathrm{Res}^W_{W'} S = 0$.

The argument in the last paragraph of the proof of Proposition 3.2.1.2 shows that $\mathrm{Ind}^W_{W'} L$ admits a surjection to any simple module in $\overline{\mathcal{O}}_{c,W',L}$, so the claim that it is a projective generator will follow as soon as we see that it is projective in $\overline{\mathcal{O}}_{c,W',L}$. For this, note that $\mathrm{Res}^W_{W'}$ and $\mathrm{Ind}^W_{W'}$ induce a biadjoint pair of functors between $\overline{\mathcal{O}}_{c,W',L}$ and the Serre subcategory of $\mathcal{O}_c(W', \mathfrak{h}_{W'})$ generated by those simple objects in the $N_W(W')$-orbit of L; indeed, that these functors are biadjoint follows from the biadjunction of $\mathrm{Res}^W_{W'}$ and $\mathrm{Ind}^W_{W'}$ as functors between $\mathcal{O}_c(W, \mathfrak{h})$ and $\mathcal{O}_c(W', \mathfrak{h}_{W'})$ and the facts that every object in $\overline{\mathcal{O}}_{c,W',L}$ is of the form $\pi(M)$, where $\pi : \mathcal{O}_{c,W',L} \to \overline{\mathcal{O}}_{c,W',L}$ is the quotient functor and where $M \in \mathcal{O}_{c,W',L}$ is such that all simple objects in its head and socle have support equal to $W\mathfrak{h}^{W'}$, that $\mathrm{Ind}^W_{W'} M'$ is such an object for every M' in the specified Serre subcategory of $\mathcal{O}_c(W', \mathfrak{h}_{W'})$, and that π is fully faithful on such objects. This Serre subcategory is semisimple by the Ext^1 statement from Proposition 3.2.1.2, and in particular L is projective in that category. Biadjoint functors preserve projectives, so $\mathrm{Ind}^W_{W'} L$ is projective in $\overline{\mathcal{O}}_{c,W',L}$ as needed.

The category $\mathcal{O}_c(W, \mathfrak{h})$ is equivalent to the category of finite dimensional modules of a finite dimensional $\mathbb{C}$-algebra [38, Theorem 5.16]. The same is true for its Serre subquotient $\overline{\mathcal{O}}_{c,W',L}$, which therefore is equivalent to the category of finite dimensional modules over the opposite endomorphism algebra of any projective generator. The last statement follows. $\square$

Through conjugation and the W-action on $\mathfrak{h}$, any $w \in W$ determines an isomorphism $H_c(W', \mathfrak{h}_{W'}) \to H_c({}^wW', \mathfrak{h}_{{}^wW'})$ and therefore also a bijection

$$\mathrm{Irr}(H_c(W', \mathfrak{h}_{W'})) \to \mathrm{Irr}(H_c({}^wW', \mathfrak{h}_{{}^wW'})), \quad M \mapsto {}^wM$$

on the sets of isomorphism classes of irreducible representations by transfer of structure. The group W therefore acts on the set of pairs (W', L), where $W' \subset W$ is a parabolic subgroup and L is a finite-dimensional irreducible representation of $H_c(W', \mathfrak{h}_{W'})$, by $w.(W', L) = ({}^wW', {}^wL)$. Note that $W' \subset W$ stabilizes the pair (W', L). This leads to the following definition:

Definition 3.2.1.5. *The* inertia group $I_W(W', L)$ *of the pair* (W', L) *is the subgroup*

$$I_W(W', L) := \{w \in W : w.(W', L) = (W', L)\}/W' \subset N_W(W')/W'.$$

The following statement follows from Propositions 3.2.1.2 and 3.2.1.4:

Corollary 3.2.1.6. *The number of isomorphism classes of simple modules in the category $\mathcal{O}_c(W, \mathfrak{h})$ labeled by the pair (W', L) in the sense of Proposition 3.2.1.2 is the number of irreducible representations of the $End_{\mathcal{O}_c(W,\mathfrak{h})}(Ind_{W'}^W L)^{opp}$. The number of isomorphism classes of simple modules in $\mathcal{O}_c(W, \mathfrak{h})$ with support variety $W\mathfrak{h}^{W'}$ is given by*

$$\sum_{\substack{L \in Irr(\mathcal{O}_c(W', \mathfrak{h}_{W'})),\\ \dim_{\mathbb{C}} L < \infty}} \frac{\# Irr(End_{\mathcal{O}_c(W,\mathfrak{h})}(Ind_{W'}^W L)^{opp})}{[N_W(W')/W' : I_W(W', L)]}.$$

Proof. Immediate from Propositions 3.2.1.2 and 3.2.1.4 and the definition of the inertia group. □

Note that Corollary 3.2.1.6 is only non-vacuous in the case $W' \neq W$, i.e. for counting isomorphism classes of simple modules with support strictly containing $W\mathfrak{h}^W = \{0\}$, i.e. for counting isomorphism classes of infinite-dimensional simple modules in $\mathcal{O}_c(W, \mathfrak{h})$. But the total number of irreducible representations in $\mathcal{O}_c(W, \mathfrak{h})$ is $\#\mathrm{Irr}(W)$, and in this way one also obtains counts of the finite-dimensional simple modules by subtracting the number of infinite-dimensional simple modules.

The inertia group $I_{W',L}$ will be important to our study of $\mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L)^{opp}$. The following is a basic result about the endomorphism algebra:

Proposition 3.2.1.7. *We have*

$$\dim_{\mathbb{C}} End_{H_c}(Ind_{W'}^W L) = \# I_W(W', L)$$

Proof. By adjunction, we have

$$\mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^W L) \cong \mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L, \mathrm{Res}_{W'}^W \mathrm{Ind}_{W'}^W L).$$

By the Mackey formula for rational Cherednik algebras at the level of Grothendieck groups [66, Lemma 2.5] and the fact that L is finite-dimensional and therefore annihilated by any nontrivial parabolic restriction functor, it follows that the class of $\mathrm{Res}_{W'}^W \mathrm{Ind}_{W'}^W L$ in $K_0(\mathcal{O}_c(W', \mathfrak{h}_{W'}))$ equals $\sum_{n \in N_W(W')/W'} [{}^n L]$. The Ext^1 vanishing statement from Proposition 3.2.1.2 therefore implies that $\mathrm{Res}_{W'}^{W}\mathrm{Ind}_{W'}^{W}L$ is isomorphic to the direct sum $\oplus_{n\in N_W(W')/W'}{}^nL$. We therefore have

$$\mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L, \mathrm{Res}_{W'}^{W}\mathrm{Ind}_{W'}^{W}L) = \mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L, \bigoplus_{n\in N_W(W')/W'} {}^nL).$$

As $\mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L, {}^nL)$ is isomorphic to $\mathbb{C}$ if $n \in I_{W',L}$ and is 0 otherwise, the claim follows. $\square$

With this dimension result in mind, our goal is to give a presentation of the algebra $\mathrm{End}_{H_c}(\mathrm{Ind}_{W'}^{W}L)^{opp}$ as a twisted extension of a Hecke algebra, with a natural basis indexed by $I_W(W', L)$, analogous to the presentation by Howlett and Lehrer [46] of endomorphism algebras of induced cuspidal representations in the representation theory of finite groups of Lie type. A first step towards this goal is to realize $\mathrm{End}_{H_c}(\mathrm{Ind}_{W'}^{W}L)^{opp}$ as a quotient of a twist of the group algebra of the fundamental group of $\mathfrak{h}_{reg}^{W'}/I_W(W', L)$ by a group 2-cocycle. This is achieved in the following section.

3.2.2 The Functor KZ_L

Notation: Throughout this section we will again fix a parabolic subgroup $W' \subset W$ and a finite-dimensional irreducible representation L of $H_c(W', \mathfrak{h}_{W'})$. The only parabolic induction and restrictions functors used will be between rational Cherednik algebras associated to W' and W, so we will omit the superscript W and subscript W' in the notation for the functors $\mathrm{Res}_{W'}^{W}$, $\mathrm{Ind}_{W'}^{W}$, and $\underline{\mathrm{Res}}_{W'}^{W}$. We will suppress the basepoint b in the notation for the fundamental group of $\mathfrak{h}_{reg}^{W'}$ and its quotients, as this basepoint plays no role in this section.

Lemma 3.2.2.1. *The restriction*

$$\underline{Res} : \mathcal{O}_{c,W'} \to (\mathcal{O}_c(W', \mathfrak{h}_{W'}) \boxtimes Loc(\mathfrak{h}_{reg}^{W'}))^{N_W(W')}$$

of the partial KZ functor to the subcategory $\mathcal{O}_{c,W'}$ of $\mathcal{O}_c(W,\mathfrak{h})$ is exact, full, and has image closed under subquotients. The same is true for its restriction to $\mathcal{O}_{c,W',L}$.

Proof. Recall that by definition $\underline{\mathrm{Res}}$ is the composition $RH \circ E \circ \psi_* \circ 1_{\mathfrak{h}_{reg}^{W'}} \circ \widehat{\ }_{W\mathfrak{h}_{reg}^{W'}}$. All the functors involved in this composition after $\widehat{\ }_{W\mathfrak{h}_{reg}^{W'}}$ are equivalences of categories, and in particular it suffices to check the claims of the lemma for the functor $\widehat{\ }_{W\mathfrak{h}_{reg}^{W'}}$. Observe that $W\mathfrak{h}_{reg}^{W'}$ is a principal open subset of $W\mathfrak{h}^{W'}$, defined by the nonvanishing of the polynomial

$$\delta_{W'} := \sum_{w \in W} w \left(\prod_{s \notin S'} \alpha_s. \right).$$

Indeed, for any $w, w' \in W$ with $w\mathfrak{h}^{W'} \neq w'\mathfrak{h}^{W'}$ the nonvanishing of the polynomial $w(\prod_{s \notin S'} \alpha_s)$ defines $w\mathfrak{h}_{reg}^{W'}$ in $w\mathfrak{h}^{W'}$ and $w'(\prod_{s \notin S'} \alpha_s)$ vanishes identically on $w\mathfrak{h}^{W'}$ ($w\mathfrak{h}^{W'}$ is the common vanishing locus of the $w\alpha_s$ for $s \in S'$, and as $w\mathfrak{h}^{W'} \neq w'\mathfrak{h}^{W'}$ it follows that S' - and hence its complement in S - is not stable under conjugation by $w^{-1}w'$, so $w^{-1}w'(\prod_{s \notin S'} \alpha_s)$ vanishes on $\mathfrak{h}^{W'}$ and hence $w'(\prod_{s \notin S'} \alpha_s)$ vanishes on $w\mathfrak{h}^{W'}$). Modules M in $\mathcal{O}_{c,W'}$ are set-theoretically supported on $W\mathfrak{h}^{W'}$, and in particular the completion functor $\widehat{\ }_{W\mathfrak{h}_{reg}^{W'}}$ is simply localization by $\delta_{W'}$.

Define the categories

$$H_c(W,\mathfrak{h})\text{-mod}_{\mathrm{Supp} \subset W\mathfrak{h}^{W'}}, \qquad H_c(W,\mathfrak{h})^{\wedge W\mathfrak{h}_{reg}^{W'}}\text{-mod}_{\mathrm{Supp} \subset W\mathfrak{h}^{W'}}$$

to be the full subcategories of modules over the respective algebras which are annihilated by sufficiently high powers of the ideal of $W\mathfrak{h}^{W'}$. The induction functor

$$\theta_* : H_c(W,\mathfrak{h})\text{-mod}_{\mathrm{Supp} \subset W\mathfrak{h}^{W'}} \to H_c(W,\mathfrak{h})^{\wedge W\mathfrak{h}_{reg}^{W'}}\text{-mod}_{\mathrm{Supp} \subset W\mathfrak{h}^{W'}}$$

given by extension of scalars along the map of algebras $\theta : H_c(W,\mathfrak{h}) \to H_c(W,\mathfrak{h})^{\wedge W\mathfrak{h}_{reg}^{W'}}$ amounts to localization by the function $\delta_{W'}$ defining $W\mathfrak{h}_{reg}^{W'}$ in $W\mathfrak{h}^{W'}$. We also have the associated pullback (restriction) functor θ^*, and as θ_* is localization by a single element we have $\theta_* \circ \theta^*$ is isomorphic to the identity. It follows that θ_* induces an equivalence of categories

$$\frac{H_c(W,\mathfrak{h})\text{-mod}_{\mathrm{Supp}\subset W\mathfrak{h}^{W'}}}{\ker\theta_*} \to H_c(W,\mathfrak{h})^{\wedge W\mathfrak{h}^{W'}_{reg}}\text{-mod}_{\mathrm{Supp}\subset W\mathfrak{h}^{W'}}.$$

As $\mathcal{O}_{c,W'}$ is a Serre subcategory of $H_c(W,\mathfrak{h})\text{-mod}_{\mathrm{Supp}\subset W\mathfrak{h}^{W'}}$ and $\ker\mathrm{Res} = (\ker\theta_*) \cap \mathcal{O}_{c,W'}$, the canonical functor

$$\overline{\mathcal{O}}_{c,W'} \to \frac{H_c(W,\mathfrak{h})\text{-mod}_{\mathrm{Supp}\subset W\mathfrak{h}^{W'}}}{\ker\theta_*}$$

is an inclusion of a full subcategory closed under subquotients, and the result follows. For identical reasons, the result holds for the restriction of $\underline{\mathrm{Res}}$ to the Serre subcategories $\mathcal{O}_{c,W',L}$. $\square$

It will be convenient for us to take a semidirect product splitting of the normalizer $N_W(W')$ as given by the following lemma:

Lemma 3.2.2.2. ([60]) *There is a subgroup $N_{W'} \subset N_W(W')$ such that there is a semidirect product decomposition $N_W(W') = W' \rtimes N_{W'}$.*

Fix a complement $N_{W'}$ to W' in $N_W(W')$ as in Lemma 3.2.2.2. Let $I_{W',L} \subset N_{W'}$ denote the stabilizer of L under the action of $N_{W'}$ on $\mathrm{Irr}(H_c(W',\mathfrak{h}_{W'}))$. The natural map $I_{W',L} \to I_W(W',L)$ is an isomorphism.

Notation: Throughout this section, in which W' and L are fixed, we will simplify the notation by denoting $I_{W',L}$ by I.

Given a set $\mathcal{L}$ of simple objects in $\mathcal{O}_c(W',\mathfrak{h}_{W'})$, let $\langle\mathcal{L}\rangle_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}$ denote the Serre subcategory of $\mathcal{O}_c(W',\mathfrak{h}_{W'})$ generated by $\mathcal{L}$. Note that the functor $\underline{\mathrm{Res}} : \mathcal{O}_{c,W',L} \to (\mathcal{O}_c(W',\mathfrak{h}_{W'}) \boxtimes \mathrm{Loc}(\mathfrak{h}^{W'}_{reg}))^{N_W(W')}$ factors through the Serre subcategory

$$\Big(\langle N_W(W').L\rangle_{\mathcal{O}_c(W',\mathfrak{h}_{W'})} \boxtimes \mathrm{Loc}(\mathfrak{h}^{W'}_{reg})\Big)^{N_W(W')} \subset \Big(\mathcal{O}_c(W',\mathfrak{h}_{W'}) \boxtimes \mathrm{Loc}(\mathfrak{h}^{W'}_{reg})\Big)^{N_W(W')}.$$

Let $H_{c,L}(W',\mathfrak{h}_{W'})$ be the quotient

$$H_{c,L}(W',\mathfrak{h}_{W'}) := H_c(W',\mathfrak{h}_{W'})/\cap_{n\in N_W(W')} \mathrm{Ann}_{H_c(W',\mathfrak{h}_{W'})}({}^nL).$$

Note that $H_{c,L}(W', \mathfrak{h}_{W'})$ is a semisimple finite-dimensional $\mathbb{C}$-algebra with $N_W(W')$-action and that the category $\langle N_W(W').L\rangle_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}$ is naturally equivalent to the category $H_{c,L}(W', \mathfrak{h}_{W'})$-mod$_{f.d.}$. In particular, choosing a point $b \in \mathfrak{h}_{reg}^{W'}$, taking the fiber of the local system at b defines an equivalence of categories

$$\mathrm{Fiber}_b : \left(\langle N_W(W').L\rangle_{\mathcal{O}_c(W',\mathfrak{h}_{W'})} \boxtimes \mathrm{Loc}(\mathfrak{h}_{reg}^{W'})\right)^{N_W(W')}$$

$$\to H_{c,L}(W', \mathfrak{h}_{W'}) \rtimes \pi_1(\mathfrak{h}_{reg}^{W'}/N_{W'})\text{-mod}_{f.d}$$

where the semidirect product is defined by the natural action of $\pi_1(\mathfrak{h}_{reg}^{W'}/N_{W'})$ on $H_{c,L}(W', \mathfrak{h}_{W'})$ through the natural projection $\pi_1(\mathfrak{h}_{reg}^{W'}/N_{W'}) \to N_{W'}$. Let e_L denote the central idempotent associated to the irreducible representation L of $H_{c,L}(W', \mathfrak{h}_{W'})$. Clearly e_L generates the unit ideal in $H_{c,L}(W', \mathfrak{h}_{W'}) \rtimes \pi_1(\mathfrak{h}_{reg}^{W'}/N_{W'})$ and the associated spherical subalgebra is given by

$$e_L(H_{c,L}(W', \mathfrak{h}_{W'}) \rtimes \pi_1(\mathfrak{h}_{reg}^{W'}/N_{W'}))e_L \cong \mathrm{End}_{\mathbb{C}}(L) \rtimes \pi_1(\mathfrak{h}_{reg}^{W'}/I).$$

In particular, multiplication by e_L defines an equivalence of categories

$$e_L : H_{c,L}(W', \mathfrak{h}_{W'}) \rtimes \pi_1(\mathfrak{h}_{reg}^{W'}/N_{W'})\text{-mod}_{f.d} \to \mathrm{End}_{\mathbb{C}}(L) \rtimes \pi_1(\mathfrak{h}_{reg}^{W'}/I)\text{-mod}_{f.d}.$$

The automorphism group of $\mathrm{End}_{\mathbb{C}}(L)$ is $PGL_{\mathbb{C}}(L)$, the projective general linear group of L, and in particular the action of I on $\mathrm{End}(L)$ defines a projective representation π_L of I on L. Let $\mu \in Z^2(I, \mathbb{C}^\times)$ be a group 2-cocycle of I with coefficients in $\mathbb{C}^\times$ representing the cohomology class associated to π_L. Let $\tilde{\mu} \in Z^2(\pi_1(\mathfrak{h}_{reg}^{W'}/I), \mathbb{C}^\times)$ be the pullback of μ along the natural projection $\pi_1(\mathfrak{h}_{reg}^{W'}/I) \to I$. Let $-\tilde{\mu}$ denote the opposite group cocycle defined by $(-\tilde{\mu})(g) = \tilde{\mu}(g)^{-1}$, and for any group 2-cocycle $\nu \in Z^2(\pi_1(\mathfrak{h}_{reg}^{W'}/I), \mathbb{C}^\times)$ let $\mathbb{C}_\nu[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]$ denote the ν-twisted group algebra of $\pi_1(\mathfrak{h}_{reg}^{W'}/I)$, i.e. the associative unital $\mathbb{C}$-algebra with $\mathbb{C}$-basis $\{e_g : g \in \pi_1(\mathfrak{h}_{reg}^{W'}/I)\}$ and multiplication $e_g e_{g'} = \nu(g, g')e_{gg'}$. Essentially by definition of μ we see that L is naturally a $\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]$-module, and in particular we have a functor

$$\mathrm{Hom}_{\mathrm{End}_{\mathbb{C}}(L)}(L, \bullet) : \mathrm{End}_{\mathbb{C}}(L) \rtimes \pi_1(\mathfrak{h}_{reg}^{W'}/I)\text{-mod}_{f.d.} \to \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]\text{-mod}_{f.d.}.$$

This functor is an equivalence of categories, with quasi-inverse $N \mapsto L \otimes_{\mathbb{C}} N$.

Remark 3.2.2.3. *Let $\widetilde{\pi}_L : I \to GL_{\mathbb{C}}(L)$ be any lift of the projective representation π_L. For any $n \in I$, the operator $\widetilde{\pi}_L(n)$ gives an isomorphism of $H_c(W', \mathfrak{h}_{W'})$-modules of L with its twist ${}^nL$ and in particular preserves and is determined by its action on the lowest $eu_{W'}$-weight space L^0 of L. It follows that π_L induces a projective representation π_{L^0} of I on L^0 with respect to which L^0 is (projectively) equivariant as a representation of W'. Furthermore, the projective representation π_L lifts to a linear representation of I, and in particular the cocycle μ is trivial, if and only if the same holds for the projective representation π_{L^0}. Indeed, given a representation $\widetilde{\pi}_{L^0}$ of I on L^0 with respect to which L^0 is an equivariant representation of W', this representation extends uniquely to a representation of I on all of L with respect to which L is I-equivariant as an $H_c(W', \mathfrak{h}_{W'})$-module. It follows that triviality of the cocycle μ can be checked at the level of the representation of W' in L^0. For example, the cocycle μ is trivial in the large class of examples in which W' splits as a direct product $W' = W_1' \times W_2'$ with respect to which L^0 is isomorphic to a tensor product $L_1^0 \otimes L_2^0$ and such that both I centralizes W_1' and also L_2^0 is the trivial representation.*

From the above discussion, we have:

Theorem 3.2.2.4. *The Hom functor $Hom_{\mathcal{O}_c(W,\mathfrak{h})}(Ind(L), \bullet)$ on $\mathcal{O}_{c,W',L}$ factors as the composition of a functor*

$$\widetilde{KZ_L} : \mathcal{O}_{c,W',L} \to \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]\text{-}mod_{f.d.}$$

followed by the forgetful functor to the category of finite-dimensional vector spaces. $\widetilde{KZ_L}$ is exact, full, has image closed under subquotients, and induces a fully faithful embedding

$$\overline{\mathcal{O}}_{c,W',L} \hookrightarrow \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]\text{-}mod_{f.d.}$$

with image closed under subquotients. In particular, there is a surjection of $\mathbb{C}$-algebras

$$\varphi_L : \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)] \to End_{\mathcal{O}_c(W,\mathfrak{h})}(Ind(L))^{opp}.$$

Proof. By adjunction, we have $\mathrm{Hom}_{\mathcal{O}_c(W,\mathfrak{h})}(\mathrm{Ind}(L), \bullet) \cong \mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L, \mathrm{Res}\bullet)$, and this latter functor can be expressed as the composition $\mathrm{Hom}_{\mathrm{End}_{\mathbb{C}}(L)}(L, \bullet) \circ e_L \circ \mathrm{Fiber}_b \circ \underline{\mathrm{Res}}$ of functors discussed above, from which the existence of the lift $\widetilde{KZ_L}$ follows. That $\widetilde{KZ_L}$ is exact, full, and has image closed under subquotients follows from Lemma 3.2.2.1 and the fact that the functors appearing after $\underline{\mathrm{Res}}$ in the composition above are equivalences of categories. That the induced functor on $\overline{\mathcal{O}}_{c,W',L}$ is faithful follows from the fact that it is represented by the projective generator $\mathrm{Ind}(L)$. By Proposition 3.2.1.4, this induced functor is identified with a fully faithful embedding

$$\mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(\mathrm{Ind}(L))^{opp}\text{-mod}_{f.d.} \to \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]\text{-mod}_{f.d.}$$

with image closed under subquotients and which is identity at the level of $\mathbb{C}$-vector spaces. Such a functor is simply restriction along some algebra homomorphism $\varphi_L : \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)] \to \mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(\mathrm{Ind}(L))^{opp}$. The image im φ_L is a $\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]$-submodule of $\mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(\mathrm{Ind}(L))^{opp}$, and because the functor in question has image closed under subquotients it follows that im φ_L is also a $\mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(\mathrm{Ind}(L))^{opp}$-submodule. As im φ_L contains 1, it follows that φ_L is surjective, as needed. □

Definition 3.2.2.5. *Let the* generalized Hecke algebra $\mathcal{H}(c, W', L, W)$ *attached to* c, W', L *and the group* W *be the quotient algebra*

$$\mathcal{H}(c, W', L, W) := \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]/\ker\varphi_L.$$

Let

$$KZ_L : \mathcal{O}_{c,W',L} \to \mathcal{H}(c, W', L, W)\text{-}mod_{f.d.}$$

be the natural factorization of the functor $\widetilde{KZ_L}$ *from Theorem 3.2.2.4 through the category* $\mathcal{H}(c, W', L, W)$*-mod*$_{f.d.}$*.*

Corollary 3.2.2.6. *KZ_L induces an equivalence of categories*

$$\overline{\mathcal{O}}_{c,W',L} \cong \mathcal{H}(c, W', L, W)\text{-}mod_{f.d.}.$$

Remark 3.2.2.7. *In the case $W' = 1$ and $L = \mathbb{C}$, the generalized Hecke algebra $\mathcal{H}(c, W', L, W)$ is precisely the Hecke algebra $\mathsf{H}_q(W)$ attached to the complex reflection group W, the functor $KZ_{\mathbb{C}}$ is precisely the KZ functor of [38], and the statements of Theorem 3.2.2.4 are well-known in that case. The equivalence in Corollary 3.2.2.6 in that case is the well-known equivalence $\mathcal{O}_c/\mathcal{O}_c^{tor} \cong \mathsf{H}_q(W)$-$mod_{f.d.}$.*

The following remark reduces the study of the algebras $\mathcal{H}(c, W', L, W)$ to the case in which W is an irreducible complex reflection group.

Remark 3.2.2.8. *Suppose the complex reflection group W and its reflection representation decompose as a product $(W, \mathfrak{h}) = (W_1 \times W_2, \mathfrak{h}_{W_1} \oplus \mathfrak{h}_{W_2})$. Let $S_i \subset W_i$ be the set of reflections in W_i for $i = 1, 2$, so that $S = S_1 \sqcup S_2$, and let c_i be the restriction of the parameter c to S_i. The rational Cherednik algebra $H_c(W, \mathfrak{h})$ decomposes naturally as the tensor product $H_{c_1}(W_1, \mathfrak{h}_{W_1}) \otimes H_{c_2}(W_2, \mathfrak{h}_{W_2})$. Let $W' \subset W$ be a parabolic subgroup, and for $i = 1, 2$ let $W_i' \subset W_i$ be the parabolic subgroups such that $W' = W_1' \times W_2'$. Let L_i be a finite-dimensional irreducible representation of $H_{c_i}(W_i', \mathfrak{h}_{W_i'})$ for $i = 1, 2$, so that $L = L_1 \otimes L_2$ is a finite-dimensional irreducible representation of $H_c(W_1' \times W_2', \mathfrak{h}_{W_1' \times W_2'}) = H_{c_1}(W_1', \mathfrak{h}_{W_1'}) \otimes H_{c_2}(W_2', \mathfrak{h}_{W_2'})$. Every finite-dimensional irreducible representation of $H_c(W_1' \times W_2', \mathfrak{h}_{W_1'} \times \mathfrak{h}_{W_2'})$ appears in this way, and all of the constructions appearing in Theorem 3.2.2.4 split naturally as well. In particular, there is a natural isomorphism of algebras*

$$\mathcal{H}(c, W', L, W) \cong \mathcal{H}(c_1, W_1', L_1, W_1) \otimes \mathcal{H}(c_2, W_2', L_2, W_2)$$

compatible with the surjections $\varphi_{L_1}, \varphi_{L_2}$ and $\varphi_{L_1 \otimes L_2}$.

3.2.3 Eigenvalues of Monodromy and Relations From Corank 1

We want to describe the generalized Hecke algebras $\mathcal{H}(c, W', L, W)$ as explicitly as possible so that we can in turn understand the subquotient categories $\overline{\mathcal{O}}_{c,W',L}$. To achieve this, we study the kernel $\ker \varphi_L$.

Let $W'' \subset W$ be another parabolic subgroup, with $W' \subset W'' \subset W$. Let $I_{W',L,W''} := I_{W',L} \cap W''$ be the (lift to $N_{W''}(W)$ via the splitting $N_W(W') = W' \rtimes N_{W'}$ of the) inertia group of L in $N_{W''}(W')$. The 2-cocycle $\mu \in Z^2(I_{W',L}, \mathbb{C}^\times)$ restricts to give a 2-cocycle of $I_{W',L,W''}$, which we will also denote by μ, and by pullback determines a 2-cocycle of $\pi_1((\mathfrak{h}_{W''})^{W'}_{reg}/I_{W',L,W''})$, which we will also denote by $\tilde{\mu}$. We may choose $b \in \mathfrak{h}^{W'}_{reg}$ such that the projection $b_{W''}$ of b to $(\mathfrak{h}_{W''})^{W'}$ with respect to the vector space decomposition $\mathfrak{h}^{W'} = (\mathfrak{h}_{W''})^{W'} \oplus \mathfrak{h}^{W''}$ lies in $(\mathfrak{h}_{W''})^{W'}_{reg}$. By Theorem 3.2.2.4, there is a surjection of $\mathbb{C}$-algebras

$$\varphi_{L,W''} : \mathbb{C}_{-\tilde{\mu}}[\pi_1((\mathfrak{h}_{W''})^{W'}_{reg}/I_{W',L,W''}, b_{W''})] \to \mathrm{End}_{\mathcal{O}_c(W'',\mathfrak{h}_{W''})}(\mathrm{Ind}_{W'}^{W''} L)^{opp}.$$

Notation: Throughout this section, the parabolic subgroup W' and finite dimensional irreducible representation L of $H_c(W', \mathfrak{h}_{W'})$ will remain fixed, and the parabolic subgroup W'' will vary. We will denote $I_{W',L}$ by I, as in the previous section, and we will denote $I_{W',L,W''}$ by $I_{W''}$. We will always take the basepoints for the fundamental groups of $\mathfrak{h}^{W'}_{reg}$ and $(\mathfrak{h}_{W''})^{W'}_{reg}$ and their quotients to be b and $b_{W''}$ as above, and we will suppress these from the notation for readability.

By the construction of the functor $\iota^*_{W'',W'}$ of Gordon and Martino recalled in Section 2.6 and the fact that $I_{W''}$ acts trivially on $\mathfrak{h}^{W''}_{reg}$, there is a natural map of fundamental groups

$$\iota_{W'',W',L} : \pi_1((\mathfrak{h}_{W''})^{W'}_{reg}/I_{W''}) \to \pi_1(\mathfrak{h}^{W'}_{reg}/I),$$

extending to a natural map of $\mathbb{C}$-algebras

$$\iota_{W'',W',L} : \mathbb{C}_{-\tilde{\mu}}[\pi_1((\mathfrak{h}_{W''})^{W'}_{reg}/I_{W''})] \to \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}^{W'}_{reg}/I)].$$

By functoriality, there is also an algebra homomorphism

$$\mathrm{End}_{\mathcal{O}_c(W'',\mathfrak{h}_{W''})}(\mathrm{Ind}_{W'}^{W''} L)^{opp} \to \mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(\mathrm{Ind}_{W'}^{W} L)^{opp}$$

induced by the functor $\mathrm{Ind}_{W''}^{W}$ and the isomorphism $\mathrm{Ind}_{W''}^{W} \circ \mathrm{Ind}_{W'}^{W''} \cong \mathrm{Ind}_{W'}^{W}$. The following lemma is then immediate from the constructions and from Gordon and Martino's transitivity result that was recalled in Theorem 2.6.0.1:

Lemma 3.2.3.1. *The following diagram commutes:*

$$\begin{tikzcd}
\mathbb{C}_{-\tilde{\mu}}[\pi_1((\mathfrak{h}_{W''})^{W'}_{reg}/I_{W''})] \arrow[r, "\iota_{W'',W',L}"] \arrow[d, "\varphi_{L,W''}"'] & \mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}^{W'}_{reg}/I)] \arrow[d, "\varphi_L"'] \\
End_{\mathcal{O}_c(W'',\mathfrak{h}_{W''})}(Ind_{W'}^{W''} L)^{opp} \arrow[r, "Ind_{W''}^{W}"] & End_{\mathcal{O}_c(W,\mathfrak{h})}(Ind_{W'}^{W} L)^{opp}
\end{tikzcd}$$

Definition 3.2.3.2. *Given a chain of parabolic subgroups $W' \subset W'' \subset W$ as above, we say that W' is of* corank r *in W'' if* $\dim \mathfrak{h}_{W''} = \dim \mathfrak{h}_{W'} + r$.

The parabolic subgroups $W'' \subset W$ containing W' in corank 1 are closely related to the reflections appearing in the action of $N_W(W')$ on $\mathfrak{h}^{W'}$:

Lemma 3.2.3.3. *Let $W'' \subset W$ be a parabolic subgroup containing W' in corank 1. The quotient $N_{W''}(W')/W'$ is cyclic and acts on $\mathfrak{h}^{W'}$ by complex reflections through the hyperplane $\mathfrak{h}^{W''} \subset \mathfrak{h}^{W'}$. Furthermore, every element $n \in N_W(W')$ that acts on $\mathfrak{h}^{W'}$ as a complex reflection lies in some parabolic subgroup $W'' \subset W$ containing W' in corank 1.*

Proof. The action of $N_{W''}(W')/W'$ on $\mathfrak{h}^{W'}$ is faithful because W' is precisely the pointwise stabilizer of $\mathfrak{h}^{W'}$ in W, and the action respects the decomposition $\mathfrak{h}^{W'} = (\mathfrak{h}_{W''})^{W'} \oplus \mathfrak{h}^{W''}$. As $N_{W''}(W')/W'$ acts trivially on $\mathfrak{h}^{W''}$ and $\dim(\mathfrak{h}_{W''})^{W'} = 1$, the first claim follows.

For the second claim, suppose $n \in N_W(W')$ acts on $\mathfrak{h}^{W'}$ as a complex reflection through the hyperplane $H \subset \mathfrak{h}^{W'}$. Let $W'' \subset W$ be the point-wise stabilizer of H in W, a parabolic subgroup. As n does not fix $\mathfrak{h}^{W'}$ but $W' \subset W''$, we have $H \subset \mathfrak{h}^{W''} \subsetneq \mathfrak{h}^{W'}$. As H is a hyperplane, it follows that $H = \mathfrak{h}^{W''}$, so W'' is a parabolic subgroup containing W' in corank 1 and $n \in N_{W''}(W)$ acts on $\mathfrak{h}^{W'}$ as a complex reflection through the hyperplane $\mathfrak{h}^{W''} \subset \mathfrak{h}^{W'}$, as needed. $\square$

It follows from Lemma 3.2.3.3 that the inertia group $I_{W''}$ considered above is cyclic and acts on $(\mathfrak{h}_{W''})^{W'}_{reg}$ through a faithful character. As W' is corank 1 in W'', we have $(\mathfrak{h}_{W''})^{W'}_{reg} \cong \mathbb{C}^\times$ as a $\mathbb{C}$-manifold, and in particular there is a canonical group isomorphism

$$\pi_1((\mathfrak{h}_{W''})^{W'}_{reg}/I_{W''}) \cong \mathbb{Z}.$$

Definition 3.2.3.4. *Let $T_{W',L,W''}$ denote the canonical generator of $\pi_1((\mathfrak{h}_{W''})^{W'}_{reg}/I_{W''})$ arising from the isomorphism above. We will also let $T_{W',L,W''}$ denote its image in $\pi_1(\mathfrak{h}^{W'}_{reg}/I)$ under the homomorphism $\iota_{W'',W',L}$, and we refer to $T_{W',L,W''}$ as the* generator of monodromy *about the hyperplane $\mathfrak{h}^{W''} \subset \mathfrak{h}^{W'}$. We call the parabolic subgroup W'' and its associated hyperplane $\mathfrak{h}^{W''} \subset \mathfrak{h}^{W'}$* L-trivial *(resp.,* L-essential*) when the inertia group $I_{W''}$ is trivial (resp., nontrivial). Let $\mathfrak{h}^{W'}_{L-reg}$ denote the complement of the arrangement of L-essential hyperplanes in $\mathfrak{h}^{W'}$, and let I^{ref} denote the subgroup of I generated by the subgroups $I_{W''}$ for all L-essential W''.*

We comment that I^{ref} is a normal subgroup of I and that the action of I^{ref} on $\mathfrak{h}^{W'}$ is generated by complex reflections through the L-essential hyperplanes. In fact, from Lemma 3.2.3.3 we see that I^{ref} is the maximal reflection subgroup of I with respect to its representation in $\mathfrak{h}^{W'}$. We have $\mathfrak{h}^{W'}_{reg} \subset \mathfrak{h}^{W'}_{L-reg}$, and $\mathfrak{h}^{W'}_{L-reg}$ is stable under the action of I on $\mathfrak{h}^{W'}$.

The second cohomology group $H^2(C, \mathbb{C}^\times)$ vanishes for all cyclic groups C, and in particular we may assume that the 2-cocycle μ on I is trivial on all subgroups $I_{W''} \subset I$ associated to parabolic subgroups $W'' \subset W$ containing W' in corank 1. Then, the twisted group algebra $\mathbb{C}_{-\tilde{\mu}}[\pi_1((\mathfrak{h}_{W''})^{W'}_{reg}/I_{W''})]$ is naturally identified with the Laurent polynomial ring $\mathbb{C}[T^{\pm 1}_{W',L,W''}]$. The kernel of the map $\varphi_{L,W''}$ appearing in the commutative diagram in Lemma 3.2.3.1 is therefore generated by a monic polynomial $P_{W',L,W''} \in \mathbb{C}[T_{W',L,W''}]$ nonvanishing at 0. It follows from Proposition 3.2.1.7 that $P_{W',L,W''}$ is a polynomial of degree $\#I_{W''}$.

Definition 3.2.3.5. *Let $P_{W',L,W''} \in \mathbb{C}[T]$ be the monic polynomial of degree $\#I_{W''}$ generating the kernel* $\ker \varphi_{W',L,W''}$.

Note that $P_{W',L,W''}$ only depends on W'' up to I-conjugacy and that $P_{W',L,W''}(0) \neq 0$.

Notation: As W' and L remain fixed, we will denote $P_{W',L,W''}$ by $P_{W''}$ and $T_{W',L,W''}$ by $T_{W''}$ when the meaning is clear.

We can now state the main result of this section.

Theorem 3.2.3.6. *The map φ_L of Theorem 3.2.2.4 factors through a surjection*

$$\overline{\varphi}_L : \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]}{\langle P_{W''}(T_{W''}) : \ W' \textit{ is corank 1 in } W'' \rangle} \to \mathcal{H}(c, W', L, W)$$

of $\mathbb{C}$-algebras. If the inequality

$$\dim \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{L-reg}^{W'}/I^{ref})]}{\langle P_{W''}(T_{W''}) : W'' \textit{ is } L\textit{-essential} \rangle} \leq \#I^{ref}$$

holds then it is an equality and $\overline{\varphi}_L$ is an isomorphism.

In particular, if W is a Coxeter group or if the group 2-cocycle $\mu \in Z^2(I, \mathbb{C}^\times)$ has cohomologically trivial restriction to I^{ref}, $\overline{\varphi}_L$ is an isomorphism.

Remark 3.2.3.7. *Note that only the L-essential hyperplanes feature in dimension bound above. The 2-cocycle $\tilde{\mu} \in Z^2(\pi_1(\mathfrak{h}_{L-reg}^{W'}/I^{ref}), \mathbb{C}^\times)$ is obtained by pulling back the 2-cocycle $\mu \in Z^2(I, \mathbb{C}^\times)$ along the natural projection. Note that by their definition, the generators of monodromy $T_{W''}$ are naturally elements of $\pi_1(\mathfrak{h}_{reg}^{W'}/I^{ref})$.*

The following lemma will be useful in the proof of Theorem 3.2.3.6

Lemma 3.2.3.8. *There is a linear character $\chi : \pi_1(\mathfrak{h}^{W'}_{reg}/I) \to \mathbb{C}^\times$ satisfying*

$$\chi(T_{W''}) = \begin{cases} -P_{W''}(0) & \text{if } W'' \text{ is } L\text{-trivial} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. As $P_{W''}(0)$ is nonzero and only depends on W'' up to I-conjugacy, it suffices to show that for any parabolic subgroup W'' containing W' in corank 1 there exists a group homomorphism $\theta : \pi_1(\mathfrak{h}^{W'}_{reg}/I) \to \mathbb{Z}$ such that a generator of monodromy T_H about one of the hyperplanes H defining $\mathfrak{h}^{W'}_{reg}/I$ is sent to 1 under θ if and only if H is I-conjugate to $\mathfrak{h}^{W''}$ and 0 otherwise. One may then compose with the group homomorphism $\mathbb{Z} \to \mathbb{C}^\times$, $1 \mapsto -P_{W''}(0)$, and take the product of such maps over all I-conjugacy classes of L-trivial parabolic subgroups $W'' \supset W'$. To construct such a homomorphism, choose a linear functional $\alpha \in (\mathfrak{h}^{W'})^*$ defining $\mathfrak{h}^{W''}$ in $\mathfrak{h}^{W'}$ and let $\delta = \prod_{n \in I} n\alpha$. Then, δ defines a continuous function $\delta : \mathfrak{h}^{W'}_{reg}/I \to \mathbb{C}^\times$ with associated map $\pi_1(\delta) : \pi_1(\mathfrak{h}^{W'}_{reg}/I) \to \pi_1(\mathbb{C}^\times) = \mathbb{Z}$. That $\pi_1(\delta)$ has the desired effect on the generators of monodromy then follows easily as in the proof of [7, Proposition 2.16]. □

Proof of Theorem 3.2.3.6. That φ_L factors as $\overline{\varphi}_L$ through the quotient above follows immediately from Lemma 3.2.3.1.

Assume the dimension inequality in the theorem statement holds. As the action of I^{ref} on $\mathfrak{h}^{W'}$ is generated by reflections, the quotient $\mathfrak{h}^{W'}/I^{ref}$ is a smooth $\mathbb{C}$-variety by the Chevalley-Shephard-Todd theorem [11, 69]. In particular, by Proposition A.1 of [7] and induction it follows that the kernel of the map $\pi_1(\mathfrak{h}^{W'}_{reg}/I^{ref}) \to \pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})$ is generated (as a normal subgroup) by the generators of monodromy $T_{W''}$ for the L-trivial W''. In particular, we have an isomorphism

$$\frac{\pi_1(\mathfrak{h}^{W'}_{reg}/I^{ref})}{\langle T_{W''} : W'' \text{ is } L\text{-trivial}\rangle} \cong \pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref}).$$

Clearly, this isomorphism respects the cocycle $\tilde{\mu}$. Let $\chi : \pi_1(\mathfrak{h}^{W'}_{reg}/I^{ref}) \to \mathbb{C}^\times$ be the composition of the natural map $\pi_1(\mathfrak{h}^{W'}_{reg}/I^{ref}) \to \pi_1(\mathfrak{h}^{W'}_{reg}/I)$ with the character from

Lemma 3.2.3.8. The assignments $g \mapsto \chi(g)g$ for $g \in \pi_1(\mathfrak{h}_{reg}^{W'}/I)$ extend to an algebra automorphism τ_χ of $\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]$ satisfying

$$\tau_\chi(P_{W''}(T_{W''})) = \begin{cases} -P_{W''}(0)(T_{W''}-1) & \text{if } W'' \text{ is } L\text{-trivial} \\ P_{W''}(T_{W''}) & \text{otherwise.} \end{cases}.$$

To see this formula in the case that W'' is L-trivial, recall that in that case $P_{W''}(T_{W''})$ is a monic linear polynomial, hence of the form $P_{W''}(T_{W''}) = T_{W''} + P_{W''}(0)$; applying τ_χ gives

$$\tau_\chi(P_{W''}(T_{W''})) = \tau_\chi(T_{W''} + P_{W''}(0))$$

$$= -P_{W''}(0)T_{W''} + P_{W''}(0) = -P_{W''}(0)(T_{W''}-1),$$

as above. The formula in the case that W'' is L-essential is clear. Composing these two isomorphisms yields an isomorphism

$$\frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{L-reg}^{W'}/I^{ref})]}{\langle P_{W''}(T_{W''}) : W'' \text{ is } L\text{-essential}\rangle} \cong \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I^{ref})]}{\langle P_{W''}(T_{W''}) :\ W' \text{ is corank 1 in } W''\rangle}.$$

By assumption, this algebra has dimension at most $\#I^{ref}$, and so the same holds for the dimension $d \leq \#I^{ref}$ of its image in

$$\frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]}{\langle P_{W''}(T_{W''}) :\ W' \text{ is corank 1 in } W''\rangle}$$

under the natural map induced by the map $\pi_1(\mathfrak{h}_{reg}^{W'}/I^{ref}) \to \pi_1(\mathfrak{h}_{reg}^{W'}/I)$ of fundamental groups. As $\pi_1(\mathfrak{h}_{reg}^{W'}/I)/\pi_1(\mathfrak{h}_{reg}^{W'}/I^{ref}) \cong I/I^{ref}$, it follows that

$$\dim \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]}{\langle P_{W''}(T_{W''}) :\ W' \text{ is corank 1 in } W''\rangle} \leq d \cdot (\#I/I^{ref}) \leq \#I.$$

But, as $\overline{\varphi}_L$ is a surjection and $\dim \mathcal{H}(c, W', L, W) = \#I$ by Proposition 3.2.1.7, it follows also that

$$\#I \leq \dim \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]}{\langle P_{W''}(T_{W''}) :\ W' \text{ is corank 1 in } W''\rangle}.$$

It follows that the dimension inequalities above are equalities and that $\overline{\varphi_L}$ is an isomorphism.

For the final statement of the theorem, first suppose the restriction of μ to I^{ref} is cohomologically trivial. Then the quotient

$$\frac{\mathbb{C}[\pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})]}{\langle P_{W''}(T_{W''}) : W'' \text{ is } L\text{-essential}\rangle}$$

is precisely a specialization to $\mathbb{C}$ of the Hecke algebra attached to the complex reflection group I^{ref} in the sense of Broué-Malle-Rouquier [7]. In that paper it was conjectured that the generic Hecke algebra is a free module of rank $\#I^{ref}$ over its ring of parameters, and this conjecture was subsequently proved in characteristic zero [27]. In particular, this algebra has dimension $\#I^{ref}$, as needed.

Now suppose W is a Coxeter group. In that case, I^{ref} is a Coxeter group as well, and its action on $\mathfrak{h}^{W'}$ is the complexification of a real reflection representation. We need to show that the algebra

$$\frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})]}{\langle P_{W''}(T_{W''}) : W'' \text{ is } L\text{-essential}\rangle}$$

has dimension at most, and hence equal to, $\#I^{ref}$. In this case, the fundamental group $\pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})$ is the Artin braid group $B_{I^{ref}}$ attached to the Coxeter group I^{ref}. If $S \subset I^{ref}$ is a set of simple reflections for this Coxeter group, the group $B_{I^{ref}}$ is generated by the generators of monodromy $\{e_s : s \in S\}$ about the hyperplanes associated to the simple reflections, with braid relations as recalled in Section 2.7. Given a finite list $\mathbf{s} = (\mathbf{s_1}, ..., \mathbf{s_n})$ of simple reflections, let $e_{\mathbf{s}}$ denote the product $e_{s_1} \cdots e_{s_n}$. By Matsumoto's Theorem (see [37, Theorem 1.2.2]), for any $w \in I^{ref}$ the element $e_{\mathbf{s}}$ does not depend on the choice of reduced expression $w = s_1 \dots s_n$, and we denote as usual the element $e_{\mathbf{s}}$ by e_w in this case. For a list $\mathbf{s}$ let $T_{\mathbf{s}}$ denote the element $e_{\mathbf{s}}$ viewed as an element of the quotient algebra above, and similarly for T_w. To show that the algebra has dimension at most $\#I^{ref}$, it suffices to check that the $\mathbb{C}$-span of the elements $\{T_w : w \in I^{ref}\}$ is closed under multiplication by

T_s for any $s \in S$. If $ss_1 \cdots s_n$ is a reduced expression with $w = s_1 \cdots s_n$ a reduced expression for w, then $T_sT_w = \mu(s,w)^{-1}T_{sw}$. Otherwise, we may choose a reduced expression $w = s_1 \cdots s_n$ for w with $s_1 = s$. The hyperplane associated to s is L-essential and $\#I_{W''} = 2$ for all L-essential W'' in this case, so T_s satisfies a quadratic relation $T_s^2 = a + bT_s$. We then have $T_sT_w = \mu(s,sw)T_s^2T_{sw} = \mu(s,sw)(a+be_s)T_{sw} = \mu(s,sw)(aT_{sw} + \mu(s,sw)^{-1}bT_w) = \mu(s,sw)aT_{sw} + bT_w$, which lies in the desired space, as needed. □

3.3 The Coxeter Group Case

In this section, suppose W is a finite Coxeter group with set of simple reflections $S \subset W$ and real reflection representation $\mathfrak{h}_{\mathbb{R}}$ with inner product $\langle \cdot, \cdot \rangle$. Let $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$ be the complexified reflection representation, let $c : S \to \mathbb{C}$ be a W-invariant function (by which we mean $c(s) = c(s')$ whenever s and s' are conjugate in W), and let $H_c(W, \mathfrak{h})$ be the associated rational Cherednik algebra (note that c extends uniquely to a W-invariant function on the set of reflections in W). We will use certain simplifications that arise in the Coxeter case, such as natural splittings of the normalizers of parabolic subgroups and the T_w-basis of the Hecke algebra $\mathsf{H}_q(W)$, to significantly improve upon the description of the generalized Hecke algebras $\mathcal{H}(c, W', L, W'')$ studied in the previous section. In particular, we will give a natural presentation for these algebras and will explain how to compute the quadratic relations $P_{W',L,W''}(T_{W',L,W''}) = 0$ for the L-essential parabolic subgroups $W'' \supset W'$ introduced in Definition 3.2.3.5.

Notation: As in previous sections, we will take W' and L to be fixed and will suppress them from the notation, e.g. write $I_{W''}$ rather than $I_{W',L}$, when the meaning is clear.

3.3.1 A Presentation of the Endomorphism Algebra

Let $J \subset S$ be a subset of the simple reflections and let $W' := \langle J \rangle$ be the parabolic subgroup of W generated by J. Let $H_c(W', \mathfrak{h}_{W'})$ be the associated rational Cherednik algebra as considered in previous sections, and let L be a finite-dimensional irreducible representation of $H_c(W', \mathfrak{h}_{W'})$. Let $\mathfrak{h}_{W',\mathbb{R}} \subset \mathfrak{h}_{\mathbb{R}}$ denote the unique W'-stable complement to $\mathfrak{h}_{\mathbb{R}}^{W'}$ in $\mathfrak{h}_{\mathbb{R}}$. We have $\mathfrak{h}^{W'} = (\mathfrak{h}_{\mathbb{R}}^{W'})_{\mathbb{C}}$ and $\mathfrak{h}_{W'} = (\mathfrak{h}_{W',\mathbb{R}})_{\mathbb{C}}$, where the subscript $\mathbb{C}$ denotes the complexification viewed as a subspace of $\mathfrak{h}$. The action of W' on $\mathfrak{h}_{W',\mathbb{R}}$ is naturally identified with the real reflection representation of the Coxeter system (W', J). Let $\mathfrak{h}_{\mathbb{R},reg}^{W'} \subset \mathfrak{h}_{\mathbb{R}}^{W'}$ denote the subset of points with stabilizer in W equal to W'. The complexification $(\mathfrak{h}_{\mathbb{R},reg}^{W'})_{\mathbb{C}}$ is a proper subspace of $\mathfrak{h}_{reg}^{W'}$.

Let $\Phi \subset \mathfrak{h}_{\mathbb{R}}$ denote the root system attached to (W, S), let $\Phi^+ \subset \Phi$ denote the set of positive roots, and for $s \in S$ let $\alpha_s \in \Phi$ denote the positive simple root defining the reflection s. For a subset $J \subset S$ of the simple roots, let $\Phi_J \subset \Phi$ denote the associated root subsystem with positive roots $\Phi_J^+ = \Phi_J \cap \Phi^+$. Let $\mathcal{C}_J := \{x \in \mathfrak{h}_{W',\mathbb{R}} : \langle \alpha_s, x \rangle > 0 \text{ for all } s \in J\} \subset \mathfrak{h}_{W',\mathbb{R}}$ be the associated open fundamental Weyl chamber. In [47, Corollary 3], Howlett explains that the normalizer $N_W(W')$ splits as a semidirect product $N_W(W') = W' \rtimes N_{W'}$, where $N_{W'}$ is the set-wise stabilizer of $\mathcal{C}_J$ in $N_W(W')$. We then take the inertia subgroup $I \subset N_{W'}$ to be the stabilizer in $N_{W'}$ of the representation L, as defined after Lemma 3.2.2.2. Recall that for each parabolic subgroup $W'' \subset W$ containing W' in corank 1 we have the associated subgroup $I_{W''} := I \cap W''$, and recall that $I^{ref} \trianglelefteq I$ is the normal subgroup generated by the $I_{W''}$ for all such W''. The subgroups $I_{W''}$ are all either trivial (for L-trivial W'') or of order 2 (for L-essential W'') acting on $\mathfrak{h}_{\mathbb{R}}^{W'}$ through orthogonal reflection through the hyperplane $\mathfrak{h}_{\mathbb{R}}^{W''} \subset \mathfrak{h}_{\mathbb{R}}^{W'}$. Therefore I^{ref} is a real reflection group with faithful reflection representation $\mathfrak{h}_{\mathbb{R}}^{W'}$ (potentially with nontrivial fixed points). Recall that I^{ref} is the maximal reflection subgroup of I.

The complement $\mathfrak{h}_{\mathbb{R},L-reg}^{W'}$ of the real hyperplanes $\mathfrak{h}_{\mathbb{R}}^{W''} \subset \mathfrak{h}_{\mathbb{R}}^{W'}$ is the locus of points in $\mathfrak{h}_{\mathbb{R}}^{W'}$ with trivial stabilizer in I^{ref}. By the standard theory of real reflection groups, I^{ref} acts simply transitively on the set of connected components of $\mathfrak{h}_{\mathbb{R},L-reg}^{W'}$. Choose a connected component $\mathcal{C} \subset \mathfrak{h}_{\mathbb{R},L-reg}^{W'}$, and let $I^{comp} \subset I$ be the set-wise stabilizer of $\mathcal{C}$ in I. Clearly, we then have the semidirect product decomposition

$$I = I^{ref} \rtimes I^{comp}.$$

Choosing the connected component $\mathcal{C}$ amounts to choosing a fundamental Weyl chamber for I^{ref}, and in particular the pair $(I^{ref}, S_{W',L})$ is a Coxeter system, where $S_{W',L}$ is the set of reflections through the walls of $\mathcal{C}$. As I^{comp} acts on $\mathcal{C}$, it follows that I^{comp} permutes the walls of C, and in particular the action of I^{comp} on I^{ref} is through diagram automorphisms of the Dynkin diagram of $(I^{ref}, S_{W',L})$.

Let $q : S_{W',L} \to \mathbb{C}^\times$, $s \mapsto q_s$, be the unique I-invariant function such that for each $s \in S_{W',L}$ the quadratic polynomial $P_{W',L,\langle W',s\rangle}$ factors as $(T-1)(T+q_s)$ after a rescaling of the indeterminate T. Let $l : I^{ref} \to \mathbb{Z}^{\geq 0}$ denote the length function of the Coxeter system $(I^{ref}, S_{W',L})$. We then have the following presentation for the generalized Hecke algebra $\mathcal{H}(c, W', L, W'')$, analogous to the presentation given in [46, Theorem 4.14] in the context of finite groups of Lie type:

Theorem 3.3.1.1. *Let $\mu \in Z^2(I, \mathbb{C}^\times)$ be the 2-cocycle appearing in Theorem 3.2.2.4. There is a basis $\{T_x : x \in I\}$ for $\mathcal{H}(c, W', L, W'')$ with multiplication law completely described by the following relations for all $x \in I$, $d \in I^{comp}$, $w \in I^{ref}$, and $s \in S_{W',L}$:*

(1) $T_d T_x = \mu(d,x)^{-1} T_{dx}$

(2) $T_x T_d = \mu(x,d)^{-1} T_{xd}$

(3) $T_s T_w = \begin{cases} \mu(s,w)^{-1} T_{sw} & \text{if } l(sw) > l(w) \\ q_s \mu(s,w)^{-1} T_{sw} + (q_s - 1) T_w & \text{if } l(sw) < l(w) \end{cases}$

(4) $T_w T_s = \begin{cases} \mu(w,s)^{-1} T_{ws} & \text{if } l(ws) > l(w) \\ q_s \mu(w,s)^{-1} T_{ws} + (q_s - 1) T_w & \text{if } l(ws) < l(w). \end{cases}$

Remark 3.3.1.2. *When the cocycle μ is trivial, Theorem 3.3.1.1 gives an isomorphism*

$$\mathcal{H}(c, W', L, W'') \cong I^{comp} \ltimes \mathsf{H}_q(I^{ref})$$

where $\mathsf{H}_q(I^{ref})$ denotes the Iwahori-Hecke algebra associated with the Coxeter system $(I^{ref}, S_{W',L})$ with parameter $q : s \mapsto q_s$ and where I^{comp} acts on $\mathsf{H}_q(I^{ref})$ by automorphisms induced by diagram automorphisms of the Dynkin diagram of $(I^{ref}, S_{W',L})$. We will see in a later section, by inspecting all possible cases, that the cocycle μ is indeed trivial in every case.

Proof. Take the base point $b \in \mathfrak{h}^{W'}_{reg}$ for the fundamental group to lie in the fundamental chamber $\mathcal{C} \subset \mathfrak{h}_{\mathbb{R},L-reg}^{W'}$ and outside of the hyperplanes $\mathfrak{h}^{W''} \subset \mathfrak{h}^{W'}$ for the L-trivial parabolic subgroups $W'' \supset W'$. As I acts freely on $\mathfrak{h}_{reg}^{W'}$, so too I^{comp} acts freely on the orbit $I^{comp}.b \subset \mathcal{C} \cap \mathfrak{h}_{reg}^{W'}$. For each $d \in I^{comp}$, choose a path $\gamma_d : [0,1] \to \mathcal{C} \cap \mathfrak{h}_{reg}^{W'}$ in $\mathcal{C} \cap \mathfrak{h}_{reg}^{W'}$ from b to $d.b$. Let $\overline{\gamma}_d$ denote the reverse path, so that the concatenation $\overline{\gamma}_d * \gamma_d$ is a loop in $\mathcal{C} \cap \mathfrak{h}_{reg}^{W'}$ with base point b. As $\mathcal{C}$ is contractible, the image of the path homotopy class of $\overline{\gamma}_d * \gamma_d$ under the homomorphism

$$\pi_1(\mathfrak{h}_{reg}^{W'}) \to \pi_1(\mathfrak{h}_{L-reg}^{W'})$$

induced by the inclusion $\mathfrak{h}_{reg}^{W'} \subset \mathfrak{h}_{L-reg}^{W'}$ is trivial. As discussed in the proof of Theorem 3.2.3.6, the kernel of this map is generated (as a normal subgroup) by the generators of monodromy $T_{W''}$ about the hyperplanes $\mathfrak{h}^{W''}$ for the L-trivial parabolic subgroups $W'' \supset W'$. It follows immediately that the image of the homotopy class $[\gamma_d]$ in the quotient

$$\frac{\pi_1(\mathfrak{h}_{reg}^{W'}/I)}{\langle T_{W''} : W'' \text{ is } L\text{-trivial}\rangle}$$

is independent of the choice of the path γ_d and that the assignments $d \mapsto [\gamma_d]$ for $d \in I^{comp}$ extends uniquely to an algebra homomorphism

$$\mathbb{C}_{-\mu}[I^{comp}] \to \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]}{\langle T_{W''} : W'' \text{ is } L\text{-trivial}\rangle}.$$

Composing with the automorphism τ_χ of $\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]$ discussed in the proof of Theorem 3.2.3.6, this yields an algebra homomorphism

$$\mathbb{C}_{-\mu}[I^{comp}] \to \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]}{\langle P_{W''}(T_{W''}) : W' \text{ is corank 1 in } W''\rangle}.$$

We also have the embedding

$$\frac{\mathbb{C}[\pi_1(\mathfrak{h}_{L-reg}^{W'}/I^{ref})]}{\langle P_{W''}(T_{W''}) : W'' \text{ is } L\text{-essential}\rangle} \to \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]}{\langle P_{W''}(T_{W''}) : W' \text{ is corank 1 in } W''\rangle}$$

from Theorem 3.2.3.6. It is clear that the map of $\mathbb{C}$-vector spaces

$$\mathbb{C}_{-\mu}[I^{comp}] \otimes_{\mathbb{C}} \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})]}{\langle P_{W''}(T_{W''}) : W'' \text{ is } L\text{-essential}\rangle}$$

$$\to \frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}^{W'}_{reg}/I)]}{\langle P_{W''}(T_{W''}) : W' \text{ is corank 1 in } W''\rangle}$$

induced by multiplication is surjective. It follows by Theorem 3.2.3.6 that this map is in fact an isomorphism of $\mathbb{C}$-vector spaces.

For a simple reflection $s \in S_{W',L} \subset I^{ref}$, let e_s denote the generator of monodromy $T_{W',L,\langle W',s\rangle}$. For any $w \in I^{ref}$, let $e_w \in \pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})$ denote the element obtained as the product $e_w := e_{s_1} \cdots e_{s_l}$ for any reduced expression $w = s_1 \cdots s_l$ of w as a product of simple reflections in $S_{W',L}$, as in the last paragraph of the proof of Theorem 3.2.3.6 (recall that the product here is taken in the fundamental group $\pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})$, i.e. without regards to the cocycle $-\tilde{\mu}$). For $w \in I^{ref}$, let T_w denote e_w viewed as an element of $\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})]$. By Theorem 3.2.3.6 and the last paragraph in its proof, the set $\{T_w : w \in I^{ref}\}$ forms a basis for the quotient

$$\frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}^{W'}_{L-reg}/I^{ref})]}{\langle P_{W''}(T_{W''}) : W'' \text{ is } L\text{-essential}\rangle}.$$

By rescaling the elements T_w appropriately using twists by characters as τ_χ was used in the proof of Theorem 3.2.3.6, we may assume that the quadratic relations the T_s satisfy are of the form

$$(T_s - 1)(T_s + q_s) = 0$$

for some uniquely determined $q_s \in \mathbb{C}^\times$. The assignments $s \mapsto q_s$ determine an I-invariant function $q : S_{W',L} \to \mathbb{C}^\times$ with $q(s) = q_s$. The computations at the end of the proof of Theorem 3.2.3.6 show that for $s \in S_{W',L}$ and $w \in I^{ref}$ we have the multiplication law

$$T_s T_w = \begin{cases} \mu(s,w)^{-1} T_{sw} & \text{if } l(sw) > l(w) \\ q_s \mu(s,w)^{-1} T_{sw} + (q_s - 1) T_w & \text{if } l(sw) < l(w) \end{cases}$$

where $l : I^{ref} \to \mathbb{Z}^{\geq 0}$ is the length function determined by the choice of simple reflections $S_{W',L}$ (note that $\mu(s, sw) = \mu(s, w)^{-1}$ because $s^2 = 1$). An entirely analogous calculation shows that

$$T_w T_s = \begin{cases} \mu(w, s)^{-1} T_{ws} & \text{if } l(ws) > l(w) \\ q_s \mu(w, s)^{-1} T_{ws} + (q_s - 1) T_w & \text{if } l(ws) < l(w). \end{cases}$$

For $d \in I^{comp}$, let

$$e_d \in \frac{\pi_1(\mathfrak{h}_{reg}^{W'}/I)}{\langle T_{W''} : \ W'' \text{ is } L\text{-trivial} \rangle}$$

be the image of $[\gamma_d]$ as constructed above. We may also regard the e_w for $w \in I^{ref}$ as elements of this quotient. Any element $x \in I$ can be written uniquely as a product $x = dw$ for some $d \in I^{comp}$ and $w \in I^{ref}$, and we may therefore define $e_x := e_d e_w$. Note that $e_d e_w e_{d^{-1}} = e_{dwd^{-1}}$ so $e_{dwd^{-1}} e_d = e_x$, so writing $x = w'd'$ with $w' \in I^{ref}$ and $d' \in I^{comp}$ and taking the element $e_{w'} e_{d'}$ defines the same element e_x. Note that $e_d e_d' = e_{dd'}$ for any $d, d' \in I^{comp}$, and hence $e_d e_x = e_{dx}$ and $e_x e_d = e_{xd}$ for any $d \in I^{comp}$ and $w \in I$. For any $x \in I$, let T_x denote the element e_x viewed as an element of the quotient algebra

$$\frac{\mathbb{C}_{-\tilde{\mu}}[\pi_1(\mathfrak{h}_{reg}^{W'}/I)]}{\langle P_{W''}(T_{W''}) : W' \text{ is corank 1 in } W'' \rangle}.$$

For $x \in I^{ref}$, the elements T_x are the images of the elements T_x considered above. The multiplication rules $T_d T_x = \mu(d, x)^{-1} T_{dx}$ and $T_x T_d = \mu(x, d)^{-1} T_{xd}$ for $d \in I^{comp}$ and $x \in I$ follow immediately from the computations with e_x and e_d above and the definition of the twisted group algebra. That $\{T_x : x \in I\}$ forms a basis for this quotient, and hence also for $\mathcal{H}(c, W', L, W'')$ by Theorem 3.2.3.6 follows immediately from the considerations above, and the theorem follows. □

3.3.2 Computing Parameters Via KZ

To compute the quadratic relations that the generators $T_s \in \mathcal{H}(c, W', L, W)$ satisfy, we will reduce the problem to certain explicit computations in the Hecke algebra $\mathsf{H}_q(W)$ of the ambient group W and its representations. This is possible thanks to the following result of Shan:

Lemma 3.3.2.1. *[65, Lemma 2.4] Let $KZ : \mathcal{O}_c(W, \mathfrak{h}) \to \mathsf{H}_q(W)$-mod$_{f.d.}$ be the KZ functor, and let $K, L : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_c(W, \mathfrak{h})$ be two right exact functors that map projective objects to projective objects. Then the natural map of vector spaces*

$$Hom(K, L) \to Hom(KZ \circ K, KZ \circ L)$$

$$f \mapsto 1_{KZ} f$$

is an isomorphism.

Assume that $(W', S') \subset (W, S)$ is a Coxeter subsystem of corank 1. The following lemma describing the canonical complement $N_{W'}$ to W' in $N_W(W')$ will be useful in what follows:

Lemma 3.3.2.2. *Either W' is self-normalizing in W or has index 2 in its normalizer $N_W(W')$. In the latter case, the longest elements w_0 and w_0' of W and W', respectively, commute, and the canonical complement $N_{W'}$ to W' in $N_W(W')$ is*

$$N_{W'} = \{1, w_0 w_0'\}.$$

Proof. The first statement follows from Lemma 3.2.3.3. Let $w \in N_{W'}$ be the nontrivial element. Let $\Phi \subset \mathfrak{h}_\mathbb{R}$ denote the root system of W, let $\Phi_{W'} \subset \Phi$ denote the root system of W', let $\alpha_1, ..., \alpha_r$ be an ordering of the simple roots in Φ so that $\alpha_1, ..., \alpha_{r-1}$ are the simple roots in $\Phi_{W'}$, and let $\Phi^+ \subset \Phi$ and $\Phi^+_{W'} \subset \Phi_{W'}$ denote the respective subsets of positive roots. By definition of $N_{W'}$, $w(\Phi^+_{W'}) = \Phi^+_{W'}$. From the proof of Lemma 3.2.3.3, we see that w acts by -1 in $\mathfrak{h}_\mathbb{R}^{W'}$. As the remaining simple root α_r is the unique simple root with nonzero component in $\mathfrak{h}_\mathbb{R}^{W'}$ with respect to the decomposition $\mathfrak{h}_\mathbb{R} = \mathfrak{h}_{\mathbb{R},W'} \oplus \mathfrak{h}_\mathbb{R}^{W'}$ and as every positive root $\alpha \in \Phi^+$ is uniquely of the form $\alpha = \sum_i n_i \alpha_i$ for some nonnegative integers $n_i \geq 0$, it follows that $w(\Phi^+ \backslash \Phi^+_{W'}) \subset \Phi^-$. It follows that the inversion set of w is precisely $\Phi^+ \backslash \Phi^+_{W'}$ and hence that $w = w_0 w_0'$. As w, w_0, and w_0' are involutions, it follows that w_0 and w_0' commute. $\square$

As in previous sections, let $c : S \to \mathbb{C}$ be a W-invariant function and let L be an irreducible finite-dimensional representation of $H_c(W', \mathfrak{h}_{W'})$. Let w_0 and w_0' denote the longest elements in W and W', respectively. We will assume that W' is not self-normalizing in W, and by Lemma 3.3.2.2 it follows that w_0 and w_0' commute, that W' is index 2 in $N_{W'}(W)$, and that $N_{W'} = \{1, w_0 w_0'\}$. We will assume that the involution $w_0 w_0'$ fixes the isomorphism class of L so that $I = N_{W'}$ and the monodromy operator $T_{W'}$ satisfies a nontrivial quadratic relation as in Theorem 3.2.3.6.

The following observation in this setting will be central to our approach as it will allow for the explicit computation, via computations in the Hecke algebra $\mathsf{H}_q(W)$, of the eigenvalues of monodromy in the local systems arising from the functors KZ_L. This lemma should be regarded as a generalization to Coxeter groups of arbitrary type of the calculation appearing in [40, Lemma 4.14].

Lemma 3.3.2.3. *Let $\mathcal{C}_{W'} \subset \mathfrak{h}_{W',\mathbb{R}}$ be the open fundamental Weyl chamber associated to (W', S'), let $\mathcal{C}_W \subset \mathfrak{h}_{\mathbb{R}}$ be the open fundamental Weyl chamber associated to (W, S), and choose a basepoint $b = (b', b'') \in \mathcal{C}_{W'} \times \mathfrak{h}_{reg}^{W'}$ lying in $\mathcal{C}_W$. As $\mathcal{C}_{W'}$ is contractible and stable under $w_0 w_0'$, the pair $(\gamma, T_{W'})$, where γ is any path in $\mathcal{C}_{W'}$ from b' to $w_0 w_0' b'$ and $T_{W'}$ is the half-loop in $\mathfrak{h}_{reg}^{W'}$ lifting the canonical generator of monodromy in $\pi_1(\mathfrak{h}_{reg}^{W'}/N_{W'})$, determines an element in $\pi_1(((\mathfrak{h}_{W'})_{reg} \times \mathfrak{h}_{reg}^{W'})/N_{W'})$ that does not depend on the choice of γ. The image of this element under the natural map*

$$\iota_{W',1} : \pi_1(((\mathfrak{h}_{W'})_{reg} \times \mathfrak{h}_{reg}^{W'})/N_{W'}) \to \pi_1(\mathfrak{h}_{reg}/W) = B_W$$

is $T_{w_0 w_0'}$.

Proof. That the element of $\pi_1(((\mathfrak{h}_{W'})_{reg} \times \mathfrak{h}_{reg}^{W'})/N_{W'})$ determined by the pair $(\gamma, T_{W'})$ does not depend on the choice of γ follows immediately from the contractibility of the fundamental Weyl chamber $\mathcal{C}_{W'}$. By definition of $\iota_{W',1}$, for any sufficiently large real number $R > 0$ the image $\iota_{W',1}(\gamma, T_{W'})$ in $\pi_1(\mathfrak{h}_{reg}/W)$ is represented by the image of the path

$$p : [0, 1] \to \mathfrak{h}_{reg}$$

$$p(t) = (\gamma(t), Re^{\pi i t} b'')$$

under the natural projection $\mathfrak{h}_{reg} \to \mathfrak{h}_{reg}/W$. As $w_0 w_0' b'' = -b''$, it follows that p is a path from the point $p(0) = (b', Rb'') \in \mathcal{C}_W$ to the point $p(1) = w_0 w_0' p(0) \in w_0 w_0' \mathcal{C}_W$. For each positive root $\alpha \in \Phi^+$, let $H_\alpha := \ker(\alpha) \subset \mathfrak{h}$ be the associated reflection hyperplane. That p represents the element $T_{w_0 w_0'}$ follows from the observation that p traverses a positively-oriented (with respect to the complex structure) half-loop about each hyperplane H_α for roots $\alpha \in \Phi^+ \backslash \Phi^+_{W'}$ while p does not encircle any of the remaining hyperplanes H_α for roots $\alpha \in \Phi^+_{W'}$. More precisely, for roots $\alpha \in \Phi^+ \backslash \Phi^+_{W'}$ the composition $\alpha \circ p : [0,1] \to \mathbb{C}^\times$ determines a path from the positive real axis $\mathbb{R}^+$ to the negative real axis $\mathbb{R}^-$ lying entirely in the upper half-space $\{z \in \mathbb{C}^\times : \mathrm{Re}(z) \geq 0\}$. For roots $\alpha \in \Phi^+_{W'}$, the composition $\alpha \circ p : [0,1] \to \mathbb{C}^\times$ determines a path lying entirely on $\mathbb{R}^+$, as $\gamma(t) \in \mathcal{C}_{W'}$ and $\alpha(b'') = 0$. The equality $\iota_{W',1}(\gamma, T_{W'}) = T_{w_0 w_0'}$ follows. $\square$

Let $\mathrm{Mon}(T_W)$ denote the isomorphism of functors

$$\mathrm{Res}^W_{W'} \to \mathrm{tw}_{w_0 w_0'} \circ \mathrm{Res}^W_{W'}$$

arising from monodromy along the generator of monodromy T_W in the local system $\underline{\mathrm{Res}}^W_{W'}$, where $\underline{\mathrm{Res}}^W_{W'}$ is the partial KZ functor recalled in Section 2.5. The following is an immediate corollary of Lemma 3.3.2.3 and the transitivity result of Gordon-Martino recalled in Theorem 2.6.0.1:

Lemma 3.3.2.4. *Multiplication by $T_{w_0 w_0'}$ defines an isomorphism*

$$T_{w_0 w_0'} : {}^H Res^W_{W'} \to {}^H tw_{w_0 w_0'} \circ {}^H Res^W_{W'}$$

of functors $\mathsf{H}_q(W')$-$mod_{f.d.} \to \mathsf{H}_q(W')$-$mod_{f.d.}$. Furthermore, $Mon(T_W)$ is the lift to $\mathcal{O}_c(W, \mathfrak{h})$ of $T_{w_0 w_0'}$ in the sense that, with respect to the identifications $KZ \circ Res^W_{W'} = {}^H Res^W_{W'} \circ KZ$ and $KZ \circ tw_{w_0 w_0'} = {}^H tw_{w_0 w_0'} \circ KZ$, we have an equality

$$1_{KZ} Mon(T_W) = T_{w_0 w_0'} 1_{KZ}$$

of isomorphisms of functors

$$KZ \circ Res^W_{W'} \to KZ \circ tw_{w_0w_0'} \circ Res^W_{W'}.$$

Remark 3.3.2.5. *When the meaning is clear, we denote the KZ functors defined for $\mathcal{O}_c(W,\mathfrak{h})$ and $\mathcal{O}_c(W',\mathfrak{h}_{W'})$ each by KZ.*

Proof of Lemma 3.3.2.4. The first statement follows from the observation that the functor ${}^H\mathrm{Res}^W_{W'}$ is represented by the $\mathsf{H}_q(W')$-$\mathsf{H}_q(W)$-bimodule $\mathsf{H}_q(W)$ and the observation that $T_{w_0w_0'}T_s = T_{\sigma_{w_0w_0'}(s)}T_{w_0w_0'}$ for all $s \in S'$, where $\sigma_{w_0w_0'}$ is the diagram automorphism of (W',S') induced by conjugation by w_0w_0' in W. The second statement is an immediate consequence of Lemma 3.3.2.3 and Theorem 2.6.0.1. □

Definition 3.3.2.6. *Let $X_{W'} \subset W$ be the set of minimal length left W'-coset representatives, and let $\{z_d\}_{d\in X_{W'}}$ be the unique set of elements $z_d \in \mathsf{H}_q(W')$ such that*

$$T^2_{w_0w_0'} = \sum_{d\in X_{W'}} z_dT_d.$$

We are interested in the elements $z_1, z_{w_0w_0'} \in \mathsf{H}_q(W')$ arising from the left W'-cosets that are also right W'-cosets. We now establish important properties that these elements satisfy:

Lemma 3.3.2.7. *The element $z_{w_0w_0'}$ commutes with $T_{w_0w_0'}$ and satisfies the relation*

$$xz_{w_0w_0'} = z_{w_0w_0'}\sigma(x)$$

for all $x \in \mathsf{H}_q(W')$, where $\sigma : \mathsf{H}_q(W') \to \mathsf{H}_q(W')$ is the automorphism of $\mathsf{H}_q(W')$ arising from the diagram automorphism of (W',S') obtained by conjugation by w_0w_0' in W. In particular, multiplication by $z_{w_0w_0'}$ defines a morphism

$$z_{w_0w_0'} : Id \to {}^H tw_{w_0w_0'}$$

of functors $\mathsf{H}_q(W')$*-mod*$_{f.d.} \to \mathsf{H}_q(W')$*-mod*$_{f.d.}$. *Furthermore,*

$$z_1 = q^{l(w_0 w_0')},$$

where, in the unequal parameter case, $q^{l(w_0w_0')}$ *denotes the product*

$$q^{l(w_0w_0')} := \prod_{s \in S/W} q_s^{l_s(w_0w_0')}$$

over the conjugacy classes of reflections in W*, where* $l_s : W \to \mathbb{Z}^{\geq 0}$ *is the length function of* W *attached to the conjugacy class of* s *and* $q_s \in \mathbb{C}$ *is the parameter associated to the conjugacy class of* s.

Proof. As w_0 and w_0' commute, we have $T_{w_0}T_{w_0'}^{-1} = T_{w_0w_0'} = T_{w_0'w_0} = T_{w_0'}^{-1}T_{w_0}$. Every simple reflection in S lies in both the right and left descent sets of w_0, and similarly every simple reflection in S' lies in both the right and left descent sets of w_0'. It follows that $T_{w_0}T_s = T_{\sigma_{w_0}(s)}T_{w_0}$ for every $s \in S$, where σ_{w_0} is the diagram automorphism of (W, S) obtained by conjugation by w_0, and similarly $T_{w_0'}T_s = T_{\sigma_{w_0'}(s)}T_{w_0'}$ for all $s \in S'$, where $\sigma_{w_0'}$ is the diagram automorphism of (W', S') obtained by conjugation by w_0'. It follows that $T_{w_0}^2$ is central in $\mathsf{H}_q(W)$ and $T_{w_0'}^2$ is central in $\mathsf{H}_q(W')$ (this is well known [6, 17]) and that $T_{w_0w_0'}x = \sigma(x)T_{w_0w_0'}$ for all $x \in \mathsf{H}_q(W')$. In particular, the element $T_{w_0w_0'}^2 = T_{w_0}^2T_{w_0'}^{-2} \in \mathsf{H}_q(W)$ considered above centralizes $\mathsf{H}_q(W')$. As multiplication by elements of $\mathsf{H}_q(W')$ on the right or left respects the decomposition of $T_{w_0w_0'}^2$ by W'-double cosets and as $T_{w_0w_0'}$ normalizes $\mathsf{H}_q(W')$, it follows that $xz_{w_0w_0'}T_{w_0w_0'} = z_{w_0w_0'}T_{w_0w_0'}x$ for all $x \in \mathsf{H}_q(W')$. But as $T_{w_0w_0'}x = \sigma(x)T_{w_0w_0'}$ and $T_{w_0w_0'}$ is invertible, it follows that $xz_{w_0w_0'} = z_{w_0w_0'}\sigma(x)$. That multiplication by $z_{w_0w_0'}$ defines a morphism of functors $\mathrm{Id} \to {}^H\mathrm{tw}_{w_0w_0'}$ follows immediately.

Similarly, note that conjugation by $T_{w_0'}$ clearly respects decomposition of elements by W'-double cosets and that conjugation by T_{w_0} stabilizes $\mathsf{H}_q(W')$ and sends minimal length W'-double-coset representatives of to minimal length W'-double-coset representatives of the same length. In particular, conjugation by $T_{w_0w_0'}$ fixes the top degree term $z_{w_0w_0'}T_{w_0w_0'}$ in the decomposition of $T_{w_0w_0'}^2$ by left W'-cosets.

As $T_{w_0w_0'}$ commutes with itself and $T_{w_0w_0'}z_{w_0w_0'} = \sigma(z_{w_0w_0'})T_{w_0w_0'}$ this implies that $z_{w_0w_0'}T_{w_0w_0'} = \sigma(z_{w_0w_0'})T_{w_0w_0'}$ and hence that $\sigma(z_{w_0w_0'}) = z_{w_0w_0'}$. In particular, $z_{w_0w_0'}$ commutes with $T_{w_0w_0'}$.

Finally, to show that $z_1 = q^{l(w_0w_0')}$, by multiplying on the left by $T_{w_0'}$ it suffices to show that the component of $T_{w_0}T_{w_0w_0'}$ lying in $\mathsf{H}_q(W')$ according to the decomposition of $\mathsf{H}_q(W)$ by left W'-cosets is $q^{l(w_0w_0')}T_{w_0'}$. This is clear from the interaction of the multiplication laws defining $\mathsf{H}_q(W)$ and the length function. $\square$

Definition 3.3.2.8. *Let ${}^C z_{w_0w_0'}$ be the unique morphism*

$$ {}^C z_{w_0w_0'} : Id \to tw_{w_0w_0'} $$

of functors $\mathcal{O}_c(W', \mathfrak{h}_{W'}) \to \mathcal{O}_c(W', \mathfrak{h}_{W'})$ lifting $z_{w_0w_0'}$ in the sense of Lemma 3.3.2.1, i.e. so that

$$ 1_{KZ}{}^C z_{w_0w_0'} = z_{w_0w_0'} 1_{KZ} $$

with respect to the identification $KZ \circ tw_{w_0w_0'} = {}^H tw_{w_0w_0'} \circ KZ$.

Definition 3.3.2.9. *Let $\mu_{w_0w_0'}$ be the isomorphism of functors*

$$ \mu_{w_0w_0'} : Id \oplus tw_{w_0w_0'} \to tw_{w_0w_0'} \circ (Id \oplus tw_{w_0w_0'}) = tw_{w_0w_0'} \oplus Id $$

defined by the matrix

$$ \mu_{w_0w_0'} := \begin{pmatrix} 0 & q^{l(w_0w_0')} \\ 1 & {}^C z_{w_0w_0'} \end{pmatrix}. $$

Recall that by the Mackey formula for rational Cherednik algebras attached to Coxeter groups (see Section 2.7), and the fact that L is finite-dimensional and hence annihilated by all restriction functors $\mathrm{Res}^{W'}_{W''}$ for proper parabolic subgroups $W'' \subsetneq W'$, that we have

$$ (\mathrm{Res}^W_{W'} \circ \mathrm{Ind}^W_{W'})L \cong (\mathrm{Id} \oplus \mathrm{tw}_{w_0w_0'})L. $$

Lemma 3.3.2.10. *With respect to the isomorphism*

$$ (Res^W_{W'} \circ Ind^W_{W'})L \cong (Id \oplus tw_{w_0w_0'})L $$

arising from the Mackey formula, the isomorphisms

$$(Mon(T_W)1_{Ind_{W'}^W})(L), \mu(L) : (Id \oplus tw_{w_0w_0'})L \to (tw_{w_0w_0'} \oplus Id)L$$

are equal.

Proof. This is an immediate consequence of the compatibility of the Mackey decomposition for rational Cherednik algebras with the KZ functor, Definition 3.3.2.6, and Lemmas 3.3.2.4 and 3.3.2.7. □

Notation: For a central element $z \in \mathsf{H}_q(W')$, let $z|_L \in \mathbb{C}$ denote the scalar by which z acts on irreducible representations of $\mathsf{H}_q(W')$ lying in the block of $\mathsf{H}_q(W')$-mod$_{f.d.}$ corresponding to the block of $\mathcal{O}_c(W', \mathfrak{h}_{W'})$ containing L under the KZ functor.

We can now give a formula for the quadratic relations satisfied by the elements T_s appearing in Theorem 3.3.1.1:

Theorem 3.3.2.11. *Let T denote the element $T_W \in \mathcal{H}(c, W', L, W)$, and let $n \in Aut_{\mathbb{C}}(L)$ denote the involution of L by which $w_0w_0' \in N_{W'}$ acts making L $N_{W'}$-equivariant as a $H_c(W', \mathfrak{h}_{W'})$-module. Then T satisfies the quadratic relation*

$$T^2 = ({}^C z_{w_0w_0'}(L) \circ n)|_L T + q^{l(w_0w_0')}$$

where $({}^C z_{w_0w_0'}(L) \circ n)|_L \in \mathbb{C}$ denotes the scalar by which the $H_c(W', \mathfrak{h}_{W'})$-module homomorphism $n \circ {}^C z_{w_0w_0'}(L)$ acts on the irreducible representation L.

Furthremore, if the diagram automorphism $\sigma = \sigma_{w_0w_0'}$ of (W', S') arising from conjugation by w_0w_0' is trivial, T satisfies the quadratic relation

$$T^2 = (z_{w_0w_0'})|_L T + q^{l(w_0w_0')}.$$

If the diagram automorphism σ_{w_0} is trivial but the diagram automorphism $\sigma_{w_0'}$ is nontrivial, T satisfies the quadratic relation

$$T^2 = (z_{w_0w_0'} T_{w_0'})|_L (T_{w_0'}^2)|_L^{-1/2} T + q^{l(w_0w_0')}.$$

Remark 3.3.2.12. *The projective representation of $N_{W'} = I \cong \mathbb{Z}/2\mathbb{Z}$ on L lifts to an linear representation of $N_{W'}$ in two ways, differing by a tensor product with the nontrivial character of $N_{W'}$, so there is a choice of sign for the action of the nontrivial element $w_0w_0' \in N_{W'}$ on L. The quadratic relations appearing in Theorem 3.3.2.11 under assumptions on the diagram automorphisms hold for an appropriate choice of sign for the operator n - choosing the other sign simply negates the linear term in the quadratic relation. The quadratic relations of the form $(T_s - 1)(T_s + q_s) = 0$ appearing in Theorem 3.3.1.1 are obtained by rescaling the generators T_W by twisting by characters of $\pi_1(\mathfrak{h}_{reg}^{W'}/I)$ as in Lemma 3.2.3.8, and the relations in this normalized form are only determined up to inverting q_s but do not depend on the choice of sign for the action of w_0w_0' on L.*

Remark 3.3.2.13. *The case in which the diagram automorphism σ_{w_0} is nontrivial but the diagram automorphism $\sigma_{w_0'}$ is trivial only appears for groups of type D. We will show later in Section 3.3.5 how to reduce the problem of computing the quadratic relations in type D to the type B case in which this complication does not arise.*

Proof. By Proposition 3.2.1.4, $\mathrm{Ind}_{W'}^{W}L$ is a projective generator of $\overline{\mathcal{O}}_{c,W',L}$, and hence it follows from Theorem 3.2.2.4 that the action of $\mathcal{H}(c, W', L, W)$ on the hom space $\mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L, \mathrm{Res}_{W'}^{W}\mathrm{Ind}_{W'}^{W}L)$ is faithful. It therefore suffices to check that the quadratic relation holds in this representation.

It follows from Lemma 3.3.2.10 that the action of T in the representation

$$\mathrm{Hom}_{\mathcal{O}_c(W',\mathfrak{h}_{W'})}(L, \mathrm{Res}_{W'}^{W}\mathrm{Ind}_{W'}^{W}L) \subset L^* \otimes_{\mathbb{C}} \mathrm{Res}_{W'}^{W}\mathrm{Ind}_{W'}^{W}L$$

is by the operator

$$n^* \otimes \mu_{w_0w_0'}(L) = n^* \otimes \begin{pmatrix} 0 & q^{l(w_0w_0')} \\ 1 & {}^{C}z_{w_0w_0'}(L) \end{pmatrix}.$$

As $n^2 = 1$, a simple calculation yields

$$T^2 = \left(n^* \otimes \begin{pmatrix} {}^C z_{w_0 w_0'}(L) & 0 \\ 0 & {}^C z_{w_0 w_0'}(L) \end{pmatrix} \right) \circ T + \begin{pmatrix} q^{l(w_0 w_0')} & 0 \\ 0 & q^{l(w_0 w_0')} \end{pmatrix}.$$

As the operator appearing in front of T on the righthand side acts on the hom space $\mathrm{Hom}_{\mathcal{O}_c(W', \mathfrak{h}_{W'})}(L, L \oplus \mathrm{tw}_{w_0 w_0'} L)$ by the scalar $({}^C z_{w_0 w_0'}(L) \circ n)|_L$, the first claim follows.

Now, suppose the diagram automorphism $\sigma_{w_0 w_0'}$ of (W', S') is trivial, so that $w_0 w_0'$ centralizes W' and acts on $\mathfrak{h}_{W'}$ trivially. In particular, $w_0 w_0'$ acts trivially on $H_c(W', \mathfrak{h}_{W'})$, and hence the trivial action of $N_{W'}$ on L makes L $N_{W'}$-equivariant, so we may take $n = \mathrm{Id}_L$. The quadratic relation for T in this case follows immediately.

Finally, suppose the diagram automorphism σ_{w_0} is trivial but the diagram automorphism $\sigma_{w_0'}$ is not. It follows that the diagram automorphisms $\sigma_{w_0'}$ and $\sigma = \sigma_{w_0 w_0'}$ are equal and that $T_{w_0'} x = \sigma(x) T_{w_0'}$ for all $x \in \mathsf{H}_q(W')$. In particular, multiplication by $T_{w_0'}$ defines a morphism

$$T_{w_0'} : \mathrm{Id} \to {}^H \mathrm{tw}_{w_0 w_0'}$$

of functors $\mathsf{H}_q(W')\text{-mod}_{f.d.} \to \mathsf{H}_q(W')\text{-mod}_{f.d.}$. Let ${}^C T_{w_0'}$ be the morphism

$${}^C T_{w_0'} : \mathrm{Id} \to \mathrm{tw}_{w_0 w_0'}$$

of functors $\mathcal{O}_c(W', \mathfrak{h}_{W'}) \to \mathcal{O}_c(W', \mathfrak{h}_{W'})$ obtained by lifting $T_{w_0'}$ by Lemma 3.3.2.1, similarly to the definition of ${}^C z_{w_0 w_0'}$ (Definition 3.3.2.8). We may then take the operator $n \in \mathrm{Aut}_{\mathbb{C}}(L)$ by which $w_0 w_0'$ acts to be the involutive operator

$$n = (T_{w_0'}^2)|_L^{-1/2} {}^C T_{w_0'}(L).$$

We then have

$$({}^C z_{w_0 w_0'}(L) \circ n)|_L = ({}^C z_{w_0 w_0'}(L) \circ (T_{w_0'}^2)|_L^{-1/2} ({}^C T_{w_0'}(L)))|_L = (z_{w_0 w_0'} T_{w_0'})|_L (T_{w_0'}^2)|_L^{-1/2}$$

and the final claim follows. □

We will see that, in the setting of Coxeter groups, the projective representation of I on L always lifts to a linear representation, and in particular the cocycle $\mu \in Z^2(I, \mathbb{C}^\times)$ is always trivial. Furthermore, the inertia group I is always as large as possible, i.e. it equals $N_{W'}$. These facts make the presentations of the algebras $\mathcal{H}(c, W', L, W)$ particularly simple:

Theorem 3.3.2.14. *Let W be a finite Coxeter group with simple reflections S, let $c : S \to \mathbb{C}$ be a class function, let W' be a standard parabolic subgroup generated by the simple reflections S', and let L be an irreducible finite-dimensional representation of the rational Cherednik algebra $H_c(W', \mathfrak{h}_{W'})$. Let $N_{W'}$ denote the canonical complement to W' in its normalizer $N_W(W')$, let $S_{W'} \subset N_{W'}$ denote the set of reflections in $N_{W'}$ with respect to its representation in the fixed space $\mathfrak{h}^{W'}$, and let $N_{W'}^{ref}$ denote the reflection subgroup of $N_{W'}$ generated by $S_{W'}$. Let $N_{W'}^{comp}$ be a complement for $N_{W'}^{ref}$ in $N_{W'}$, given as the stabilizer of a choice of fundamental Weyl chamber for the action of $N_{W'}^{ref}$ on $\mathfrak{h}^{W'}$. Then there is a class function $q_{W',L} : S_{W'} \to \mathbb{C}^\times$ and an isomorphism of algebras*

$$\mathcal{H}(c, W', L, W) \cong N_{W'}^{comp} \ltimes \mathsf{H}_{q_{W',L}}(N_{W'}^{ref})$$

where the semidirect product is defined by the action of $N_{W'}^{comp}$ on $\mathsf{H}_{q_{W',L}}(N_{W'}^{ref})$ by diagram automorphisms arising from the conjugation action of $N_{W'}^{comp}$ on $N_{W'}^{ref}$.

Theorem 3.3.2.14 is proved by case-by-case analysis of the inertia subgroups I and their 2-cocycles μ, to which the rest of this chapter is dedicated. The class function $q_{W',L}$ can be explicitly computed using the corank-1 methods developed in Section 3.3.2. We compute this class function in many cases, leading to complete lists of the irreducible finite-dimensional representations of the algebras $H_c(W, \mathfrak{h})$ in many new cases in exceptional types.

Remark 3.3.2.15. *In all cases that we have computed explicitly, the class function $q_{W',L}$ depends only on the parabolic subgroup W', and not on the finite-dimensional irreducible representation L. It would be interesting to have a conceptual explanation for this fact.*

Notation For a Coxeter group W, corank 1 parabolic subgroup $W' \subset W$, and finite-dimensional irreducible representation L of $H_c(W', \mathfrak{h}_{W'})$, let $q(c, W', L, W) \in \mathbb{C}^\times$ denote a scalar such that the element $T_{W',L,W} \in \mathcal{H}(c, W', L, W)$, after an appropriate rescaling, satisfies the quadratic relation

$$(T - 1)(T + q(c, W', L, W)) = 0.$$

Note that $q(c, W', L, W)$ is determined only up to taking an inverse. The calculations in the proof of Theorem 3.3.2.11 show that the constant term of the monic quadratic relation satisfied by the canonical (up to sign) element $T_{W',L,W}$ is $q^{l(w_0 w_0')}$, and the linear term in the monic quadratic relation satisfied by $T_{W',L,W}$ can therefore be recovered, again up to sign, from $q^{l(w_0 w_0')}$ and $q(c, W', L, W)$. Note also that the ambiguity of $q(c, W', L, W)$ up to inverse has no impact on the isomorphism class of any Iwahori-Hecke algebra for which $q(c, W', L, W)$ is a parameter, and an explicit isomorphism can be obtained by scaling the corresponding generators by $-q(c, W', L, W)^{-1}$.

3.3.3 Type A

Let $n \geq 1$ be a positive integer, S_n be the symmetric group on n letters with irreducible reflection representation $\mathfrak{h} = \{(z_1, ..., z_n) \in \mathbb{C}^n : \sum_i z_i = 0\}$, $c \in \mathbb{C}$ be a complex number, and let $H_c(S_n, \mathfrak{h})$ be the associated rational Cherednik algebra. As a first illustration of the results obtained in the previous sections, let us now recover the following result of Wilcox describing the subquotients of the filtration of category $\mathcal{O}_c(S_n, \mathfrak{h})$ by the dimension of supports:

Theorem 3.3.3.1. (Wilcox, [73, Theorem 1.8]) *Suppose $c = r/e > 0$ is a positive rational number with r and e relatively prime positive integers. The subquotient category of $\mathcal{O}_c(S_n, \mathfrak{h})$ obtained as the quotient of the full subcategory of modules in $\mathcal{O}_c(S_n, \mathfrak{h})$ supported on the subvariety $S_n \mathfrak{h}^{S_e^k}$, where $k \geq 0$ is a nonnegative integer, modulo the Serre subcategory of modules with strictly smaller support, is equivalent to the category of finite-dimensional modules over the algebra $\mathbb{C}[S_k] \otimes \mathsf{H}_q(S_{n-ke})$, where $\mathsf{H}_q(S_{n-ke})$ is the Hecke algebra of S_{n-ke} with parameter $q = e^{-2\pi i c}$.*

Let $c = r/e$ be as in Theorem 3.3.3.1. By [3, Theorem 1.2], the only parabolic subgroups W' of S_n such that $H_c(W', \mathfrak{h}_{W'})$ has nonzero finite-dimensional representations are the conjugates of parabolic subgroups of the form S_e^k for some nonnegative integer $k \leq n/e$, and the unique irreducible finite-dimensional representation of $H_c(S_e^k, \mathfrak{h}_{S_e^k})$, up to isomorphism, is $L := L(\text{triv})$, where triv denotes the trivial representation of S_e^k. It follows that the subquotient category appearing in Theorem 3.3.3.1 is the subquotient $\overline{\mathcal{O}}_{c,S_e^k,L}$. By Theorem 3.2.2.4, to prove Theorem 3.3.3.1 it suffices to give an isomorphism of algebras

$$\mathcal{H}(c, S_e^k, L(\text{triv}), S_n) \cong \mathbb{C}[S_k] \otimes \mathsf{H}_q(S_{n-ek}).$$

This follows from Theorems 3.3.1.1 and 3.3.2.11, as follows.

The fixed space $\mathfrak{h}^{S_e^k}$ is

$$\mathfrak{h}^{S_e^k} = \{(z_1, ..., z_n) \in \mathfrak{h} : \text{ for } 0 \leq l < k, \ z_{le+i} = z_{le+j} \text{ for } 1 \leq i < i \leq e\}.$$

Take coordinates $x_1, ..., x_k, y_1, ..., y_{n-ek}$ for $\mathfrak{h}^{S_e^k}$, where $x_l = z_{(l-1)e+i}$ for $1 \leq l \leq k$ and $1 \leq i \leq e$ and $y_j = z_{ek+j}$ for $1 \leq j \leq n - ek$. These coordinates satisfy the relation $\sum_i x_i + e\sum_j y_j = 0$. The complement $N_{S_e^k}$ to S_e^k in its normalizer is isomorphic to $S_k \times S_{n-ek}$, with the action of S_k on $\mathfrak{h}^{S_e^k}$ given by permuting the x_i coordinates and the action of S_{n-ek} given by permuting the y_j coordinates. The action on $H_c(S_e^k, \mathfrak{h}_{S_e^k}) \cong H_c(S_e, \mathfrak{h}_{S_e})^{\otimes k}$ is by permuting the tensor factors, and in particular the inertia group $I_{S_e^k,L}$ is maximal, i.e. equals $N_{S_e^k}$. Clearly the action of $N_{S_e^k}$ on $\mathfrak{h}^{S_e^k}$ is generated by reflections, so we have $I^{ref}_{S_e^k,L} = N_{S_e^k}$ and $I^{comp}_{S_e^k,L} = 1$. The trivial action of $N_{S_e^k}$ on the trivial representation triv of S_e^k makes triv equivariant. In particular, by Remark 3.2.2.3, the 2-cocycle $\mu \in Z^2(I_{S_e^k}, \mathbb{C}^\times)$ is trivial.

There are three distinct $N_{S_e^k}$-orbits of hyperplanes defining $\mathfrak{h}_{reg}^{S_e^k} \subset \mathfrak{h}^{S_e^k}$, given by (1) $x_i = x_j$ for $1 \leq i < j \leq k$, (2) $x_i = y_j$ for $1 \leq i \leq k$ and $1 \leq j \leq n - ek$, and (3) $y_i = y_j$ for $1 \leq i < j \leq n - ek$. The stabilizers in S_n of the $x_i = y_j$ hyperplanes are those parabolic subgroups containing S_e^k and conjugate to $S_e^{k-1} \times S_{e+1}$, and S_e^k is self-normalizing in these groups. The stabilizers of the hyperplanes $x_i = x_j$ are those parabolic subgroups of S_n containing S_e^k and conjugate to $S_{2e} \times S_e^{k-2}$, and the stabilizers of the hyperplanes $y_i = y_j$ are of those parabolic subgroups of S_n containing S_e^k and conjugate to $S_e^k \times S_2$. Note that S_e^k is not self-normalizing in either of these types of parabolic subgroups, and in particular the space $\mathfrak{h}_{L-reg}^{S_e^k}$ is the complement of the hyperplanes $x_i = x_j$ and $y_i = y_j$. It follows already from Theorem 3.3.1.1 that there is an isomorphism of algebras $\mathcal{H}(c, S_e^k, L(\mathrm{triv}), S_n) \cong \mathsf{H}_{q_1}(S_k) \otimes \mathsf{H}_{q_2}(S_{n-ek})$, for some complex parameters $q_1, q_2 \in \mathbb{C}^\times$. To show Theorem 3.3.3.1, it therefore suffices to show that $q_1 = 1$ and $q_2 = e^{-2\pi i c}$.

By Remark 3.2.2.8, the parameter q_1 can be computed by studying the inclusion $S_e^2 \subset S_{2e}$ and the parameter q_2 can be computed by studying the inclusion $1 \subset S_2$. In the latter case, the associated central element $z_{T_1} \in \mathsf{H}_q(1) = \mathbb{C}$ is $1-q = 1-e^{-2\pi i c}$, and therefore by Theorem 3.3.2.11 the associated quadratic relation is $T^2 = (1-q)T + q$, so $q_2 = q = e^{-2\pi i c}$.

To obtain the parameter q_1, we need to analyze the inclusion $S_e^2 \subset S_{2e}$ and the associated element $z_{w_0 w_0'} \in \mathsf{H}_q(S_e^2)$, where w_0 is the longest element of S_{2e} and w_0' is the longest element of S_e^2.

Proposition 3.3.3.2. *The decomposition of the element $T_{w_0 w_0'}^2$ in the T_w-basis of $\mathsf{H}_q(S_{2e})$ is given by*

$$T_{w_0 w_0'}^2 = \sum_{w \in X_e} (1-q)^{a(w)} q^{b(w)} T_w$$

where $X_e \subset S_{2e}$ is the subset of elements $w \in S_{2e}$ such that the three conditions

(1) $w^2 = 1$

(2) $w(i) = i$ or $w(i) > e$ for $1 \leq i \leq e$

(3) $w(i) = i$ or $w(i) \leq e$ for $m < i \leq 2e$,

hold and where the functions $a, b : X_e \to \mathbb{Z}^{\geq 0}$ are defined by

$$a(w) = \#\{i \in [1, e] : w(i) > e\}$$

and

$$b(w) = -\#\{(i,j) : 1 \le i < j \le e, w(i) > w(j)\} + \sum_{i=1}^{e} \begin{cases} e & w(i) = i \\ 2e - w(i) & w(i) > e. \end{cases}$$

In particular, the element $z_{w_0 w_0'} \in \mathsf{H}_q(S_e^2) = \mathsf{H}_q(S_e)^{\otimes 2}$ *is given by*

$$z_{w_0 w_0'} = (1-q)^e q^{\binom{e}{2}} \sum_{w \in S_e} q^{-l(w)} T_w \otimes T_{w^{-1}}.$$

Proof. The expression for $T^2_{w_0 w_0'}$ can be obtained by a simple inductive argument using the reduced expression

$$w_0 w_0' = (s_e \cdots s_{2e-1})(s_{e-1} \cdots s_{2e-2}) \cdots (s_1 \cdots s_e)$$

and the relations defining the Hecke algebra $\mathsf{H}_q(S_{2e})$, from which the expression for $z_{w_0 w_0'}$ follows immediately. □

The image of the Verma module $\Delta_c(\mathrm{triv})$ in $\mathcal{O}_c(S_e^2, \mathfrak{h}_{S_e^2})$ under the KZ functor is the 1-dimensional representation $KZ(\Delta_c(\mathrm{triv}))$ on which all of the generators $T_i \in \mathsf{H}_q(S_e^2)$ act by the identity. In particular, the element $z_{w_0 w_0'}$ acts by the scalar $(1-q)q^{\binom{e}{2}} \sum_{w \in S_e} q^{-l(w)} = (1-q)q^{\binom{e}{2}} P_{S_e}(q^{-1})$, where P_{S_e} is the Poincaré polynomial of S_e. By the well known identity

$$P_{S_e} = \prod_{i=1}^{e} \frac{1-q^i}{1-q}$$

and the fact that $q = e^{-2\pi i c}$ is a primitive e^{th} root of unity, it follows that $z_{w_0 w_0'}$ acts by 0 on $KZ(\Delta_c(\mathrm{triv}))$, and hence it follows that $z_{w_0 w_0'}$ acts by 0 on all simple objects in the block of $\mathsf{H}_q(S_e^2)$ containing $KZ(\Delta_c(\mathrm{triv}))$. In particular, ${}^C z_{w_0 w_0'}(L(\mathrm{triv})) = 0$. Note also that $l(w_0 w_0') = e^2$, so $q^{l(w_0 w_0')} = 1$. By Theorem 3.3.2.11, it follows that $T^2_{S_e^2, L, S_{2e}} = 1$. Therefore, the parameter q_1 is 1, and the isomorphism

$$\mathcal{H}(c, S_e^k, L, S_n) \cong \mathbb{C}[S_k] \otimes \mathsf{H}_q(S_{n-ek})$$

follows, as needed.

3.3.4 Type B

In this section we will illustrate our results and check Theorem 3.3.2.14 in the setting of the type B Coxeter groups. The results we obtain in type B follow from the work of Shan-Vasserot [66].

Recall that the Coxeter group B_n is the semidirect product $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where S_n acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permutation, and that it acts in its reflection representation $\mathfrak{h} = \mathbb{C}^n$ by permutations and sign changes of the coordinates. There are two conjugacy classes of reflections, the first associated to the coordinate hyperplanes $z_i = 0$ and the second associated to the hyperplanes $z_i = \pm z_j$. A class function c on the set of reflections therefore amounts to a choice of parameter $c_1 \in \mathbb{C}$ for the $z_i = 0$ hyperplanes and a choice of parameter $c_2 \in \mathbb{C}$ for the $z_i = \pm z_j$ hyperplanes. Let $H_c(B_n, \mathfrak{h})$ be the associated rational Cherednik algebra. We will choose the set of simple reflections $s_0, s_1, ..., s_{n-1}$ so that s_0 is the reflection through the hyperplane $z_1 = 0$, negating the first coordinate, and so that s_i, for $0 < i < n$, is the reflection through the hyperplane $z_i = z_{i+1}$, transposing the i^{th} and $(i+1)^{st}$ coordinates.

Recall that the irreducible representations of the Coxeter group B_n are naturally labeled by pairs of partitions, or *bipartitions*, $\lambda = (\lambda_1, \lambda_2) \vdash n$ of n (see, for example, [37]). In particular, the simple objects in $\mathcal{O}_c(B_n, \mathfrak{h})$ are also labeled by bipartitions, with the bipartition λ corresponding to the irreducible representation $L(\lambda) := L(V_\lambda)$, where V_λ is the associated irreducible representation of B_n and $L(V_\lambda)$ is the unique simple quotient of the Verma module $\Delta_c(V_\lambda)$ attached to V_λ. The representation theory of $H_c(B_n, \mathfrak{h})$ is much richer than that of $H_c(S_n, \mathfrak{h}_{S_n})$, and in particular the latter algebra may admit many nonisomorphic irreducible finite-dimensional representations.

Any parabolic subgroup of B_n is conjugate to a unique parabolic subgroup of the form $B_l \times S_{n_1} \times \cdots \times S_{n_k}$ for some nonnegative integers $k, l \geq 0$ and positive integers $n_1 \geq \cdots \geq n_k > 0$ with $l + \sum_i n_i \leq n$. By [3, Theorem 1.2], the only such parabolic subgroups whose rational Cherednik algebras admit nonzero finite-dimensional representations are those of the form $B_l \times S_e^k$. Let W' be such a parabolic subgroup, and let L be an irreducible finite-dimensional representation of $H_c(W', \mathfrak{h}_{W'})$. Then L is of the form $L(V_\lambda \otimes \mathrm{triv}_e^{\otimes k})$, where λ is a bipartition of l, V_λ is the associated irreducible representation of B_l, and triv_e denotes the trivial representation of S_e. If $k > 0$ and such a finite-dimensional irreducible representation exists, the parameter c_2 must be of the form r/e for some integer r relatively prime to e.

The fixed space $\mathfrak{h}^{W'} \subset \mathbb{C}^n$ consists of those points $(z_1, ..., z_n)$ such that both $z_i = 0$ for $1 \leq i \leq l$ and also for $0 \leq m < k$ we have $z_{l+1+me+i} = z_{l+1+me+j}$ for $1 \leq i, j \leq e$. For $1 \leq i \leq k$ let x_i denote the coordinate of $z_{l+1+ie+1}$ in $\mathfrak{h}^{W'}$, and for $1 \leq j \leq n-l-ke$ let y_j denote the coordinate z_{j+l+ke}, so that $\mathfrak{h}^{W'}$ is identified with $\mathbb{C}^k \oplus \mathbb{C}^{n-l-ke}$ where the x_i give the standard coordinates for $\mathbb{C}^k$ and the y_j give the standard coordinates for $\mathbb{C}^{n-l-ke}$. The natural complement $N_{W'}$ to W' in its normalizer $N_{B_n}(W')$ is isomorphic to $B_k \times B_{n-l-ek}$ compatibly with the natural reflection representation of the latter group on $\mathbb{C}^k \oplus \mathbb{C}^{n-l-ke}$. In particular, $N_{W'} = N_{W'}^{ref}$. Each parabolic subgroup $W'' \subset B_n$ containing W' in corank 1 is conjugate to a unique parabolic subgroup appearing among the five following cases; the form of the fixed hyperplane $\mathfrak{h}^{W''} \subset \mathfrak{h}^{W'}$ is listed after the parabolic subgroup W'':

(1) $B_{l+k} \times S_e^{k-1}$	$x_i = 0$
(2) $B_l \times S_{2e} \times S_e^{k-2}$	$x_i = \pm x_j$
(3) $B_{l+1} \times S_e^k$	$y_i = 0$
(4) $B_l \times S_e^k \times S_2$	$y_i = \pm y_j$
(5) $B_l \times S_{e+1} \times S_e^{k-1}$	$x_i = \pm y_j$.

The only such parabolic subgroups in which W' is self-normalizing are those of type (5). Furthermore, as the longest element of any Coxeter group of type B acts by -1 on its reflection representation, it follows that endowing $V_\lambda \otimes \mathrm{triv}_e^{\otimes k}$ with the trivial representation of $N_{W'}$ gives equivariant structure to $L(V_\lambda \otimes \mathrm{triv}_e^{\otimes k})$. In particular, $I = N_{W'} = N_{W'}^{ref}$, the cocycle $\mu \in Z^2(I, \mathbb{C}^\times)$ is trivial, and $\mathfrak{h}_{L-reg}^{W'}$ is the complement in $\mathfrak{h}^{W'}$ of the hyperplanes of the forms (1) - (4). In particular, by Theorem 3.3.1.1 we have an isomorphism of algebras

$$\mathcal{H}(c, B_l \times S_e^k, L(V_\lambda \otimes \mathrm{triv}_e^{\otimes k}), B_n) \cong \mathsf{H}_{q_1,q_2}(B_k) \otimes \mathsf{H}_{q_3,q_4}(B_{n-l-ke}),$$

where $q_i \in \mathbb{C}^\times$ is the complex number such that the monodromy operator (appropriately scaled) associated to the hyperplanes of type (i), as listed above, satisfies the quadratic relation $(T-1)(T+q_i) = 0$. Here q_1 and q_3 are associated to the reflections through hyperplanes $x_i = 0$ and $y_i = 0$, respectively, and q_2 and q_4 are associated to the reflections through hyperplanes $x_i = \pm x_j$ and $y_i = \pm y_j$, respectively. In particular, Theorem 3.3.2.14 holds in type B.

By Remark 3.2.2.8 the parameter q_2 associated to the inclusion $B_l \times S_e^k \subset B_l \times S_{2e} \times S_e^{k-2}$ can be computed using the inclusion $S_e^2 \subset S_{2e}$ and the finite-dimensional irreducible representation $L(\mathrm{triv}_e^{\otimes 2})$ of $H_c(S_e^2, \mathfrak{h}_{S_e^2})$. This case was treated in Section 3.3.3, and we have $q_2 = 1$. Similarly, q_4 can by computed using the inclusion $1 \subset S_2$, where S_2 is generated by a reflection in B_n associated to a hyperplane $z_i = \pm z_j$ for any $i, j > l+ke$, giving $q_4 = e^{-2\pi i c_2}$. The computation of the parameters q_1 and q_3 reduce to computing the parameters $q(c, B_k \times S_e, L(V_\lambda \otimes \mathrm{triv}_e), B_{k+e})$ and $q(c, B_k, L(V_\lambda), B_{k+1})$, respectively, which in turn can be computed using Theorem 3.3.2.11. The parameter $q_1 = q(c, B_k \times S_e, L(V_\lambda \otimes \mathrm{triv}_e), B_{k+e})$ was computed explicitly in [59, Theorem 2.12], and we have $q_1 = -(-e^{-2\pi i c_1})^e$, with no dependence on k or λ.

We now explain how to compute the parameter $q_3 = q(c, B_n, L(V_\lambda), B_{n+1})$ from only the parameter c and the bipartition $\lambda \vdash n$. Let $p = e^{-2\pi i c_1}$ and $q = e^{-2\pi i c_2}$, so that $\mathsf{H}_{p,q}(B_n)$ is the Hecke algebra appearing in the KZ functor for $\mathcal{O}_c(B_n, \mathfrak{h}_{B_n})$, where the parameter p is associated with reflections through hyperplanes $z_i = 0$ and q is associated with reflections through hyperplanes $z_i = \pm z_j$. Let w_0 denote the longest element of B_{n+1}, let w_0' denote the longest element of B_n, and let $z_{w_0 w_0'}$ denote the associated central element in $\mathsf{H}_{p,q}(B_n)$. For $1 \leq i \leq n$, let $t_i \in B_n$ denote the reflection $t_i := s_{i-1} \cdots s_1 s_0 s_1 \cdots s_{i-1}$ negating the i^{th} coordinate.

Fix a bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \vdash n$ of n, and for $i = 1, 2$ let $\lambda_1^{(i)} \geq \cdots \geq \lambda_{l_i}^{(i)} > 0$ be the parts of the partition $\lambda^{(i)}$. Recall that we may view λ as a pair of Young diagrams in the following way. Refer to an element $b = (x, y, i) \in \mathbb{Z}^{>0} \times \mathbb{Z}^{>0} \times \{1, 2\}$ as a *box*. A finite subset $Y \subset \mathbb{Z}^{>0} \times \mathbb{Z}^{>0} \times \{1, 2\}$ is called a *Young diagram* if whenever Y contains the box (x, y, i) it also contains all boxes of the form (x', y', i) for positive integers x', y' satisfying $1 \leq x' \leq x$ and $1 \leq y' \leq y$. Let $YD(\lambda) \subset \mathbb{Z}^{>0} \times \mathbb{Z}^{>0} \times \{1, 2\}$ be the Young diagram consisting of those boxes (x, y, i) such that $y \leq l_i$ and $x \leq \lambda_y^{(i)}$. Define the *content*, with respect to the parameters p and q, of a box $b = (x, y, i)$ to be $q^{x-y}p^{-1}$ if $i = 1$ and to be $-q^{x-y}$ if $i = 2$. Denote the content of b by $ct_{p,q}(b)$.

Definition 3.3.4.1. *Given a bipartition $\lambda \vdash n$ of n and parameters $p, q \in \mathbb{C}^\times$ for the Hecke algebra $\mathsf{H}_{p,q}(B_n)$, define the scalar $z_{p,q}(\lambda) \in \mathbb{C}$ by*

$$z_{p,q}(\lambda) := (1 - p)q^n + (1 - q)^2 q^{n-1} p \sum_{b \in YD(\lambda)} ct_{p,q}(b).$$

Remark 3.3.4.2. *Our definition of content differs slightly from the definition of content appearing in [37, Section 10.1.4] because we choose a different convention for the quadratic relations satisfied by the generators T_i of $\mathsf{H}_{p,q}(B_n)$, i.e. that the quadratic relations should be divisible by $(T - 1)$ rather than $(T + 1)$. This is natural from the perspective of the KZ functor. Our algebra $\mathsf{H}_{p,q}(B_n)$ is isomorphic to the algebra $\mathsf{H}_{p^{-1},q^{-1}}(B_n)$ appearing in [37] under the isomorphism induced by the assignments $T_s \mapsto q_s^{-1} T_s$. The inversion of the parameters explains the discrepancy in the definition of content.*

Proposition 3.3.4.3. *In the notation of Lemma 3.3.2.7, $q^{l(w_0 w_0')} = pq^{2n}$, and the central element $z_{w_0 w_0'} \in \mathsf{H}_{p,q}(B_n)$ has the following expansion in the T_w-basis:*

$$z_{w_0 w_0'} = (1 - p)q^n + (1 - q)^2 \sum_{i=1}^{n} q^{n-i} T_{t_i}.$$

Moreover, $z_{w_0 w_0'}$ acts on any irreducible representation of $\mathsf{H}_{p,q}(B_n)$ lying in the block of $\mathsf{H}_{p,q}(B_n)$-mod$_{f.d.}$ corresponding via the KZ functor to the block of $\mathcal{O}_c(B_n, \mathfrak{h}_{B_n})$ containing $L(V_\lambda)$ by the scalar $z_{p,q}(\lambda)$ defined in Definition 3.3.4.1.

Proof. The expressions for $z_{w_0 w_0'}$ and $q^{l(w_0 w_0')}$ follow immediately from standard calculations in $\mathsf{H}_{p,q}(B_{n+1})$ using the reduced expression $s_n \cdots s_1 s_0 s_1 \cdots s_n$ for $w_0 w_0'$, where s_0 is the simple reflection through the hyperplane $z_1 = 0$ and, for $i > 0$, s_i is the simple reflection through the hyperplane $z_i = z_{i+1}$. To show that $z_{w_0 w_0'}$ acts by $z_{p,q}(\lambda)$ on the irreducibles in the block of $\mathsf{H}_{p,q}(B_n)$-mod$_{f.d.}$ appearing in the theorem, it suffices to show that $z_{w_0 w_0'}$ acts by $z_{p,q}(\lambda)$ on $KZ(\Delta_c(V_\lambda))$. By a standard deformation argument, it suffices to prove this for generic parameters p, q, c. For generic parameters, $KZ(\Delta_c(V_\lambda))$ is isomorphic to the irreducible representation $V_\lambda^{p,q}$ of $\mathsf{H}_{p,q}(B_n)$ described in [37, Theorem 10.1.5] in terms of Hoefsmit's matrices, and that $z_{w_0 w_0'}$ acts by the scalar $z_{p,q}(\lambda)$ on $V_\lambda^{p,q}$ then follows immediately from the explicit description of the diagonal action of the elements T_{t_i} on the standard Young tableau basis of $V_\lambda^{p,q}$. $\square$

In particular, by Theorem 3.3.2.11, the canonical generator $T_{B_n, L(V_\lambda), B_{n+1}}$ of the algebra $\mathcal{H}(c, B_n, L(V_\lambda), B_{n+1})$ satisfies the same quadratic relation as the matrix

$$\begin{pmatrix} 0 & pq^{2n} \\ 1 & z_{p,q}(\lambda) \end{pmatrix},$$

i.e.

$$T^2 = z_{p,q}(\lambda) T + pq^{2n}.$$

Rescaling appropriately, one obtains the parameter $q(c, B_n, L(V_\lambda), B_{n+1})$. Note that when q is a primitive e^{th} root of unity, this quadratic relation depends only on the e-cores of the components of λ.

3.3.5 Type D

In this section we will show that Theorem 3.3.2.14 holds in type D and that the study of the generalized Hecke algebras $\mathcal{H}(c, W', L, W)$ when W is of type D largely reduces to the case in which W is of type B.

Recall that for $n \geq 4$ the reflection group D_n of type D and rank n is the subgroup of B_n of index 2 consisting of those elements acting on $\mathbb{C}^n$ with an even number of sign changes. D_n is an irreducible reflection group with reflection representation $\mathbb{C}^n$ in this way, generated by reflections through the hyperplanes $z_i = \pm z_j$ for $1 \leq i < j \leq n$.

If $s_0, ..., s_{n-1}$ are the simple reflections for B_n introduced in the previous section, then the reflections $s_1', s_1, s_2, ..., s_{n-1}$, where $s_1' = s_0 s_1 s_0$ is the reflection through the hyperplane $z_1 = -z_2$, form a system of simple reflections for D_n with respect to which it is a Coxeter group.

The irreducible complex representations of D_n are easily described in terms of those of B_n recalled in the previous section. In particular, when $\lambda = (\lambda^1, \lambda^2)$ is a bipartition of n for which $\lambda^1 \neq \lambda^2$, then the restriction of V_λ to D_n is irreducible, and $V_{(\lambda^1,\lambda^2)}$ and $V_{(\lambda^2,\lambda^1)}$ are isomorphic as representations of D_n. When $\lambda^1 = \lambda^2$, i.e. when the bipartition λ is *symmetric*, the restriction of V_λ to D_n splits as a direct sum of two non-isomorphic irreducible representations V_λ^+ and V_λ^-. All irreducible representations of D_n appear in this way, and the only isomorphisms among these representations are those of the form $V_{(\lambda^1,\lambda^2)} \cong V_{(\lambda^2,\lambda^1)}$.

All reflections in D_n are conjugate, so a parameter for the rational Cherednik algebra of type D is determined by a single number $c \in \mathbb{C}$. It follows immediately from the definition by generators and relations that the rational Cherednik algebra $H_c(D_n, \mathbb{C}^n)$ embeds naturally in the type B algebra $H_{0,c}(B_n, \mathbb{C}^n)$, where in the latter algebra the parameter takes value 0 on reflections through hyperplanes $z_i = 0$ and value c on reflections through hyperplanes $z_i = \pm z_j$. Let $q = e^{-2\pi i c}$ be the parameter for the Hecke algebra $\mathsf{H}_q(D_n)$ whose category of finite-dimensional modules is the target of the KZ functor. Note similarly that the Hecke algebra $\mathsf{H}_q(D_n)$ embeds naturally as a subalgebra of the Hecke algebra $H_{1,q}(B_n)$ compatibly with the T_w bases (see [37, Section 10.4.1]); note that $T_{s_0}^2 = 1$. This embedding is compatible with the KZ functors in the obvious way. It is shown in [68] that when the bipartition λ is symmetric the irreducible representations $L_c(V_\lambda^\pm)$ are always infinite dimensional. In particular, the finite-dimensional irreducible representations of $H_c(D_n, \mathbb{C}^n)$ always extend to irreducible representations of $H_{0,c}(B_n, \mathbb{C}^n)$, although not uniquely.

Suppose $W' \subset D_n$ is a standard parabolic subgroup such that $H_c(W', \mathfrak{h}_{W'})$ admits a finite-dimensional irreducible representation L. The irreducible parabolic subgroups of D_n are of types A and D. We will now describe a procedure for producing a presentation of the algebra $\mathcal{H}(c, W', L, D_n)$ in the form appearing in Theorem 3.3.2.14, and in particular we will see that Theorem 3.3.2.14 holds in type D. Clearly, by tensoring with the sign character of D_n, we may assume $c > 0$.

First suppose that the decomposition of W' into a product of irreducible parabolic subgroups involves no factors of type D. Then by [3, Theorem 1.2] we can assume that $c = r/e$ for positive relatively prime integers $r \geq 1, e \geq 2$, that W' is of the form S_e^k for some integer $k > 0$ such that $ke \leq n$, and that $L = L(\mathrm{triv}_e^{\otimes k})$. It follows from Remark 3.2.2.3 that the inertia group I equals the complement $N_{W'}$ to W' in $N_{D_n}(W')$ and that the cocycle $\mu \in Z^2(N_{W'}, \mathbb{C}^\times)$ is trivial. In particular, Theorem 3.3.2.14 holds in this case. A detailed description of the group $N_{W'}$ and its maximal reflection subgroup $N_{W'}^{ref}$ (typically a proper subgroup of $N_{W'}$) may be found in [47]. As usual, the parameter $q_{W',L}$ associated to the Hecke algebra $\mathsf{H}_{q_{W',L}}(N_{W'}^{ref})$ can be computed using Theorem 3.3.2.11. The parabolic subgroups $W'' \subset D_n$ containing W' in corank 1 and in which W' is not self-normalizing are of the form (1) $S_e^k \times S_2$, (2) $S_e^{k-2} \times S_{2e}$, (3) (in the case $e = 2$) $S_2^{k-3} \times D_4$, and (4) (in the case $e = 4$) $S_4^{k-1} \times D_4$. By Remark 3.2.2.8, parameter computations in these cases reduce to the cases, respectively, (1) $1 \subset S_2$, (2) $S_e^2 \subset S_{2e}$, (3) $S_2^3 \subset D_4$, and (4) $S_4 \subset D_4$. As discussed in Section 3.3.3 about type A, the quadratic relation in case (1) is $(T-1)(T+q) = 0$ with $q = e^{-2\pi i c}$, and the quadratic relation in case (2) is $T^2 = 1$. To compute the quadratic relation in case (3), we use Theorem 3.3.2.11 again. In particular, letting w_0 denote the longest element of D_4 and w_0' the longest element of S_2^3, we have $q^{l(w_0 w_0')} = (-1)^9 = -1$, and computations in the computer algebra package CHEVIE in GAP3 [36, 58] show that the central element $z_{w_0 w_0'}$ acts on the trivial representation of $H_{-1}(S_2)^{\otimes 3}$ by the scalar 2. By Theorem 3.3.2.11 the quadratic relation appearing in case (3) is $T^2 = 2T - 1$, i.e. $(T-1)^2 = 0$. Similarly, one obtains the quadratic relation $(T-1)^2 = 0$ in remaining case (4) as well.

Now, consider the remaining case in which the decomposition of W' into a product of irreducible parabolic subgroups involves a factor of type D. Then again by [3, Theorem 1.2] we can assume that W' is conjugate to a parabolic subgroup of the form $D_l \times S_e^k$ for some integers $e \geq 2, l \geq 4, k \geq 0$ such that $l + ke \leq n$, and the finite-dimensional irreducible representation L is isomorphic to $L_c(V_\lambda \otimes \mathrm{triv}_e^{\otimes k})$

for some bipartition $\lambda = (\lambda^1, \lambda^2)$ with $\lambda^1 \neq \lambda^2$. The parameter c must again be of the form $c = r/e$ for some positive integer r relatively prime to e. Furthermore, it follows from [55, Lemma 4.2, Corollary 4.3] that $H_c(D_l, \mathbb{C}^l)$ admits no nonzero finite-dimensional representations when e is odd, so we may assume that e is even. To see this, consider a parameter $c = r/e$ for the rational Cherednik algebra $H_c(D_n, \mathbb{C}^n)$, where $e > 2$ is an odd positive integer and r an integer relatively prime to e. The algebra $H_c(D_n, \mathbb{C}^n)$ admits nonzero finite-dimensional representations if and only if the algebra $H_{0,c}(B_n, \mathbb{C}^n)$ admits nonzero finite-dimensional representations. In the notation from [55, Section 4], in this case we have $\kappa = -r/e$ and $(s_1, s_2) = (0, -\frac{e}{2r})$. Indexes 1 and 2 are not equivalent under the equivalence relation $\sim_{(0,c)}$ as we have $s_2 - s_1 = -\frac{e}{2r} \notin \frac{1}{r}\mathbb{Z} = \kappa^{-1}\mathbb{Z} + \mathbb{Z}$, so by [55, Lemma 4.2] the category $\mathcal{O}_{(0,c)}(B_n, \mathbb{C}^n)$ decomposes as a direct sum of outer tensor products of categories $\mathcal{O}$ associated to reflection groups S_k with reflection representation $\mathbb{C}^k$, for various k, in a manner preserving supports [55, Corollary 4.3]. As the rational Cherednik algebras associated to the reducible reflection representations $(S_k, \mathbb{C}^k)$ have no nonzero finite-dimensional representations for any parameter values, it follows that the rational Cherednik algebra $H_{r/e}(D_n, \mathbb{C}^n)$ also has no nonzero finite-dimensional representations. We will therefore assume that $e > 1$ is a positive even integer.

As the fixed space of W' equals the fixed space of the parabolic subgroup $B_l \times S_e^k$ of $B_n \supset D_n$, it follows that $N_{D_n}(W') = D_n \cap N_{B_n}(B_l \times S_e^k)$. As $\lambda_1 \neq \lambda_2$, the representation $V_\lambda \otimes \mathrm{triv}_e^{\otimes k}$ of W' extends to a representation of $B_l \times S_e^k$, and we've seen in Section 3.3.4 that such a representation extends to a representation of $N_{B_n}(B_l \times S_e^k)$. In particular, it follows that the inertia group I is maximal, i.e. equals $N_{W'}$, and that the cocycle $\mu \in Z^2(N_{W'}, \mathbb{C}^\times)$ is trivial, so Theorem 3.3.2.14 holds in this remaining case in type D. As discussed in "Case 1" of the section "Type D" of [47], in this case $N_{W'}$ equals $N_{W'}^{ref}$ and is isomorphic to $B_k \times B_{n-l-ke}$ as a reflection group acting on $(\mathbb{C}^n)^{W'} \cong \mathbb{C}^k \oplus \mathbb{C}^{n-l-ke}$ in a manner completely analogous to the discussion in Section 3.3.4. In particular, in this case we have

$$\mathcal{H}(c, D_l \times S_e^k, L_c(V_\lambda \otimes \mathrm{triv}_e^{\otimes k}), D_n) \cong \mathsf{H}_{q_1,1}(B_k) \otimes \mathsf{H}_{q_2,q}(B_{n-l-ke})$$

where q_1 and q_2 are associated to the short roots of B_k and B_{n-l-ke}, respectively, $q = e^{-2\pi ic}$, $q_1 = q(c, D_l \times S_e, L_c(V_\lambda \otimes \mathrm{triv}_e), D_{l+e})$, and $q_2 = q(c, D_l, L_c(V_\lambda), D_{l+1})$. The following result reduces the computation of these parameters to the type B:

Proposition 3.3.5.1. *In the setting of the previous paragraph, we have*

$$q(c, D_n, L_c(V_\lambda), D_{n+1}) = q((0, c), B_n, L_{(0,c)}(V_\lambda), B_{n+1})$$

and

$$q(c, D_n \times S_e, L_c(V_\lambda \otimes triv_e), D_{n+e}) = q((0, c), B_n \times S_e, L_{(0,c)}(V_\lambda \otimes triv_e), B_{n+e}).$$

Proof. We consider $q(c, D_n, L_c(V_\lambda), D_{n+1})$ first. Let l_D denote the length function on D_{n+1} with respect to the simple reflections $s_1, s_1', ..., s_n$ introduced above, and let l_B denote the length function on B_{n+1} with respect to the simple reflections $s_0, s_1, ..., s_n$. Let $w_{0,D}, w_{0,B}, w_{0,D}'$, and $w_{0,B}'$ denote the longest elements of the Coxeter groups D_{n+1}, B_{n+1}, D_n and B_n, respectively. Then $w_{0,D}w_{0,D}' = s_0 w_{0,B} w_{0,B}' = w_{0,B} w_{0,B}' s_0$. Regard $\mathsf{H}_q(D_n)$ as a subalgebra of $H_{1,q}(B_n)$ via the T_w-bases. We then have $T_{w_{0,D}w_{0,D}'} = T_{s_0} T_{w_{0,B}w_{0,B}'} = T_{w_{0,B}w_{0,B}'} T_{s_0}$ and $T_{w_0'}^2 = 1$. In particular, $T_{w_{0,D}w_{0,D}'}^2 = T_{w_{0,B}w_{0,B}'}^2$ and it follows from Proposition 3.3.4.3 and the definitions that $z_{w_{0,D}w_{0,D}'} = T_{s_0} z_{w_{0,B}w_{0,B}'} = z_{w_{0,B}w_{0,B}'} T_{s_0}$. As the representation V_λ extends to B_n, we can choose the operator by which $w_{0,B}w_{0,B}'$ acts on L as in Theorem 3.3.2.11 to be T_{s_0}. By Theorem 3.3.2.11, the generator of monodromy $T \in \mathcal{H}(c, D_n, L_c(V_\lambda), D_{n+1})$ satisfies the quadratic relation

$$T^2 = (T_0 z_{w_{0,D}w_{0,D}'})|_L T + q^{l_D(w_{0,D}w_{0,D}')} = z_{1,q}(\lambda) T + q^{2n}.$$

This is precisely the quadratic relation obtained for the generator of monodromy generating the algebra $T \in \mathcal{H}(c, D_n, L_c(V_\lambda), D_{n+1})$, as shown in Section 3.3.4, and the first equality follows.

The second equality follows by a similar argument. $\square$

3.3.6 Parameters for Generalized Hecke Algebras in Exceptional Types

We will now describe the parameters arising for the generalized Hecke algebras in exceptional type. In each row of the following Table 3.1, W is an irreducible finite Coxeter group and $W' \subset W$ is a corank-1 parabolic subgroup of W that is not self-normalizing in W. The complex number c is a parameter for the rational Cherednik algebra $H_c(W, \mathfrak{h})$ such that $H_c(W', \mathfrak{h}_{W'})$ admits nontrivial finite-dimensional representations, and λ is an irreducible representation of W' such that $L(\lambda)$ is a finite-dimensional irreducible representation of $H_c(W', \mathfrak{h}_{W'})$. For a given W', all c of the form $1/d$ such that $H_c(W', \mathfrak{h}_{W'})$ admits a finite-dimensional irreducible representation are given, and for each W', c a complete list of lowest weights λ with $\dim L(\lambda) < \infty$ is given. Finally, $q(c, W', L, W)$ is a complex number such that the monodromy operator T associated to the tuple $(c, W', L(\lambda), W)$, after an appropriate rescaling if necessary, satisfies the quadratic relation

$$(T - 1)(T + q(c, W', L, W)) = 0.$$

Where appropriate, $q(c, W', L, W)$ is given as a power of the "KZ parameter" $q = e^{-2\pi i c}$. Table 3.1 includes every case needed to give presentations for the generalized Hecke algebras arising in types E, H and I; this data was obtained by using Theorem 3.3.2.11 and computations with the computer algebra package CHEVIE in GAP3 [36, 58] as well as SAGE. Type F can be handled by these same methods, although the description of the relevant parameters c and irreducible finite-dimensional representations for the parabolic subgroups of types B and C arising in this case is more complicated to display in a table. We will give the counts of modules of given support in $\mathcal{O}_c(F_4, \mathfrak{h})$ in the unequal parameter case later in Section 3.3.9.

In Table 3.1 and below, we will list the parameter for groups B_n in the form $(c_1, c_2) \in \mathbb{C}^2$, where c_1 specifies the value of the parameter on the short roots. In the last row of the table, the parameter $(1/2, c_2)$ for the even dihedral group $I_2(2m)$ indicates that the parameter takes value $1/2$ on those reflections conjugate to the nontrivial element of the chosen parabolic subgroup A_1 and arbitrary value $c_2 \in \mathbb{C}$ on the remaining parameters. The relevant parameter values and lists of finite-dimensional irreducible representations for the groups of type D are obtained by a standard reduction to type B (as in Section 3.3.5), where these lists are easily produced using the methods of [55]. The labeling used for irreducible representations of the exceptional groups is compatible with that appearing in [45]; in particular, we denote the trivial representation by triv, the reflection representation by V (and its Galois conjugate in type H by $\widetilde{V}$), and other representations are denoted in the form $\varphi_{x,y}$ where x indicates the dimension of the representation and y indicates its b-invariant, i.e. the lowest degree in the grading of the coinvariant algebra in which the representation appears. The labels $\varphi_{x,y}$ are compatible with the labels appearing in the GAP3 computer algebra package. This is a different labeling system than appears in some standard references, e.g. [37], although it is simple to convert between this labeling system and others using the tables appearing in [37, Appendix C]. Irreducible representations of groups of type D are labeled by (unordered) pairs of partitions, in the standard way.

Remark 3.3.6.1. *In all cases listed except the case of E_7 at parameter $1/10$, the associated monodromy operator T has an eigenvalue equal to 1, and in particular no rescaling was needed to list the parameter $q(c, W', L(\lambda), W)$; the monodromy operator T associated to irreducible representation $L_{1/10}(V)$ of $H_{1/10}(E_7, \mathfrak{h})$ associated to the inclusion $E_7 \subset E_8$ satisfies the quadratic relation $(T + e^{\pi i/5})^2 = 0$, which is of the form $(T-1)^2 = 0$ after rescaling T. In the cases in which T has an eigenvalue equal to 1, the parameter $q(c, W', L, W)$ is necessarily equal to $q^{l(w_0 w_0')}$, where q is the parameter appearing in the relevant KZ functor and $q^{l(w_0 w_0')}$ is as in Theorem 3.3.2.11, and this covers all other cases in the table. We remark that there are other cases, not relevant for the exceptional groups, in which T does not have an eigenvalue equal to 1 before rescaling; for example, $L_{(-1/6,1/3)}(\mathrm{triv})$ is a finite-dimensional irreducible representation of $H_{(-1/6,1/3)}(B_2)$ and the quadratic relation associated to the inclusion $B_2 \subset B_3$ is $(T+p)^2 = 0$, where $p = e^{-\pi i/3}$.*

Remark 3.3.6.2. *In all cases we have computed, the parameter $q(c, W', L, W)$ depends only on c, W', and W, and notably not on the finite-dimensional irreducible representation L. This fact is reflected in Table 3.1, where we list all relevant lowest weights λ for each pair (W', c) in the same row. It would be interesting to have a conceptual explanation for this fact.*

Table 3.1: Parameters for Generalized Hecke Algebras

W'	c	λ	W	$q(c, W', L(\lambda), W)$
$A_n \times A_n$	$1/(n+1)$	triv	A_{2n+1}	1
A_1^3	$1/2$	triv	D_4	-1
A_3	$1/4$	triv	D_4	-1
D_4	$1/6$	triv	D_5	q^2
D_4	$1/4$	triv	D_5	1
D_4	$1/2$	triv, $(3,1)$	D_5	1
A_5	$1/6$	triv	D_6	-1
D_5	$1/8$	triv	D_6	q^2
$D_4 \times A_1$	$1/2$	triv $\otimes$ triv, $(3,1) \otimes$ triv	D_6	-1
A_3^2	$1/4$	triv	D_7	-1
D_6	$1/10$	triv	D_7	q^2
D_6	$1/6$	triv	D_7	1
D_6	$1/2$	triv, $(0,3^2), (1,5), (2,4)$	D_7	1
A_5	$1/6$	triv	E_6	-1
A_6	$1/7$	triv	E_7	1
D_6	$1/10$	triv	E_7	q^3
D_6	$1/6$	triv	E_7	-1
D_6	$1/2$	triv, $(0,3^2), (1,5), (2,4)$	E_7	-1
E_6	$1/12$	triv	E_7	q^3
E_6	$1/9$	triv	E_7	1
E_6	$1/6$	triv, V	E_7	-1

Table 3.1: (continued)

E_6	$1/3$	$\mathrm{triv}, V, \Lambda^2 V$	E_7	1
A_7	$1/8$	triv	E_8	-1
D_7	$1/12$	triv	E_8	-1
D_7	$1/4$	$\mathrm{triv}, (2,5)$	E_8	-1
E_7	$1/18$	triv	E_8	q^3
E_7	$1/14$	triv	E_8	q
E_7	$1/10$	V	E_8	-1
E_7	$1/6$	$\mathrm{triv}, V, \varphi_{15,7}, \varphi_{21,6}$	E_8	-1
E_7	$1/2$	$\mathrm{triv}, V, \varphi_{15,7}, \varphi_{21,6}$		
		$\varphi_{27,2}, \varphi_{35,13}, \varphi_{189,5}$	E_8	-1
A_2	$(c_1, 1/3)$	triv	B_3	$e^{-6\pi i c_1}$
A_1^2	$(1/2, 1/2)$	triv	B_3	-1
A_2	$1/3$	triv	H_3	1
$A_1 \times A_1$	$1/2$	triv	H_3	-1
$I_2(5)$	$1/5$	triv	H_3	1
H_3	$1/10$	triv	H_4	-1
H_3	$1/6$	triv	H_4	-1
H_3	$1/2$	$\mathrm{triv}, V, \widetilde{V}$	H_4	-1
A_3	$1/4$	triv	H_4	-1
A_1	$(1/2, c_2)$	triv	$I_2(2m)$	$(-1)^{m-1} e^{-2mc_2\pi i}$

3.3.7 Type E

Generalized Hecke Algebras for E_6

The following table list all of the generalized Hecke algebras arising from the rational Cherednik algebra $H_c(E_6, \mathfrak{h}_{E_6})$ of type E_6 for parameters of the form $c = 1/d$ such that $\mathcal{O}_c(E_6, \mathfrak{h}_{E_6})$ is not semisimple, i.e. for those integers $d > 1$ dividing one of the fundamental degrees $2, 5, 6, 8, 9$ and 12 of E_6. The first column indicates the parameter value c. The second column, labeled W', lists a unique representative of each conjugacy class of parabolic subgroups of E_6 for which a nonzero finite-dimensional representation appears at the parameter value specified in the title of the table; if a conjugacy class is missing, no nonzero finite-dimensional irreducible representations exist for that class. The column labeled λ gives a complete list of the lowest weights λ of the finite-dimensional irreducible representations $L_c(\lambda)$ of $H_c(W', \mathfrak{h}_{W'})$. By inspection, we see that in each case the inertia group is maximal, the 2-cocycle μ is trivial, and in particular Theorem 3.3.2.14 holds for algebras of type E_6. Furthermore, from Table 3.1 we see that the parameters for the generalized Hecke algebra does not depend on the choice of lowest weight for the finite-dimensional representation; therefore in the column labeled $\mathcal{H}$ we give the generalized Hecke algebra $\mathcal{H}(c, W', \lambda, E_6)$ common to each of the λ appearing in a given row. In the final column labeled #Irr, we give the number of irreducible representations with support labeled by W', obtained as the product of the number of λ appearing in a given row with the number of irreducible representations of the corresponding generalized Hecke algebra. As there are 25 irreducible representations of the group E_6, these numbers add to 25 for each parameter value.

Throughout, q denotes the "KZ parameter" $q := e^{-2\pi i c}$. The exact descriptions of the generalized Hecke algebras follow easily from parameters in Table 3.1 and Howlett's detailed descriptions of the groups $N_{W'}^{ref}$ and $N_{W'}^{comp}$ appearing in [47]; any semidirect products appearing in the description of the algebras $\mathcal{H}$ are given by the diagram automorphisms indicated in [47]. By convention, we list the parameters of 2-parameter Hecke algebras by giving the parameter for the short roots first, e.g. $\mathsf{H}_{p,q}(B_3)$ indicates that parameter p is associated with the 3 reflections given by short roots and that parameter q is associated with the remaining 6 reflections given by long roots.

The finite-dimensional irreducible representations of $H_c(E_6, \mathfrak{h}_{E_6})$, if they exist, appear in the rows labeled by E_6. For all parameters except $c = 1/2$, the list of the lowest weights of the finite-dimensional irreducible representations of $H_c(E_6, \mathfrak{h}_{E_6})$ is obtained from results of Norton [61]. For $c = 1/2$, our table shows that there are no such finite-dimensional irreducible representations.

Table 3.2: Refined Filtration by Supports for E_6

c	W'	λ	$\mathcal{H}$	#Irr
$1/12$	1	triv	$\mathsf{H}_q(E_6)$	24
	E_6	triv	$\mathbb{C}$	1
$1/9$	1	triv	$\mathsf{H}_q(E_6)$	24
	E_6	triv	$\mathbb{C}$	1
$1/8$	1	triv	$\mathsf{H}_q(E_6)$	24
	D_5	triv	$\mathbb{C}$	1
$1/6$	1	triv	$\mathsf{H}_q(E_6)$	20
	D_4	triv	$\mathsf{H}_{q^2}(A_2)$	2
	A_5	triv	$\mathsf{H}_{-1}(A_1)$	1
	E_6	triv, V	$\mathbb{C}$	2
$1/5$	1	triv	$\mathsf{H}_q(E_6)$	23
	A_4	triv	$\mathsf{H}_q(A_1)$	2
$1/4$	1	triv	$\mathsf{H}_q(E_6)$	19
	A_3	triv	$\mathsf{H}_{-1,q}(B_2)$	3
	D_4	triv	$\mathsf{H}_1(A_2)$	3
$1/3$	1	triv	$\mathsf{H}_q(E_6)$	13
	A_2	triv	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathsf{H}_q(A_2^2)$	5
	A_2^2	triv	$\mathsf{H}_{1,q}(G_2)$	4
	E_6	$\mathrm{triv}, V, \Lambda^2 V$	$\mathbb{C}$	3
$1/2$	1	triv	$\mathsf{H}_q(E_6)$	8
	A_1	triv	$\mathsf{H}_q(A_5)$	4
	A_1^2	triv	$\mathsf{H}_{1,-1}(B_3)$	4
	A_1^3	triv	$\mathsf{H}_{-1}(A_1) \otimes \mathsf{H}_1(A_2)$	3
	D_4	$\mathrm{triv}, (3,1)$	$\mathsf{H}_1(A_2)$	6

Generalized Hecke Algebras for E_7

In this section we produce a table for E_7 analogous to Table 3.2, following the same conventions. Again, we only list those parameters $c = 1/d$ for positive integers $d > 1$ dividing a fundamental degree of E_7 - the degrees of E_7 are $2, 6, 8, 10, 12, 14$, and 18. The group E_6 has 60 irreducible representations. By inspection, we see that Theorem 3.3.2.14 holds in type E_7.

Note that there are two distinct conjugacy classes of parabolic subgroups of E_7 of type A_5, while inside E_6 and E_8 there is only 1. We denote a representative of the conjugacy class of parabolic subgroups of type A_5 appearing already in E_6 by A_5', and we denote a representative of the remaining conjugacy class by A_5''.

There are also two distinct conjugacy classes of parabolic subgroups of E_7 of type A_1^3, although one of these is absent in Howlett's table [47]. The two classes can be distinguished by containment in parabolic subgroups of type D_4; we denote the class of parabolic subgroups of type A_1^3 contained in a parabolic subgroup of D_4 by $(A_1^3)'$ and the remaining class by $(A_1^3)''$. The class $(A_1^3)'$ is treated in Howlett's paper; the complement $N_{(A_1^3)'}$ to $(A_1^3)'$ in its normalizer in E_7 acts on $\mathfrak{h}_{(A_1^3)'}$ as a reflection group of type $B_3 \times A_1$. In the remaining case $(A_1^3)''$, the complement in the normalizer acts in the fixed space as a reflection group of type F_4. This can be seen as follows. Computations in GAP3 [36, 58] show that this complement has order 1152. This group has a decomposition as a semidirect product $N^{comp} \ltimes N^{ref}$, where N^{ref} is a real reflection group of rank at most 4 and N^{comp} is a finite group acting on N^{ref} by diagram automorphisms. From the classification of finite reflection groups, the only possibilities giving rise to groups of order 1152 are $N^{ref} = F_4$ and $N^{comp} = 1$, or $N^{ref} = D_4$ and $N^{comp} = S_3$ where S_3 acts on D_4 by the full group of diagram automorphisms. As $(A_1^3)''$ is contained in parabolic subgroups of different types A_1^4 and $A_1 \times A_3$ in which it is not self-normalizing, it follows that the hyperplanes in the reflection representation of N^{ref} cannot all be conjugate. This rules out $N^{ref} = D_4$, and we conclude that the representation of the complement in the space $\mathfrak{h}_{(A_1^3)''}$ is identified with the reflection representation of F_4.

When the denominator of c is greater than 2, the lowest weights listed for the finite-dimensional irreducible representations of E_7 were determined by Norton [61], and we list those representations in the table below. When the denominator of c equals 2, we see that there are exactly 7 isomorphism classes of finite-dimensional irreducible representations of $H_c(E_7, \mathfrak{h})$. In [45], all but 7 possible lowest weights for these irreducible representations were ruled out. In particular, these 7 representations are in fact finite-dimensional, and this completes the classification of finite-dimensional irreducible representations of the rational Cherednik algebras of type E_7.

Table 3.3: Refined Filtration by Supports for E_7

c	W'	λ	$\mathcal{H}$	#Irr
1/18	1	triv	$\mathsf{H}_q(E_7)$	59
	E_7	triv	$\mathbb{C}$	1
1/14	1	triv	$\mathsf{H}_q(E_7)$	59
	E_7	triv	$\mathbb{C}$	1
1/12	1	triv	$\mathsf{H}_q(E_7)$	58
	E_6	triv	$\mathsf{H}_{q^3}(A_1)$	2
1/10	1	triv	$\mathsf{H}_q(E_7)$	57
	D_6	triv	$\mathsf{H}_{q^3}(A_1)$	2
	E_7	V	$\mathbb{C}$	1
1/9	1	triv	$\mathsf{H}_q(E_7)$	58
	E_6	triv	$\mathsf{H}_1(A_1)$	2
1/8	1	triv	$\mathsf{H}_q(E_7)$	56
	D_5	triv	$\mathsf{H}_{q^2}(A_1) \otimes \mathsf{H}_q(A_1)$	4
1/7	1	triv	$\mathsf{H}_q(E_7)$	58
	A_6	triv	$\mathsf{H}_1(A_1)$	2
1/6	1	triv	$\mathsf{H}_q(E_7)$	43
	D_4	triv	$\mathsf{H}_{q,q^2}(B_3)$	6
	A_5	triv	$\mathsf{H}_{-1}(A_1^2)$	1
	A_5'	triv	$\mathsf{H}_{-1,q}(G_2)$	3

	D_6	triv	$\mathsf{H}_{-1}(A_1)$	1
	E_6	triv, V	$\mathsf{H}_{-1}(A_1)$	2
	E_7	triv, $V, \varphi_{15,7}, \varphi_{21,6}$	$\mathbb{C}$	4
1/5	1	triv	$\mathsf{H}_q(E_7)$	54
	A_4	triv	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathsf{H}_q(A_2)$	6
1/4	1	triv	$\mathsf{H}_q(E_7)$	40
	A_3	triv	$\mathsf{H}_{-1,q}(B_3) \otimes \mathsf{H}_q(A_1)$	10
	D_4	triv	$\mathsf{H}_{q,1}(B_3)$	10
1/3	1	triv	$\mathsf{H}_q(E_7)$	32
	A_2	triv	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathsf{H}_q(A_5)$	14
	A_2^2	triv	$\mathsf{H}_{1,q}(G_2) \otimes \mathsf{H}_1(A_1)$	8
	E_6	triv, $V, \Lambda^2 V$	$\mathsf{H}_1(A_1)$	6
1/2	1	triv	$\mathsf{H}_{-1}(E_7)$	12
	A_1	triv	$\mathsf{H}_{-1}(D_6)$	6
	A_1^2	triv	$\mathsf{H}_{1,-1}(B_4) \otimes \mathsf{H}_{-1}(A_1)$	6
	$(A_1^3)'$	triv	$\mathsf{H}_{-1,1}(B_3) \otimes \mathsf{H}_{-1}(A_1)$	3
	$(A_1^3)''$	triv	$\mathsf{H}_{1,-1}(F_4)$	9
	A_1^4	triv	$\mathsf{H}_{-1,1}(B_3)$	3
	D_4	triv, $(3,1)$	$\mathsf{H}_{-1,1}(B_3)$	6
	$D_4 \times A_1$	triv, $(3,1) \otimes$ triv	$\mathsf{H}_{-1,1}(B_2)$	4
	D_6	triv, $(0,3^2), (1,5), (2,4)$	$\mathsf{H}_{-1}(A_1)$	4
	E_7	triv, $V, \varphi_{15,7}, \varphi_{21,6}$		
		$\varphi_{27,2}, \varphi_{35,13}, \varphi_{189,5}$	$\mathbb{C}$	7

Generalized Hecke Algebras for E_8

Next we produce a table describing the generalized Hecke algebras arising from E_8. Again, we only list those parameters $c = 1/d$ for positive integers $d > 1$ dividing one of the fundamental degrees $2, 8, 12, 14, 18, 20, 24$, and 30 of E_8. There are 112 irreducible representations of the group E_8. By inspection, we see that Theorem

3.3.2.14 holds in type E_8 as well.

When the denominator of c is 2, we see that there are 12 isomorphism classes of finite-dimensional irreducible representations. Comparing with the lists of "potential" lowest weights appearing in [45], we are again able to give a complete list of the finite-dimensional irreducible representations in his case. By similar comparisons with the results of [45], we are able to obtain such lists in all cases except when the denominator of c is 3, 4 or 18. In those cases, we give the list of "potential" lowest weights from [45] and the number of those which are in fact finite-dimensional. In each of these three cases, we see that exactly one of these "potential" finite-dimensional representations is in fact infinite-dimensional. Furthermore, this problem is resolved fairly easily when the denominator of c is not 3. For denominator 18, Rouquier [63] proved that the representation $L_{1/18}(V)$ is finite-dimensional, and therefore the remaining "potential" finite-dimensional representation $L_{1/18}(\varphi_{28,8})$, is in fact infinite-dimensional. That $L_{1/18}(\varphi_{28,8})$ is infinite-dimensional can be seen independently by the observation that its lowest eu-weight, 2/3, is not a nonpositive integer. When the denominator of c is 4, the entire decomposition matrix for the block of $\mathcal{O}_{1/4}(E_8, \mathfrak{h})$ containing the simple object $L_{1/4}(\varphi_{28,8})$ can be easily produced following methods of Norton [61, Lemmas 3.5, 3.6], yielding the equality

$$[L_{1/4}(\varphi_{28,8})]$$

$$= [\Delta_{1/4}(\varphi_{28,8})] - [\Delta_{1/4}(\varphi_{700,16})] + [\Delta_{1/4}(\varphi_{1344,19})] - [\Delta_{1/4}(\varphi_{700,28})] + [\Delta_{1/4}(\varphi_{28,68})]$$

in the Grothendieck group of $\mathcal{O}_{1/4}(E_8, \mathfrak{h})$. It follows that $\mathrm{Supp}(L_{1/4}(\varphi_{28,8})) = E_8\mathfrak{h}^{D_4}$ and in particular that $L_{1/4}(\varphi_{28,8})$ is infinite-dimensional, ruling out this "potential" finite-dimensional representation.

Table 3.4: Refined Filtration by Supports for E_8

c	W'	λ	$\mathcal{H}$	#Irr
1/30	1	triv	$\mathsf{H}_q(E_8)$	111
	E_8	triv	$\mathbb{C}$	1
1/24	1	triv	$\mathsf{H}_q(E_8)$	111

Table 3.4: continued

	E_8	triv	$\mathbb{C}$	1
$1/20$	1	triv	$\mathsf{H}_q(E_8)$	111
	E_8	triv	$\mathbb{C}$	1
$1/18$	1	triv	$\mathsf{H}_q(E_8)$	109
	E_7	triv	$\mathsf{H}_{q^3}(A_1)$	2
	E_8	V	$\mathbb{C}$	1
$1/15$	1	triv	$\mathsf{H}_q(E_8)$	110
	E_8	triv, V	$\mathbb{C}$	2
$1/14$	1	triv	$\mathsf{H}_q(E_8)$	110
	E_7	triv	$\mathsf{H}_q(A_1)$	2
$1/12$	1	triv	$\mathsf{H}_q(E_8)$	102
	E_6	triv	$\mathsf{H}_{q^3,q}(G_2)$	5
	D_7	triv	$\mathsf{H}_{-1}(A_1)$	1
	E_8	triv, $\varphi_{28,8}, \varphi_{35,2}, \varphi_{50,8}$	$\mathbb{C}$	4
$1/10$	1	triv	$\mathsf{H}_q(E_8)$	104
	D_6	triv	$\mathsf{H}_{q^2,q^3}(B_2)$	4
	E_7	V	$\mathsf{H}_{-1}(A_1)$	1
	E_8	triv, $V, \varphi_{28,8}$	$\mathbb{C}$	3
$1/9$	1	triv	$\mathsf{H}_q(E_8)$	106
	E_6	triv	$\mathsf{H}_{1,q}(G_2)$	6
$1/8$	1	triv	$\mathsf{H}_q(E_8)$	100
	D_5	triv	$\mathsf{H}_{q^2,q}(B_3)$	9
	A_7	triv	$\mathsf{H}_{-1}(A_1)$	1
	E_8	triv, $\varphi_{160,7}$	$\mathbb{C}$	2
$1/7$	1	triv	$\mathsf{H}_q(E_8)$	108
	A_6	triv	$\mathsf{H}_{1,q}(A_1^2)$	4
$1/6$	1	triv	$\mathsf{H}_q(E_8)$	75
	D_4	triv	$\mathsf{H}_{q^2,q}(F_4)$	13

Table 3.4: continued

	A_5	triv	$\mathsf{H}_{-1,q}(G_2) \otimes \mathsf{H}_{-1}(A_1)$	3
	D_6	triv	$\mathsf{H}_{1,-1}(B_2)$	2
	E_6	triv, V	$\mathsf{H}_{-1,q}(G_2)$	6
	E_7	triv, $V, \varphi_{15,2}, \varphi_{21,6}$	$\mathsf{H}_{-1}(A_1)$	4
	E_8	triv, $V, \varphi_{28,8}, \varphi_{35,2}, \varphi_{50,8}$		
		$\varphi_{56,19}, \varphi_{175,12}, \varphi_{300,8}, \varphi_{210,4}$	$\mathbb{C}$	9
1/5	1	triv	$\mathsf{H}_q(E_8)$	96
	A_4	triv	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathsf{H}_q(A_4)$	12
	E_8	triv, $V, \varphi_{28,8}, \varphi_{56,19}$	$\mathbb{C}$	4
1/4	1	triv	$\mathsf{H}_q(E_8)$	69
	A_3	triv	$\mathsf{H}_{-1,q}(B_5)$	14
	D_4	triv	$\mathsf{H}_{1,q}(F_4)$	20
	A_3^2	triv	$\mathsf{H}_{1,-1}(B_2)$	2
	D_7	triv, $(2,5)$	$\mathsf{H}_{-1}(A_1)$	2
	E_8	triv, $\varphi_{35,2}, \varphi_{50,8}$		
		$\varphi_{210,4}, \varphi_{350,14}$	$\mathbb{C}$	5
1/3	1	triv	$\mathsf{H}_q(E_8)$	52
	A_2	triv	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathsf{H}_q(E_6)$	26
	A_2^2	triv	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathsf{H}_{1,q}(G_2)^{\otimes 2}$	14
	E_6	triv, $V, \Lambda^2 V$	$\mathsf{H}_{1,q}(G_2)$	12
	E_8	exactly 8 of triv, V		
		$\varphi_{28,8}, \varphi_{35,2}, \varphi_{50,8}, \varphi_{160,7}$		
		$\varphi_{175,12}, \varphi_{300,8}, \varphi_{840,13}$	$\mathbb{C}$	8
1/2	1	triv	$\mathsf{H}_{-1}(E_8)$	23
	A_1	triv	$\mathsf{H}_{-1}(E_7)$	12
	A_1^2	triv	$\mathsf{H}_{1,-1}(B_6)$	12
	A_1^3	triv	$\mathsf{H}_{-1,1}(F_4) \otimes \mathsf{H}_{-1}(A_1)$	9
	A_1^4	triv	$\mathsf{H}_{-1,1}(B_4)$	5

Table 3.4: continued

D_4	triv, $(3,1)$	$\mathsf{H}_{1,-1}(F_4)$	18
$D_4 \times A_1$	triv, $(3,1) \otimes$ triv	$\mathsf{H}_{-1,1}(B_3)$	6
D_6	triv, $(0,3^2),(1,5),(2,4)$	$\mathsf{H}_{-1,1}(B_2)$	8
E_7	triv, $V, \varphi_{15,7}, \varphi_{21,6}$		
	$\varphi_{27,2}, \varphi_{35,13}, \varphi_{189,5}$	$\mathsf{H}_{-1}(A_1)$	7
E_8	triv, $V, \varphi_{28,8}, \varphi_{35,2},$		
	$\varphi_{50,8}, \varphi_{175,12}, \varphi_{300,8}, \varphi_{210,4},$		
	$\varphi_{560,5}, \varphi_{840,14}, \varphi_{1050,10}, \varphi_{1400,8}$	$\mathbb{C}$	12

3.3.8 Type H

Generalized Hecke Algebras for H_3

Next we produce a table describing the generalized Hecke algebras arising from H_3. Again, we only list those parameters $c = 1/d$ for positive integers $d > 1$ dividing one of the fundamental degrees $2, 6$ and 10 of H_3. There are 10 irreducible representations of the group H_3. By inspection, we see that Theorem 3.3.2.14 holds in type H_3 as well.

Table 3.5: Refined Filtration by Supports for H_3

c	W'	λ	$\mathcal{H}$	#Irr
$1/10$	1	triv	$\mathsf{H}_q(H_3)$	9
	H_3	triv	$\mathbb{C}$	1
$1/6$	1	triv	$\mathsf{H}_q(H_3)$	9
	H_3	triv	$\mathbb{C}$	1
$1/5$	1	triv	$\mathsf{H}_q(H_3)$	8
	$I_2(5)$	triv	$\mathsf{H}_1(A_1)$	2
$1/3$	1	triv	$\mathsf{H}_q(H_3)$	8
	A_2	triv	$\mathsf{H}_1(A_1)$	2
$1/2$	1	triv	$\mathsf{H}_q(H_3)$	5

Table 3.5: continued

A_1	triv	$\mathsf{H}_{-1}(A_1^2)$	1
A_1^2	triv	$\mathsf{H}_{-1}(A_1)$	1
H_3	triv, $V, \widetilde{V}$	$\mathbb{C}$	3

Generalized Hecke Algebras for H_4

Next we produce a table describing the generalized Hecke algebras arising from H_4. Again, we only list those parameters $c = 1/d$ for positive integers $d > 1$ dividing one of the fundamental degrees $2, 12, 20$ and 30 of H_4. There are 34 irreducible representations of the group H_4. By inspection, we see that Theorem 3.3.2.14 holds in type H_4 as well.

When the denominator of c is not equal to 2, the list of lowest weights λ giving the finite-dimensional irreducible representations of $H_c(H_4, \mathfrak{h})$ are obtained from results of Norton [61]. When the denominator is 2, our count below shows that there are exactly 6 isomorphism classes of finite-dimensional irreducible representations of $H_c(H_4, \mathfrak{h})$. In [45, Section 5.6], all except 6 of the possible lowest weights λ for the finite-dimensional irreducible representations $L_c(\lambda)$ are ruled out, and in particular those "potential" lowest weights are in fact precisely the lowest weights of the finite-dimensional $L_c(\lambda)$. This confirms a conjecture of Norton [61] about the classification of these representations.

Table 3.6: Refined Filtration by Supports for H_4

c	W'	λ	$\mathcal{H}$	#Irr
1/30	1	triv	$\mathsf{H}_q(H_4)$	33
	H_4	triv	$\mathbb{C}$	1
1/20	1	triv	$\mathsf{H}_q(H_4)$	33
	H_4	triv	$\mathbb{C}$	1
1/15	1	triv	$\mathsf{H}_q(H_4)$	32
	H_4	triv, $\widetilde{V}$	$\mathbb{C}$	2

Table 3.6: continued

$1/12$	1	triv	$\mathsf{H}_q(H_4)$	33
	H_4	triv	$\mathbb{C}$	1
$1/10$	1	triv	$\mathsf{H}_q(H_4)$	29
	H_3	triv	$\mathsf{H}_{-1}(A_1)$	1
	H_4	triv, $V, \widetilde{V}, \varphi_{9,6}$	$\mathbb{C}$	4
$1/6$	1	triv	$\mathsf{H}_q(H_4)$	30
	H_3	triv	$\mathsf{H}_{-1}(A_1)$	1
	H_4	triv, $V, \widetilde{V}$	$\mathbb{C}$	3
$1/5$	1	triv	$\mathsf{H}_q(H_4)$	24
	$I_2(5)$	triv	$\mathsf{H}_{1,q}(I_2(10))$	6
	H_4	triv, $V, \varphi_{9,2}, \varphi_{16,6}$	$\mathbb{C}$	4
$1/4$	1	triv	$\mathsf{H}_q(H_4)$	30
	A_3	triv	$\mathsf{H}_{-1}(A_1)$	1
	H_4	triv, $\varphi_{9,2}, \varphi_{9,6}$	$\mathbb{C}$	3
$1/3$	1	triv	$\mathsf{H}_q(H_4)$	26
	A_2	triv	$\mathsf{H}_{1,q}(G_2)$	4
	H_4	triv, $V, \widetilde{V}, \varphi_{16,3}$	$\mathbb{C}$	4
$1/2$	1	triv	$\mathsf{H}_{-1}(H_4)$	18
	A_1	triv	$\mathsf{H}_{-1}(H_3)$	5
	A_1^2	triv	$\mathsf{H}_{1,-1}(B_2)$	2
	H_3	triv, $V, \widetilde{V}$	$\mathsf{H}_{-1}(A_1)$	3
	H_4	triv, $V, \widetilde{V}, \varphi_{9,2}, \varphi_{9,6}, \varphi_{25,4}$	$\mathbb{C}$	6

3.3.9 Type F_4 with Unequal Parameters

In this section we will both prove Theorem 3.3.2.14 for rational Cherednik algebras of type F_4 and count the number of irreducible representations in $\mathcal{O}_{c_1,c_2}(F_4, \mathfrak{h})$ of each possible support for all values of the parameters c_1, c_2, including in the case of unequal parameters. As a corollary, comparing with the results of [45], we classify the irreducible finite-dimensional representations of $H_{1/2,1/2}(F_4, \mathfrak{h})$. This confirms a conjecture of Norton [61] about these representations and completes the classification of the irreducible finite-dimensional representations of rational Cherednik algebras of type F_4 with equal parameters, the other equal parameter cases having been treated by Norton. We expect that comparing our counts with the results of [45] completes the classification of the finite-dimensional irreducible representations for many other parameter values as well, although we do not perform this comparison here.

First, we fix some notation for the Coxeter group F_4. Our convention will be that the simple roots of F_4 are labeled s_1, s_2, s_3, s_4 where s_1, s_2 are short simple roots (with parameter $c_1 \in \mathbb{C}$) and s_3, s_4 are long simple roots (with parameter $c_2 \in \mathbb{C}$), and that s_2, s_3 are adjacent in the Dynkin diagram. The KZ parameters are given by $p := e^{-2\pi i c_1}$ and $q := e^{-2\pi i c_2}$. We label standard parabolic subgroups, one from each conjugacy class, as follows:

$A_1' := \langle s_1 \rangle$

$A_1 := \langle s_4 \rangle$

$A_1' \times A_1 := \langle s_1, s_4 \rangle$

$A_2' := \langle s_1, s_2 \rangle$

$A_2 := \langle s_3, s_4 \rangle$

$B_2 := \langle s_2, s_3 \rangle$

$C_3 := \langle s_1, s_2, s_3 \rangle$

$B_3 := \langle s_2, s_3, s_4 \rangle$

$A_1' \times A_2 := \langle s_1, s_3, s_4 \rangle$

$A_2' \times A_1 := \langle s_1, s_2, s_4 \rangle$

Theorem 3.3.2.14 follows in the case $W = F_4$ by application of Remark 3.2.2.3 to the cases in which $W' \subset W$ is one of the standard parabolic subgroups appearing above. In particular, when W' is a product of Coxeter groups of type A, the only irreducible representations λ of W' which can appear as the lowest weights of a finite-dimensional representation $L_c(\lambda)$ are linear characters, i.e. tensor products of either trivial or sign representations. These representations always extend to representations of $N_{F_4}(W')$, and then the statement Theorem 3.3.2.14 follows from Remark 3.2.2.3.

The remaining parabolic subgroups W' are B_2, B_3, and C_3. In each of these cases, Howlett [47] has shown that the normalizer $N_{F_4}(W')$ splits as a direct product $W' \times W''$, and Theorem 3.3.2.14 follows by Remark 3.2.2.3 in this case as well.

The classification of the irreducible finite-dimensional representations of the rational Cherednik algebras $H_c(W', \mathfrak{h}_{W'})$ where $W' \subset F_4$ is one of the proper nontrivial parabolic subgroups appearing above is well known. Applying our results to each of these cases, we can count the number of irreducible representations in $\mathcal{O}_c(F_4, \mathfrak{h})$ with support labeled by any of these parabolic subgroups. This information is presented in the following tables; the left column of each table specifies certain conditions on the parameters c_1, c_2 in terms of the KZ parameters $p = e^{-2\pi i c_1}$ and $q = e^{-2\pi i c_2}$, and the right column of each table gives the number of irreducible representations in $\mathcal{O}_{c_1,c_2}(F_4, \mathfrak{h})$ with support labeled by the parabolic subgroup appearing in the title of the table. Here Φ_d denotes the d^{th} cyclotomic polynomial.

Table 3.7: Simple Modules for $H_{c_1,c_2}(F_4)$ labeled by A_1'

condition	# simples labeled
$p = -1$ and $\Phi_1(q)\Phi_2(q)\Phi_3(q)\Phi_4(q) \neq 0$	9
$p = -1$ and $\Phi_1(q)\Phi_2(q) = 0$	3
$p = -1$ and $\Phi_3(q) = 0$	6
$p = -1$ and $\Phi_4(q) = 0$	8
otherwise	0

Table 3.8: Simple Modules for $H_{c_1,c_2}(F_4)$ labeled by A_1

condition	# simples labeled
$q = -1$ and $\Phi_1(p)\Phi_2(p)\Phi_3(p)\Phi_4(p) \neq 0$	9
$q = -1$ and $\Phi_1(p)\Phi_2(p) = 0$	3
$q = -1$ and $\Phi_3(p) = 0$	6
$q = -1$ and $\Phi_4(p) = 0$	8
otherwise	0

Table 3.9: Simple Modules labeled by $A_1' \times A_1$

condition	# simples labeled
$p = q = -1$	1
otherwise	0

Table 3.10: Simple Modules labeled by A_2'

condition	# simples labeled
$\Phi_3(p) = 0$ and $\Phi_2(q)\Phi_3(q)\Phi_6(q)\Phi_{12}(q) \neq 0$	6
$\Phi_3(p) = 0$ and $\Phi_2(q)\Phi_6(q) = 0$	3
$\Phi_3(p) = 0$ and $\Phi_3(q) = 0$	4
$\Phi_3(p) = 0$ and $\Phi_{12}(q) = 0$	5
otherwise	0

Table 3.11: Simple Modules labeled by A_2

condition	# simples labeled
$\Phi_3(q) = 0$ and $\Phi_2(p)\Phi_3(p)\Phi_6(p)\Phi_{12}(p) \neq 0$	6
$\Phi_3(q) = 0$ and $\Phi_2(p)\Phi_6(p) = 0$	3
$\Phi_3(q) = 0$ and $\Phi_3(p) = 0$	4
$\Phi_3(q) = 0$ and $\Phi_{12}(p) = 0$	5
otherwise	0

Table 3.12: Simple Modules labeled by B_2

condition	# simples labeled
$p = -1$ and $q = -1$	2
$p^{\pm 1} = -q$ and $\Phi_3(q)\Phi_6(q) = 0$	2
$p^{\pm 1} = -q$ and $\Phi_3(q)\Phi_6(q) \neq 0$	4

Table 3.12: continued

otherwise	0

Table 3.13: Simple Modules labeled by B_3

condition	# simples labeled
$p = -q^{\pm 2}$ and $\Phi_1(q)\Phi_6(q)\Phi_{12}(q) \neq 0$	2
$p = -q^{\pm 2}$ and $\Phi_6(q)\Phi_{12}(q) = 0$	1
$p = -1 = -q^{\pm 2}$ and $q = 1$	3
$p = -1$ and $\Phi_3(q) = 0$	1
otherwise	0

Table 3.14: Simple Modules labeled by C_3

condition	# simples labeled
$q = -p^{\pm 2}$ and $\Phi_1(p)\Phi_6(p)\Phi_{12}(p) \neq 0$	2
$q = -p^{\pm 2}$ and $\Phi_6(p)\Phi_{12}(p) = 0$	1
$q = -1 = -p^{\pm 2}$ and $p = 1$	3
$q = -1$ and $\Phi_3(p) = 0$	1
otherwise	0

Table 3.15: Simple Modules labeled by $A_1' \times A_2$

condition	# simples labeled
$p = -1$ and $\Phi_3(q) = 0$	1
otherwise	0

Table 3.16: Simple Modules labeled by $A_2' \times A_1$

condition	# simples labeled
$\Phi_3(p) = 0$ and $q = -1$	1
otherwise	0

To count the irreducible finite-dimensional representations for given parameters (c_1, c_2), we need first to count the irreducible representations in $\mathcal{O}_{c_1,c_2}(F_4, \mathfrak{h})$ of full support, which is achieved by counting the number of irreducible representations of the Hecke algebra $\mathsf{H}_{p,q}(F_4)$. The following table, produced by computations in CHEVIE in GAP3 [36, 58] and in Mathematica, gives this number of irreducible representations in all cases up to obvious symmetries. In particular, the isomorphism class as a $\mathbb{C}$-algebra of the Hecke algebra $\mathsf{H}_{p,q}(F_4)$ does not change when the order of the parameters p, q is reversed or when either of the parameters is replaced with its inverse. The following table then gives the number of irreducible representations of $\mathsf{H}_{p,q}(F_4)$ for a complete set of parameters $p, q \in \mathbb{C}^\times$ up to these symmetries. The first column specifies a hypersurface in the p, q-plane at which the Hecke algebra $\mathsf{H}_{p,q}(F_4)$ is not semisimple, and the second column specifies additional conditions on q, and the final column gives the corresponding number of irreducible representations of $\mathsf{H}_{p,q}(F_4)$.

Table 3.17: Irreducible Representations of $\mathsf{H}_{p,q}(F_4)$

condition	q condition	# simples
$p = -1$	$q = 1$	9
	$q = -1$	8
	$q =$ roots of unity order 3	11
	$q =$ roots of unity of order 4 or 6	14
	otherwise	15
$p^{\pm 1} + q = 0$	$q = \pm 1$	9
	$q =$ roots of unity of order 3 or 6	13
	$q =$ roots of unity of order 8	18

Table 3.17: continued

	otherwise	19
$p^{\pm 1} + q^2 = 0$	$q = 1$	9
	$q = -1$	8
	$q =$ roots of unity of order 3	13
	$q =$ roots of unity of order 4 or 6	20
	$q =$ roots of unity of order 9	22
	$q =$ roots of unity of order 10	21
	$q =$ roots of unity of order 12	17
	otherwise	23
$p^{\pm 1} = iq$	$q = 1$ or $q = -i$	20
	$q = -1$	14
	$q =$ roots of unity of order 3	17
	$q = i$	14
	$q =$ roots of unity of order 6	23
	$q \in \{e^{2\pi i/8}, e^{5\cdot 2\pi i/8}\}$	18
	$q \in \{e^{2\pi i/12}, e^{5\cdot 2\pi i/12}\}$	17
	$q \in \{e^{7\cdot 2\pi i/12}, e^{11\cdot 2\pi i/12}\}$	23
	$q \in \{e^{7\cdot 2\pi i/24}, e^{11\cdot 2\pi i/24}, e^{19\cdot 2\pi i/24}, e^{23\cdot 2\pi i/24}\}$	23
	otherwise	24
$\Phi_3(p) = 0$	$q = -1$	11
	$q =$ roots of unity of order 3	15
	$q =$ roots of unity of order 6	13
	$q =$ roots of unity of order 12	17
	otherwise	19
$\Phi_6(p) = 0$	$q = 1$	22
	$q = -1$	14
	$q =$ roots of unity of order 3	13
	$q =$ roots of unity of order 6	20

Table 3.17: continued

	$q =$ roots of unity of order 12	23
	otherwise	24
$p^{\pm 1} = e^{2\pi i/6} q$	$q = 1$	22
	$q = -1$	11
	$q = e^{2\pi i/3}$	11
	$q = e^{2\cdot 2\pi i/3}$	13
	$q = e^{2\pi i/6}$	13
	$q = e^{5\cdot 2\pi i/6}$	22
	$q \in \{e^{2\pi i/9}, e^{4\cdot 2\pi i/9}, e^{7\cdot 2\pi i/9}\}$	22
	$q \in \{e^{2\pi i/18}, e^{7\cdot 2\pi i/18}, e^{13\cdot 2\pi i/18}\}$	22
	$q \in \{e^{2\pi i/24}, e^{7\cdot 2\pi i/24}, e^{13\cdot 2\pi i/24}, e^{19\cdot 2\pi i/24}\}$	23
	otherwise	24
otherwise		25

The following table consolidates the information in the previous tables and counts the number of irreducible finite-dimensional representations of $H_{c_1,c_2}(F_4, \mathfrak{h})$ for all parameters c_1, c_2 in terms of the KZ parameters p and q, up to the same symmetries mentioned above, i.e. up to exchanging and inverting p and q, for which the category $\mathcal{O}_{c_1,c_2}(F_4, \mathfrak{h})$ is not semisimple. The first column specifies conditions on the parameters p and q. The remaining columns give the counts of the number of irreducible representations in $\mathcal{O}_{c_1,c_2}(F_4, \mathfrak{h})$, for any parameters c_1, c_2 satisfying $p = e^{-2\pi i c_1}$ and $q = e^{-2\pi i c_2}$, with support labeled by the parabolic subgroup appearing at the top of the column. In particular, the last column, labeled by F_4, counts finite-dimensional representations. This last column is obtained from the others by subtracting their sum from 25, the number of irreducible complex representations of the Coxeter group F_4. For those conditions on p and q specifying a certain finite set of points, we give a defining ideal for these points; for example, $(\Phi_2(p), \Phi_3(q))$ specifies that $p = -1$ and that q is a primitive cube root of unity. The remaining conditions specify certain curves with certain exceptional points removed.

Table 3.18: Modules of Given Support in $H_{c_1,c_2}(F_4)$

condition	1	A_1'	A_1	$A_1' \times A_1$	A_2'	A_2	B_2	B_3	C_3	$A_1' \times A_2$	$A_2' \times A_1$	F_4
$(\Phi_1(p), \Phi_2(q))$	9	.	3	.	.	.	4	.	3	.	.	6
$(\Phi_1(p), \Phi_4(q))$	20	.	.	.	.	.	.	2	.	.	.	3
$(\Phi_1(p), \Phi_6(q))$	22	.	.	.	.	.	.	.	.	.	.	3
$(\Phi_2(p), \Phi_2(q))$	8	3	3	1	.	.	2	2	2	.	.	4
$(\Phi_2(p), \Phi_3(q))$	11	6	.	.	.	3	.	1	.	1	.	3
$(\Phi_2(p), \Phi_4(q))$	14	8	.	.	.	.	.	.	.	.	.	3
$(\Phi_2(p), \Phi_6(q))$	14	9	.	.	.	.	.	.	.	.	.	2
$(\Phi_3(p), \Phi_3(q))$	15	.	.	.	4	4	.	.	.	.	.	2
$(\Phi_3(p), \Phi_6(q))$	13	.	.	.	3	.	2	.	2	.	.	5
$(\Phi_3(p), \Phi_{12}(q))$	17	.	.	.	5	.	.	1	.	.	.	2
$(\Phi_4(p), \Phi_4(q))$	19	.	.	.	.	.	4	.	.	.	.	2
$(\Phi_6(p), \Phi_6(q))$	20	.	.	.	.	.	.	1	1	.	.	3
$(\Phi_6(p), \Phi_{12}(q))$	23	.	.	.	.	.	.	.	.	.	.	2
$(\Phi_8(p), \Phi_8(q),$ $\Phi_1(pq)\Phi_1(pq^{-1}))$	24	.	.	.	.	.	.	.	.	.	.	1
$(\Phi_8(p), \Phi_8(q),$ $\Phi_2(pq)\Phi_2(pq^{-1}))$	18	.	.	.	.	.	4	.	.	.	.	3

Table 3.18: continued

$(\Phi_9(p), \Phi_{18}(q),$ $\Phi_2(p^2q)\Phi_2(p^2q^{-1}))$	22	.	.	.	.	.	.	.	2	.	.	1
$(\Phi_{10}(p), \Phi_{10}(q),$ $\Phi_2(pq^2)\Phi_2(p^{-1}q^2))$	21	.	.	.	.	.	.	2	2	.	.	.
$(\Phi_{12}(p), \Phi_{12}(q),$ $\Phi_1(pq)\Phi_1(p^{-1}q))$	24	.	.	.	.	.	.	.	.	.	.	1
$(\Phi_{24}(p), \Phi_{24}(q),$ $\Phi_4(pq), \Phi_6(p^{-1}q))$	23	.	.	.	.	.	.	.	.	.	.	2
$\Phi_2(p) = 0$ and $(q^6-1)\Phi_4(q) \neq 0$	15	9	.	.	.	.	.	.	.	.	.	1
$\Phi_3(p) = 0$ and $\Phi_2(q)\Phi_3(q)\Phi_6(q)\cdot$ $\Phi_{12}(q) \neq 0$	19	.	.	.	6	.	.	.	.	.	.	.
$\Phi_6(p) = 0$ and $(q^6-1)\Phi_{12}(q) \neq 0$	24	.	.	.	.	.	.	.	.	.	.	1
$\Phi_2(pq) = 0$ and $(q^6-1)\Phi_8(q) \neq 0$	19	.	.	.	.	.	4	.	.	.	.	2
$\Phi_2(pq^2) = 0$ and												

Table 3.18: continued

$(q^{12}-1)\Phi_9(q)\cdot$ $\Phi_{10}(q) \neq 0$	23	.	.	.	.	.	.	2	.	.	.	.
$\Phi_4(pq)=0$ and $(q^{12}-1)\Phi_2(pq^{-1})\cdot$ $\Phi_6(pq^{-1}) \neq 0$	24	.	.	.	.	.	.	.	.	.	.	1
$\Phi_6(pq)=0$ and $(q^6-1)\Phi_4(pq^{-1})\cdot$ $\Phi_2(pq^{-2})\cdot$ $\Phi_2(p^{-2}q) \neq 0$	24	.	.	.	.	.	.	.	.	.	.	1

Table 3.18 shows in particular that there are exactly 4 distinct irreducible finite-dimensional representations of the rational Cherednik algebra $H_{1/2,1/2}(F_4, \mathfrak{h})$. In notation compatible with the labeling of the irreducible representations of the Coxeter group F_4 in GAP3, it is shown in [45, Section 5.7.2] that with equal parameters $c_1 = c_2 = 1/2$ the lowest weights λ such that the representation $L_{1/2,1/2}(\lambda)$ is finite-dimensional are among the representations $\varphi_{1,0}, \varphi'_{2,4}, \varphi''_{2,4}$, and $\varphi_{9,2}$. In particular, we have the following corollary, confirming Norton's conjecture on the classification of the irreducible finite-dimensional representations of $H_{1/2,1/2}(F_4, \mathfrak{h})$:

Corollary 3.3.9.1. *The set of irreducible representations λ of the Coxeter group F_4 for which the irreducible lowest weight representation $L_{1/2,1/2}(F_4, \mathfrak{h})$ is finite dimensional is*

$$\{\varphi_{1,0}, \varphi'_{2,4}, \varphi''_{2,4}, \varphi_{9,2}\}.$$

3.3.10 Type I

The only remaining case in which Theorem 3.3.2.14 needs to be verified is the case of the irreducible Coxeter groups of type I, i.e. the dihedral groups. For $d \geq 3$, let $I_2(d)$ denote the dihedral Coxeter group with $2d$ elements. Chmutova [13] has computed character formulas for all irreducible representations $L_c(\lambda)$ in the category $\mathcal{O}_c(I_2(d), \mathfrak{h})$ for all parameters c, and in particular the supports of all irreducible representations in $\mathcal{O}_c(I_2(d), \mathfrak{h})$ are known in this case, so we will not produce tables counting the number of irreducible representations of given support in this case.

To see that Theorem 3.3.2.14 holds in type I, we need only consider the case of rank 1 parabolic subgroups, i.e. those of type A_1. Let $d \geq 3$ be an integer and let $W' \subset I_2(d)$ be a parabolic subgroup of type A_1. If the parameter c_1 attached to the reflection generating A_1 does not satisfy $c_1 \in 1/2 + \mathbb{Z}$, the rational Cherednik algebra $H_{c_1}(W', \mathfrak{h}_{W'})$ does not admit any nonzero finite-dimensional representations. Otherwise, there is a unique finite-dimensional representation L of $H_{c_1}(W', \mathfrak{h}_{W'})$ with lowest weight triv or sgn, depending on whether $c_1 > 0$ or $c_1 < 0$, respectively. In either case, L is $N_{I_2(d)}(W')$-equivariant by Remark 3.2.2.3. This completes the proof of Theorem 3.3.2.14.

Chapter 4

The Dunkl Weight Function

4.1 Introduction

In addition to characters, another interesting class of invariants attached to irreducible representations of rational Cherednik algebras are the *signature characters*. When the parameter c is *real*, i.e. $c(s) = \overline{c(s^{-1})}$ for all reflections $s \in S$, each standard module $\Delta_c(\lambda)$ admits a W-invariant, graded, *contravariant* Hermitian form $\beta_{c,\lambda}$, unique up to scaling by a positive real number. The kernel of the form $\beta_{c,\lambda}$ coincides with the unique maximal proper submodule of $\Delta_c(\lambda)$, and in particular $\beta_{c,\lambda}$ descends to the irreducible quotient $L_c(\lambda)$. Similarly to the definition of the character of $L_c(\lambda)$, the *signature character* of $L_c(\lambda)$ is the generating function recording the signature of the form $\beta_{c,\lambda}$ in each finite-dimensional graded component of $L_c(\lambda)$. Determining explicit formulas for these signature characters and describing the set of parameters c such that $\beta_{c,\lambda}$ is unitary has proved very difficult, with complete results available in only a limited collection of cases, e.g. [33] and [70] for the type A case and [44] for the classical and cyclotomic types. The study of the forms $\beta_{c,\lambda}$ can be viewed as an analogue for rational Cherednik algebras of the study of unitarizability (or, more generally, of signatures of invariant Hermitian forms) of representations of real reductive groups considered in detail by Adams, van Leeuwen, Trapa, and Vogan [1].

When W is a finite real reflection group, the rational Cherednik algebra contains a natural $\mathfrak{sl}_2$-triple $\mathbf{e}, \mathbf{f}, \mathbf{h} \in H_c(W, \mathfrak{h})$. The element $\mathbf{f}$ acts by a degree -2 operator, and hence nilpotently, on any representation $M \in \mathcal{O}_c(W, \mathfrak{h})$. In particular, its exponential $\exp(\mathbf{f})$ is well-defined and preserves the natural filtration on any $M \in \mathcal{O}_c(W, \mathfrak{h})$. In particular, the determination of the signatures of the *Gaussian inner product* $\gamma_{c,\lambda}(v, v') := \beta_{c,\lambda}(\exp(\mathbf{f})v, \exp(\mathbf{f})v')$, considered in [10] and [33, Definition 4.5], in the filtered pieces of $\Delta_c(\lambda)$ is equivalent to the determination of the signature character of $L_c(\lambda)$. As it happens, the Gaussian inner product $\gamma_{c,\lambda}$ appears easier to study than the contravariant form $\beta_{c,\lambda}$ itself.

The main purpose of this chapter, generalizing previous results of Dunkl [19, 20, 21], is to introduce a fundamental object, the *Dunkl weight function* $K_{c,\lambda}$, for the study of the Gaussian inner product $\gamma_{c,\lambda}$. In fact, it is natural to consider a slightly more general context in which the parameter c is not required to be real; for general c, $\gamma_{c,\lambda}$ is a sesquilinear pairing

$$\gamma_{c,\lambda} : \Delta_c(\lambda) \times \Delta_{c^\dagger}(\lambda) \to \mathbb{C},$$

where $c^\dagger$ is the parameter given by $c^\dagger(s) = \overline{c(s^{-1})}$. By the PBW theorem for rational Cherednik algebras [29, Theorem 1.3], the standard module $\Delta_c(\lambda)$ can be identified with $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ as a $\mathbb{C}[\mathfrak{h}] \rtimes \mathbb{C}W$-module, independently of c, where $\mathbb{C}[\mathfrak{h}]$ denotes the algebra of polynomial functions on the reflection representation $\mathfrak{h}$. With respect to this identification, and having chosen a W-invariant positive-definite Hermitian form $(v_1, v_2) \mapsto v_2^\dagger v_1$ on λ, we have the following result, where $\mathfrak{h}_\mathbb{R}$ denotes the real reflection representation of W:

Theorem. *For any finite Coxeter group W and irreducible representation λ of W, there is a unique family $K_{c,\lambda}$, holomorphic in $c \in \mathfrak{p}$, of $End_\mathbb{C}(\lambda)$-valued tempered distributions on $\mathfrak{h}_\mathbb{R}$ such that the following integral representation of the Gaussian pairing $\gamma_{c,\lambda}$ holds for all $c \in \mathfrak{p}$:*

$$\gamma_{c,\lambda}(P, Q) = \int_{\mathfrak{h}_\mathbb{R}} Q(x)^\dagger K_{c,\lambda}(x) P(x) e^{-|x|^2/2} dx \qquad \textit{for all } P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda.$$

This theorem, along with additional properties of $K_{c,\lambda}$, appears in Theorem 4.3.3.1

and is proved in Sections 4.3.4-4.3.6. The family of tempered distributions $K_{c,\lambda}$ appearing in the theorem above will be called the *Dunkl weight function*. The case in which λ is the trivial representation, including the extension of the weight function to arbitrary c as a tempered distribution, has been studied previously in detail by Etingof [28], leading to the complete determination of the parameters c for which $L_c(\text{triv})$ is finite-dimensional for Coxeter groups W. For dihedral groups W, Dunkl has provided explicit formulas for the matrix entries of $K_{c,\lambda}$ for c small and real.

Additionally, for real parameters $c = c^\dagger$, the Dunkl weight function $K_{c,\lambda}$ provides a bridge between the study of invariant Hermitian forms on representations of rational Cherednik algebras and of associated finite Hecke algebras via the Knizhnik-Zamolodchikov (KZ) functor. The KZ functor, introduced by Ginzburg, Guay, Opdam, and Rouquier [38], is an exact functor $KZ : \mathcal{O}_c(W, \mathfrak{h}) \to \mathsf{H}_q(W)\text{-mod}_{f.d.}$ from the category $\mathcal{O}_c(W, \mathfrak{h})$ to the category $\mathsf{H}_q(W)\text{-mod}_{f.d.}$ of finite-dimensional representations of the Hecke algebra $\mathsf{H}_q(W)$ attached to the Coxeter group W at a particular parameter q. The construction of the KZ functor depends on a choice of point x_0 in the complement $\mathfrak{h}_{reg}$ of the reflection hyperplanes in $\mathfrak{h}$, with any two points giving rise to isomorphic functors. Let KZ_{x_0} denote the functor associated to the point $x_0 \in \mathfrak{h}_{reg}$. As a vector space, $KZ_{x_0}(\Delta_c(\lambda))$ is naturally identified with the vector space λ itself. In Theorem 4.3.3.1 we will see that the restriction of the distribution $K_{c,\lambda}$ to $\mathfrak{h}_{\mathbb{R},reg} := \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{h}_{reg}$ is given by integration against an analytic function with values in $\text{End}_{\mathbb{C}}(\lambda)$, and, with respect to the identification $KZ_{x_0}(\Delta_c(\lambda)) \cong_{\mathbb{C}} \lambda$, the value $K_{c,\lambda}(x_0)$ at any $x_0 \in \mathfrak{h}_{\mathbb{R},reg}$ determines a braid group invariant Hermitian form on $KZ_{x_0}(\Delta_c(\lambda))$. In Section 4.4, we exploit this relationship between invariant Hermitian forms on $\Delta_c(\lambda)$ and $KZ_{x_0}(\Delta_c(\lambda))$ to show that the KZ functor preserves signatures, and hence unitarizability, in an appropriate sense made precise in Theorem 4.4.0.1.

This chapter is organized as follows. In Section 4.2, we recall the definition of signature characters, introduce Janzten filtrations on standard modules $\Delta_c(\lambda)$, and use the Jantzen filtrations to prove that the shifted signature characters are rational functions, allowing for the definition of the *asymptotic signature* $a_{c,\lambda} \in \mathbb{Q} \cap [-1, 1]$

recording the limiting behavior of the signatures of $\beta_{c,\lambda}$ in graded components of high degree. In Section 4.3, we state and prove the main theorem on the existence of the Dunkl weight function. Finally, in Section 4.4, we use the Dunkl weight function and its properties established in Section 4.3 to prove Theorem 4.4.0.1 on the compatibility of the KZ functor with signatures. In Section 4.5, we discuss related conjectures and further directions.

4.2 Signature Characters and the Jantzen Filtration

4.2.1 Jantzen Filtrations on Standard Modules

Let W be a finite complex reflection group with reflection representation $\mathfrak{h}$. Let $S \subset W$ be the set of complex reflections in W, and let $\mathfrak{p}$ be the $\mathbb{C}$-vector space of W-invariant functions $c : S \to \mathbb{C}$. For any $c \in \mathfrak{p}$, let $c^\dagger \in \mathfrak{p}$ be defined by $c^\dagger(s) = \overline{c(s^{-1})}$. Refer to any $c \in \mathfrak{p}$ satisfying $c = c^\dagger$ as *real*, and let $\mathfrak{p}_\mathbb{R}$ be the $\mathbb{R}$-vector space $\mathfrak{p}_\mathbb{R} := \{c \in \mathfrak{p} : c = c^\dagger\}$.

Recall that $\mathfrak{p}$ is the parameter space for the family of rational Cherednik algebras attached to $(W, \mathfrak{h})$; we denote the rational Cherednik algebra attached to $(W, \mathfrak{h})$ and parameter $c \in \mathfrak{p}$ by $H_c(W, \mathfrak{h})$. For each irreducible representation $\lambda \in \mathrm{Irr}(W)$, we have the associated standard module

$$\Delta_c(\lambda) := H_c(W, \mathfrak{h}) \otimes_{\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^*]} \lambda$$

over $H_c(W, \mathfrak{h})$. As a $\mathbb{Z}^{\geq 0}$-graded $\mathbb{C}$-vector space, $\Delta_c(\lambda)$ is naturally identified with $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ independently of c.

Fix a W-invariant positive definite Hermitian form $(\cdot, \cdot)_\lambda$ on λ - we will take the convention that Hermitian forms are conjugate-linear in the second factor. Such a form is uniquely determined up to $\mathbb{R}^{>0}$-scaling. Similarly, fix a W-invariant positive-definite Hermitian form $(\cdot, \cdot)_\mathfrak{h}$ on $\mathfrak{h}$. This determines the conjugate-linear isomorphism $T : \mathfrak{h} \to \mathfrak{h}^*$ given by

$$T(y)(x) := (x, y)_\mathfrak{h}.$$

When the parameter $c \in \mathfrak{p}$ is *real*, i.e. $c = c^\dagger$, the standard module $\Delta_c(\lambda)$ admits a unique W-invariant Hermitian form $\beta_{c,\lambda}$ [33, Proposition 2.2] such that the contravariance condition

$$\beta_{c,\lambda}(yv, v') = \beta_{c,\lambda}(v, T(y)v)$$

holds for all $v, v' \in \Delta_c(\lambda)$ and $y \in \mathfrak{h}$ and that coincides with $(\cdot,\cdot)_\lambda$ in degree 0.

As above, regarding the standard modules $\Delta_c(\lambda)$ for various c as the same $\mathbb{Z}^{\geq 0}$-graded vector space $\Delta(\lambda)$, we may view the forms $\beta_{c,\lambda}$ on $\Delta(\lambda)$ as an algebraic family of Hermitian forms $\beta_{c,\lambda}[d]$, parameterized by $c \in \mathfrak{p}_\mathbb{R}$ in a polynomial manner, on each finite-dimensional graded component $\Delta(\lambda)[d]$. In particular, we are naturally led to consider Jantzen filtrations, as follows.

Let $c_0, c_1 \in \mathfrak{p}_\mathbb{R}$ be real parameters such that there exists $\delta > 0$ such that $\beta_{c(t),\lambda}$ is nondegenerate for all $c(t) := c_0 + tc_1$ with $t \in (-\delta, \delta)\backslash\{0\}$. The finite-dimensional $\mathbb{C}$-vector spaces $\Delta(\lambda)[d]$ for $n \geq 0$ along with the polynomial families of Hermitian forms $\beta_{c(t),\lambda}[d]$ satisfy the conditions of [71, Definition 3.1]. In particular, for each $d \geq 0$ define the finite descending filtration

$$\Delta(\lambda)[d] = \Delta(\lambda)[d]^{\geq 0} \supset \Delta(\lambda)[d]^{\geq 1} \supset \cdots \supset \Delta(\lambda)[d]^{\geq N} = 0$$

on $\Delta(\lambda)[d]$ as follows (to simplify the notation, we do not include the choice of c_0, c_1 in the notation for the filtration, but of course this filtration and its properties may be dependent on this choice). Let $\Delta(\lambda)[d]^{\geq k}$ consist of those vectors $v \in \Delta(\lambda)[d]$ such that there is some $\epsilon > 0$ and an analytic function $f_v : (-\epsilon, \epsilon) \to \Delta(\lambda)[d]$ satisfying $f_v(0) = v$ and such that the analytic function

$$t \mapsto \beta_{c(t),\lambda}[d](f_v(t), v')$$

vanishes at least to order k for all $v' \in \Delta_c(\lambda)[d]$ (clearly, one may equivalently consider only polynomial functions f_v). For each $k \geq 0$, define the Hermitian form $\beta_{c_0,\lambda}[d]^{\geq k}$

on $\Delta(\lambda)[d]^{\geq k}$ by

$$\beta_{c_0,\lambda}[d]^{\geq k}(v, v') = \lim_{t \to 0} \frac{1}{t^k} \beta_{c(t),\lambda}[d](f_v(t), f_{v'}(t)),$$

where $v, v' \in \Delta(\lambda)[d]^{\geq k}$ and $f_v, f_{v'}$ are any analytic functions as above (this limit does not depend on the choice of such $f_v, f_{v'}$). We then have the following theorem:

Theorem 4.2.1.1. *(Jantzen [50, 5.1], Vogan [71, Theorem 3.2]) The radical of the Hermitian form $\beta_{c_0,\lambda}[d]^{\geq k}$ is precisely $\Delta(\lambda)[d]^{\geq k+1}$.*

In particular, the Hermitian form $\beta_{c_0,\lambda}[d]^{\geq k}$ descends to a nondegenerate Hermitian form on each filtration subquotient $\Delta(\lambda)[d]^{(k)} := \Delta(\lambda)[d]^{\geq k}/\Delta(\lambda)[d]^{\geq k+1}$. Denote this induced nondegenerate Hermitian form by $\beta_{c_0,\lambda}[d]^{(k)}$. For all $k, d \geq 0$, let $p_d^{(k)}$ (resp. $q_d^{(k)}$) be the dimension of a maximal positive definite (resp., negative definite) subspace of $\Delta(\lambda)[d]^{(k)}$ with respect to the form $\beta_{c_0,\lambda}[d]^{(k)}$. Note that for any fixed $d \geq 0$ we have $p_d^{(k)} = q_d^{(k)} = 0$ for all sufficiently large k.

We will take the convention that the *signature* of a Hermitian form β on a finite dimensional $\mathbb{C}$-vector space V is the integer $p - q$, where p (respectively, q) is the dimension of any maximal positive-definite (respectively, negative-definite) subspace of V with respect to β. For example, in the context of the previous paragraph, $p_d^{(k)} - q_d^{(k)}$ is the signature of the form $\beta_{c_0,\lambda}[d]^{(k)}$. If the form β is non-degenerate then the dimension of V and the quantity $p - q$ determines the tuple (p, q) considered to be the signature of β in some references.

Proposition 4.2.1.2. *(Vogan [71, Proposition 3.3])*

(a) For all small positive t (i.e. $t \in (0, \delta)$) and any $d \geq 0$, the signature of the nondegenerate Hermitian form $\beta_{c(t),\lambda}[d]$ on $\Delta(\lambda)[d]$ is

$$\sum_{k \geq 0} p_d^{(k)} - \sum_{k \geq 0} q_d^{(k)}.$$

(b) Similarly, for all small negative t (i.e. $t \in (-\delta, 0)$) and any $d \geq 0$, the signature

of $\beta_{c(t),\lambda}[d]$ *is*

$$\left(\sum_{k\ even} p_d^{(k)} + \sum_{k\ odd} q_d^{(k)}\right) - \left(\sum_{k\ odd} p_d^{(k)} + \sum_{k\ even} q_d^{(k)}\right).$$

Define the descending filtration

$$\Delta(\lambda) = \Delta(\lambda)^{\geq 0} \supset \Delta(\lambda)^{\geq 1} \supset \cdots$$

by

$$\Delta(\lambda)^{\geq k} := \bigoplus_{d\geq 0} \Delta(\lambda)[d]^{\geq k} \subset \bigoplus_{d\geq 0} \Delta(\lambda)[d] = \Delta(\lambda).$$

Lemma 4.2.1.3. *The filtration of* $\Delta_{c_0}(\lambda)$ *by the subspaces* $\Delta(\lambda)^{\geq k}$ *is a filtration by* $H_{c_0}(W, \mathfrak{h})$*-submodules. We have* $\Delta(\lambda)^{\geq k} = 0$ *for sufficiently large* k*.*

Proof. Let $v \in \Delta(\lambda)[d]^{\geq k}$, and let $f_v : (-\epsilon, \epsilon) \to \Delta(\lambda)[d]$ be as in the definition of the filtration, exhibiting that $v \in \Delta(\lambda)[d]^{\geq k}$. Then for any homogeneous $h \in H_{c_0}(W, \mathfrak{h})$ of degree d', viewing $h \in H_{c(t)}(W, \mathfrak{h})$ for all $t \in \mathbb{R}$ via the PBW basis, the path hf_v exhibits hv as an element of $\Delta(\lambda)[d + d']^{\geq k}$. In particular, $\Delta(\lambda)^{\geq k}$ is a $H_{c_0}(W, \mathfrak{h})$-submodule of $\Delta(\lambda)$. By the finite-length property of $H_{c_0}(W, \mathfrak{h})$-modules in category $\mathcal{O}_{c_0}(W, \mathfrak{h})$, it follows that the filtration $\Delta(\lambda)^{\geq k}$ stabilizes in k. For any fixed d we have $\Delta(\lambda)[d]^{\geq k} = 0$ for any k sufficiently large, and it follows that $\Delta(\lambda)^{\geq k} = 0$ for all sufficiently large k. $\square$

We refer to the filtration of $\Delta_{c_0}(\lambda)$ appearing in Lemma 4.2.1.3 as the *Jantzen filtration* of $\Delta_{c_0}(\lambda)$. Note that this Jantzen filtration depends on the choice of the additional parameter $c_1 \in \mathfrak{p}_{\mathbb{R}}$ determining the direction for the deformation.

4.2.2 Hermitian Duals

In this section we will introduce Hermitian duals in the setting of rational Cherednik algebras, analogous to the Hermitian duals considered by Vogan [71] in the Lie-theoretic setting. First we will briefly recall contragredient duals. Let $c \in \mathfrak{p}$ be any parameter for the rational Cherednik algebra attached to $(W, \mathfrak{h})$. Let $\overline{c} \in \mathfrak{p}$ be the parameter defined by $\overline{c}(s) = c(s^{-1})$. As explained in [31, Section 3.11], there is a natural isomorphism

$$\gamma : H_c(W, \mathfrak{h})^{opp} \to H_{\overline{c}}(W, \mathfrak{h}^*)$$

acting trivially on $\mathfrak{h}$ and $\mathfrak{h}^*$ and sending $w \mapsto w^{-1}$ for all $w \in W$. For any $M \in \mathcal{O}_c(W, \mathfrak{h})$, the restricted dual $M^\dagger := \bigoplus_{z \in \mathbb{C}} M_z^*$ is naturally a $H_c(W, \mathfrak{h})^{opp}$-module; by transfer of structure along γ, we regard $M^\dagger$ as a $H_{\overline{c}}(W, \mathfrak{h}^*)$-module. We have:

Proposition 4.2.2.1. *([31, Proposition 3.32]) The assignment $M \mapsto M^\dagger$ determines a $\mathbb{C}$-linear equivalence of categories $^\dagger : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_{\overline{c}}(W, \mathfrak{h}^*)^{opp}$.*

Given a $\mathbb{C}$-algebra A, let $\overline{A}$ denote the $\mathbb{C}$-algebra that is equal to A as a ring and as a ring is equal to the complex conjugate vector space of A. In other words, the identity map $\mathrm{Id} : A \to \overline{A}$ is an isomorphism of rings and satisfies $z\mathrm{Id}(a) = \mathrm{Id}(\overline{z}a)$. Clearly $\overline{A}$ is a $\mathbb{C}$-algebra, with unit $\overline{\eta}$ satisfying $\overline{\eta}(z) = \eta(\overline{z})$ for all $z \in \mathbb{C}$, where $\eta : \mathbb{C} \to A$ is the unit map for the $\mathbb{C}$-algebra A. Similarly, for any A-module M the complex conjugate vector space $\overline{M}$ is naturally an $\overline{A}$-module, and this clearly defines an conjugate-linear equivalence of categories A-mod $\to \overline{A}$-mod.

The complex conjugate of a rational Cherednik algebra is again a rational Cherednik algebra. More precisely:

Lemma 4.2.2.2. *Fix a nondegenerate W-invariant Hermitian form $(\cdot, \cdot)_{\mathfrak{h}}$ on $\mathfrak{h}$, and let $T : \mathfrak{h} \to \mathfrak{h}^*$ be the conjugate-linear isomorphism introduced in Section 4.2.1. Then the mappings $y \mapsto Ty$ for $y \in \mathfrak{h}$, $x \mapsto T^{-1}x$ for $x \in \mathfrak{h}^*$, and $w \mapsto w$ for $w \in W$ extend uniquely to an isomorphism of $\mathbb{C}$-algebras*

$$\omega : H_{\overline{c}}(W, \mathfrak{h}^*) \to \overline{H_{c^\dagger}(W, \mathfrak{h})}.$$

Proof. Regarded as a map $\mathfrak{h} \to \overline{\mathfrak{h}^*}$, T is an isomorphism of complex representations of W, and similarly for $T^{-1} : \mathfrak{h}^* \to \overline{\mathfrak{h}}$. It follows that the assignments in the lemma extend uniquely to a $\mathbb{C}$-linear isomorphism $\mathbb{C}W \ltimes T(\mathfrak{h}^* \oplus \mathfrak{h}) \to \overline{\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)}$ which determines a map $\mathbb{C}W \ltimes T(\mathfrak{h}^* \oplus \mathfrak{h}) \to \overline{H_{c^\dagger}(W, \mathfrak{h})}$. As $\mathfrak{h}$ is sent to $\mathfrak{h}^*$ and $\mathfrak{h}^*$ to $\mathfrak{h}$ under this map, the commutators $[x, x']$ and $[y, y']$ for $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$ are sent to 0 by this map. Let $\langle\cdot,\cdot\rangle$ denote the natural pairing of $\mathfrak{h}$ with $\mathfrak{h}^*$. By definition of T, we have, for any $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$,

$$\langle Ty, T^{-1}x\rangle = (T^{-1}x, y)_{\mathfrak{h}} = \overline{(y, T^{-1}x)_{\mathfrak{h}}} = \overline{\langle x, y\rangle}.$$

In particular, under the map $\mathbb{C}W \ltimes T(\mathfrak{h}^* \oplus \mathfrak{h}) \to \overline{H_{c^\dagger}(W,\mathfrak{h})}$, for any $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$ the image of the element

$$[x,y] - \langle x,y\rangle + \sum_{s\in S} \overline{c}_s \langle x, \alpha_s^\vee\rangle\langle y, \alpha_s\rangle s$$

in $\overline{H_{c^\dagger}(W,\mathfrak{h})}$ is

$$[T^{-1}x, Ty] - \overline{\langle x,y\rangle} + \sum_{s\in S} c_s^\dagger \overline{\langle x, \alpha_s^\vee\rangle\langle y,\alpha_s\rangle} s$$
$$= [T^{-1}x, Ty] - \langle T^{-1}x, Ty\rangle + \sum_{s\in S} c_s^\dagger \langle T^{-1}x, T\alpha_s^\vee\rangle\langle Ty, T^{-1}\alpha_s\rangle s.$$

As $T\alpha_s^\vee \in \mathfrak{h}^*$ and $T\alpha_s \in \mathfrak{h}$ are eigenvectors for s with nontrivial eigenvalues and

$$\langle T\alpha_s^\vee, T^{-1}\alpha_s\rangle = \overline{\langle \alpha_s, \alpha_s^\vee\rangle} = \overline{2} = 2,$$

the expression above is 0 in $H_{c^\dagger}(W,\mathfrak{h})$. It follows that there is an induced map of $\mathbb{C}$-algebras $H_{\overline{c}}(W,\mathfrak{h}^*) \to \overline{H_{c^\dagger}(W,\mathfrak{h})}$, and clearly this map is an isomorphism. $\square$

Let $\sigma : \overline{H_c(W,\mathfrak{h})^{opp}} \to H_{c^\dagger}(W,\mathfrak{h})$ be the isomorphism of $\mathbb{C}$-algebras obtained by compositing γ and ω. Its action on generators is given by $\sigma(x) = T^{-1}x$ for $x \in \mathfrak{h}^*$, $\sigma(y) = Ty$ for $y \in \mathfrak{h}$, and $\sigma(w) = w^{-1}$ for $w \in W$. As T depends on the choice of the form $(\cdot,\cdot)_{\mathfrak{h}}$, so does σ.

For any $M \in \mathcal{O}_c(W,\mathfrak{h})$, the *Hermitian dual* $M^h := \overline{M^\dagger}$ is naturally an $\overline{H_c(W,\mathfrak{h})^{opp}}$-module, and by transfer of structure along σ we regard M^h as a $H_{c^\dagger}(W,\mathfrak{h})$-module. Clearly $M^h \in \mathcal{O}_{c^\dagger}(W,\mathfrak{h})$, so we have:

Lemma 4.2.2.3. *The assignment $M \mapsto M^h$ defines a conjugate-linear equivalence of categories*

$${}^h : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_{c^\dagger}(W,\mathfrak{h})^{opp}.$$

Definition 4.2.2.4. *Given modules $M \in \mathcal{O}_c(W, \mathfrak{h})$ and $N \in \mathcal{O}_{c^\dagger}(W, \mathfrak{h})$, a sesquilinear pairing*

$$\beta : M \times N \to \mathbb{C}$$

will be called contravariant *if*

$$\beta(hm, n) = \beta(m, \sigma(h)n) \qquad \text{for all } h \in H_c(W, \mathfrak{h}), m \in M, n \in N.$$

As $\sigma(w) = w^{-1}$ for all $w \in W$ and as σ sends the grading element of $H_c(W, \mathfrak{h})$ to the grading element of $H_{c^\dagger}(W, \mathfrak{h})$, it follows that any contravariant pairing β is automatically graded and W-invariant.

We will be particularly concerned with contravariant Hermitian forms on modules $M \in \mathcal{O}_c(W, \mathfrak{h})$ for real parameters $c \in \mathfrak{p}_{\mathbb{R}}$. We will use the following lemma later:

Lemma 4.2.2.5. *Suppose the parameter $c \in \mathfrak{p}$ is real, i.e. $c = c^\dagger$. Let $M \in \mathcal{O}_c(W, \mathfrak{h})$ be equipped with a nondegenerate W-invariant contravariant Hermitian form β. Then the assignment*

$$m \mapsto \beta(\cdot, m)$$

defines an isomorphism $M \cong M^h$ of $H_c(W, \mathfrak{h})$-modules.

Proof. That the map in question is a map of $H_c(W, \mathfrak{h})$-modules follows from the observation that the statement that β is W-invariant and contravariant means precisely that $\beta(hv, v') = \beta(v, \sigma(h)v')$ for all $v, v' \in M$ and $h \in H_c(W, \mathfrak{h}) = H_{c^\dagger}(W, \mathfrak{h})$. That the map is an isomorphism follows from the nondegeneracy of β. □

4.2.3 Characters and Signature Characters

Let us now briefly recall the definition of characters and signature characters.

Definition 4.2.3.1. *Let $M \in \mathcal{O}_c(W, \mathfrak{h})$. Then the* character *of M [31, Section 3.9], denoted $ch(M)$, is the formal series*

$$ch(M)(w, t) := \sum_{z \in \mathbb{C}} t^z \, Tr_{M_z}(w), \quad w \in W.$$

In particular, taking $w = 1$, one obtains the graded dimension of M:

$$\mathrm{ch}(M)(1,t) = \sum_{z\in\mathbb{C}} t^z \dim M_z.$$

Definition 4.2.3.2. *When M is a lowest weight module with lowest weight λ, we define the* shifted character *$ch_0(M)$ by*

$$ch_0(M)(w,t) := t^{-h_c(\lambda)} ch(M)(w,t).$$

The character of a standard module $\Delta_c(\lambda)$ is given by [31, Proposition 3.27]

$$\mathrm{ch}(\Delta_c(\lambda))(w,t) = \frac{\chi_\lambda(w)t^{h_c(\lambda)}}{\det_{\mathfrak{h}^*}(1-tw)}.$$

Furthermore, it is a standard result (e.g. [31, Section 3.13]) that the set of classes

$$\{[\Delta_c(\lambda)] : \lambda \in \mathrm{Irr}(W)\}$$

of the standard modules form a basis of the integral Grothendieck group $K_0(\mathcal{O}_c(W,\mathfrak{h}))$ and that whenever

$$[M] = \sum_{\lambda\in\mathrm{Irr}(W)} n_\lambda[\Delta_c(\lambda)]$$

in $K_0(\mathcal{O}_c(W,\mathfrak{h}))$ we have

$$\mathrm{ch}(M) = \sum_{\lambda\in\mathrm{Irr}(W)} n_\lambda \mathrm{ch}(\Delta_c(\lambda)).$$

In particular, it follows from standard facts about Hilbert series that for any lowest weight module M we have $\mathrm{ch}_0(M)(1,t) = (1-t)^{-r}p_M(t)$, where r is the dimension of support of M and where $p_M(t)$ is a polynomial in t with integer coefficients such that $p_M(t)(1) \neq 0$.

When $M \in \mathcal{O}_c(W,\mathfrak{h})$ is equipped with a graded Hermitian form, we may similarly define the signature character of the tuple (M,β):

Definition 4.2.3.3. *Let $M \in \mathcal{O}_c(W, \mathfrak{h})$ be equipped with a graded Hermitian form β. Then the* signature character *$sch(M, \beta)$ is the formal series*

$$sch(M, \beta)(t) := \sum_{z \in \mathbb{C}} t^z sign(\beta_z)$$

where, for each $z \in \mathbb{C}$, $sign(\beta_z)$ denotes the signature of the restriction β_z of the form β to the weight space M_z.

When $M \in \mathcal{O}_c(W, \mathfrak{h})$ is a lowest weight module and the parameter c is real, we will often write $\mathrm{sch}(M)$ rather than $\mathrm{sch}(M, \beta)$, where it is implicit that the Hermitian form β is W-invariant, contravariant, and positive definite in the lowest weight space. In this setting, we also define the shifted signature character:

Definition 4.2.3.4. *When $M \in \mathcal{O}_c(W, \mathfrak{h})$ is a lowest weight module with lowest weight λ, we define the* shifted signature character *$sch_0(M)$ by*

$$sch_0(M)(t) := t^{-h_c(\lambda)} sch(M)(t).$$

4.2.4 Rationality of Signature Characters

The following rationality result for the shifted signature characters $\mathrm{sch}_0(L_c(\lambda))$ generalizes to arbitrary finite complex reflection groups W a corresponding result for signature characters in type A due to Venkateswaran [70, Corollary 1.3]:

Proposition 4.2.4.1. *For any irreducible complex representation $\lambda \in Irr(W)$ and real parameter $c \in \mathfrak{p}_{\mathbb{R}}$, the shifted signature character $sch_0(L_c(\lambda))$ is of the form*

$$sch_0(L_c(\lambda))(t) = (1-t)^{-r} p_{L_c(\lambda)}(t)$$

for some polynomial $p_{L_c(\lambda)}(t)$ with integer coefficients, where $r = \dim Supp(L_c(\lambda))$.

The proof of Proposition 4.2.4.1 is an inductive argument relying on the following lemmas:

Lemma 4.2.4.2. *Let $\lambda \in Irr(W)$, and let $c_0, c_1 \in \mathfrak{p}_{\mathbb{R}}$ be real parameters so that the Jantzen filtration of $\Delta_{c_0}(\lambda)$ is defined. Let $c(s) = c_0 + sc_1$. For sufficiently small $s > 0$, we have*

$$sch_0(\Delta_{c(s)}(\lambda))(t) = sch_0(\Delta_{c(-s)}(\lambda))(t) + 2t^{-h_{c_0}(\lambda)} \sum_{k \text{ odd}} sch(\Delta_{c_0}(\lambda)^{(k)}, \beta^{(k)}_{c_0,\lambda})(t)$$

and

$$sch_0(L_{c_0}(\lambda))(t) = sch_0(\Delta_{c(s)}(\lambda))(t) - t^{-h_{c_0}(\lambda)} \sum_{k \geq 1} sch(\Delta_{c_0}(\lambda)^{(k)}, \beta^{(k)}_{c_0,\lambda})(t).$$

Proof. This is an immediate consequence of Proposition 4.2.1.2 and the definition of signature characters. □

The following lemma (and its proof) is a reformulation for rational Cherednik algebras of Vogan's [71, Lemma 3.9].

Lemma 4.2.4.3. *Suppose $c = c^\dagger$ and $M \in \mathcal{O}_c(W, \mathfrak{h})$ admits a W-invariant contravariant nondegenerate Hermitian form β. Suppose*

$$[M] = \sum_{i=1}^{n} [L_i]$$

in $K_0(\mathcal{O}_c(W, \mathfrak{h}))$ for some simple modules $L_1, ..., L_n$. Then there are $\epsilon_1, ..., \epsilon_n \in \{\pm 1\}$ such that

$$sch(M, \beta) = \sum_{i=1}^{n} \epsilon_i sch(L_i).$$

Proof. The proof is by induction on the length of M. The case $M = 0$ (or $M = 1$) is trivial, so suppose $M > 0$ and let $L \subset M$ be a simple submodule. If the restriction of β to L is nondegenerate, then $M = L \oplus L^\perp$ and $L^\perp$ is an $H_c(W, \mathfrak{h})$-submodule of M on which β is nondegenerate. As all W-invariant contravariant Hermitian forms on L are proportional, we have $\mathrm{sch}(L, \beta|_L) = \pm \mathrm{sch}(L)$, and the claim follows by induction.

If the restriction of β to L is degenerate, it is 0. The inclusion $L \subset M$ gives rise to a surjection $M^h \to L^h$. By Lemma 4.2.2.5, we have an isomorphism $M \to M^h$,

$m \mapsto \beta(\cdot, m)$. The resulting composition $\varphi : M \to L^h$ satisfies

$$\varphi(m)(l) = \beta(l, m)$$

for all $l \in L$ and $m \in M$. As $\beta|_L = 0$, we have $L^\perp = \ker \varphi \supset L$, and in particular the form β descends to a nondegenerate W-invariant contravariant Hermitian form on the subquotient $N := \ker \varphi / L$. Now we have that $L \cong L^h$ so

$$M = N + L + L^h = N + 2L$$

in $K_0(\mathcal{O}_c(W, \mathfrak{h}))$. Furthermore it follows from [71, Sublemma 3.18] that $\mathrm{sch}(M) = \mathrm{sch}(N)$ and hence $\mathrm{sch}(M) = \mathrm{sch}(N) + \mathrm{sch}(L) - \mathrm{sch}(L)$, and the claim follows by induction. $\square$

Proof of Proposition 4.2.4.1. Let $c \in \mathfrak{p}_{\mathbb{R}}$. It follows from [31, Proposition 3.35] that there are only finitely many $s \in [0, 2]$, say $\{s_1, ..., s_N\}$ for some $N \geq 0$, such that $\mathcal{O}_{sc}(W, \mathfrak{h})$ is not semisimple. Furthermore, at $s = 0$ we have both that $\mathcal{O}_{sc}(W, \mathfrak{h})$ is semisimple and that every simple module $L_0(\lambda) = \Delta_0(\lambda)$ is unitary. In particular, for all λ we have $\mathrm{sch}_0(L_0(\lambda)) = (1-t)^{-l} \dim \lambda$ where $l = \dim \mathfrak{h}$, so the proposition holds for $c = 0$. Furthermore, note that the signature character $\mathrm{sch}_0(\Delta_{c(s)}(\lambda))$ does not depend on s for those s in a fixed interval (s_i, s_{i+1}). So, by induction, we may assume that the proposition holds for all sc with $s \in [0, 1)$ and we need then only prove that it holds for all sc with $s \in [1, 1 + \delta)$ for some $\delta > 0$.

For any $\lambda \in \mathrm{Irr}(W)$ such that $\Delta_{c_0}(\lambda)$ is simple, i.e. $L_{c_0}(\lambda) = \Delta_{c_0}(\lambda)$, the signature character $\mathrm{sch}_0(\Delta_{(1+s)c_0}(\lambda))$ does not depend on s for $|s|$ sufficiently small. In particular, the proposition holds for those $\lambda \in \mathrm{Irr}(W)$ minimal in any highest weight ordering $\leq_{c_0}$ for $\mathcal{O}_{c_0}(W, \mathfrak{h})$. By induction, we may then assume that the proposition holds for those lowest weights $\mu \in \mathrm{Irr}(W)$ strictly lower than λ with respect to $\leq_{c_0}$. Taking $c_1 := c_0$ in Lemma 4.2.4.2, it then follows from Lemmas 4.2.4.2 and 4.2.4.3 that $\mathrm{sch}_0(L_{c_0}(\lambda))$ and all $\mathrm{sch}_0(\Delta_{(1+s)c_0}(\lambda))$ for sufficiently small $s > 0$ are of the form $(1-t)^{-l} p(t)$ for some polynomial $p(t)$ with integer coefficients. As the absolute value of the coefficients of the series $\mathrm{sch}_0(L_{c_0}(\lambda))$ is bounded by the coefficients of $\mathrm{ch}_0(L_{c_0}(\lambda))$, it follows that $\mathrm{sch}_0(L_{c_0}(\lambda))$ is of the form $(1-t)^{-r}p(t)$ for some polynomial p with integer coefficients, as needed. □

4.2.5 Asymptotic Signatures

In this section we will introduce the *asymptotic signature* $a_{c,\lambda}$ of the irreducible representation $L_c(\lambda)$, generalizing to arbitrary finite complex reflection groups the corresponding notion in type A studied by Venkateswaran [70], and prove that it is a rational number in the interval $[-1,1]$. Later, in the case that $\mathrm{Supp} L_c(\lambda) = \mathfrak{h}$, we will see in Theorem 4.4.0.1 that, in the Coxeter case, $a_{c,\lambda}$ is in fact a normalization of the signature of an invariant Hermitian form on $KZ(L_c(\lambda))$, generalizing to arbitrary finite Coxeter groups a corresponding result in type A due to Venkateswaran [70, Theorem 1.4].

Lemma 4.2.5.1. *Let $g(t) = \sum_{n=0}^\infty g_n t^n$ be a rational function regular for $|t| < 1$ and with a pole of maximal order at $t = 1$. Let $p(t)$ and $q(t)$ be polynomials with $q(1) \neq 0$, and let $s(t) = p(t)g(t) = \sum_{n=0}^\infty s_n t^n$ and $d(t) = q(t)g(t) = \sum_{n=0}^\infty d_n t^n$. Then the limits*

$$\lim_{t\to 1^-} \frac{s(t)}{d(t)}, \quad \lim_{N\to\infty} \frac{\sum_{n\le N} s_n}{\sum_{n\le N} d_n}$$

both exist and equal $p(1)/q(1)$.

Proof. That the first limit exists and equals $p(1)/q(1)$ is clear. Let r be the order of the pole of g at $t = 1$. We have

$$\sum_{N=0}^\infty \left(\sum_{n\le N} s_n\right) t^N = p(t)g(t)/(1-t) = (p(t)-p(1))g(t)/(1-t) + p(1)g(t)/(1-t).$$

As $g(t)/(1-t)$ has a pole of order $r+1$ at $t = 1$, greater than the order of any other pole of $g(t)/(1-t)$ or of any pole of $(p(t)-p(1))g(t)/(1-t)$, it follows from a consideration of partial fractions that $\lim_{N\to\infty}(\sum_{n\le N} s_n)/N^r = p(1)C$, where $C \neq 0$ is a nonzero constant depending only on the rational function $g(t)$. Similarly, we have

$\lim_{N\to\infty}(\sum_{n\leq N} d_n)/N^r = q(1)C$, and the claim follows. ☐

We will use Lemma 4.2.5.1 in the special cases that $g(t) = (1-t)^{-l}$ when working signature characters and $g(t) = \prod_{i=1}^{l}(1-t^{d_i})^{-1}$, where $d_1, ..., d_l$ are the fundamental degrees of the reflection group W, when working with the *isotypic signature characters* introduced in Section 4.2.6.

Lemma 4.2.5.2. *Let $c \in \mathfrak{p}_{\mathbb{R}}$ be a real parameter and let $\lambda \in Irr(W)$, so that the irreducible lowest weight module $L_c(\lambda)$ admits a W-invariant contravariant nondegenerate Hermitian form $\beta_{c,\lambda}$, normalized to be positive definite on λ. For any $n \geq 0$, let $\beta_{c,\lambda}^{\leq n}$ denote the restriction of β to the space $L_c(\lambda)^{\leq n} := \bigoplus_{k\leq n} L_c(\lambda)[k]$. Then the limits*

$$\lim_{n\to\infty} \frac{sign(\beta_{c,\lambda}^{\leq n})}{\dim L_c(\lambda)^{\leq n}}, \quad \lim_{t\to 1^-} \frac{sch_0(L_c(\lambda))(t)}{ch_0(L_c(\lambda))(1,t)}$$

exist and are equal to the same rational number $a_{c,\lambda} \in [-1,1]$. If

$$r := \dim Supp(L_c(\lambda)) > 0,$$

then the limit

$$\lim_{n\to\infty} \frac{sign(\beta_{c,\lambda}[n])}{\dim L_c(\lambda)[n]}$$

also exists and equals $a_{c,\lambda}$.

Proof. In the case $r = 0$ the claim is clear, so we may assume $r > 0$. Using Proposition 4.2.4.1, write $\mathrm{ch}_0(L_c(\lambda))(1,t) = \sum_{n=0}^{\infty} d_n t^n = (1-t)^{-r}q(t)$ and $\mathrm{sch}_0(L_c(\lambda)) = \sum_{n=0}^{\infty} s_n t^n = (1-t)^{-r}p(t)$, where $p(t)$ and $q(t)$ are polynomials with integer coefficients satisfying $q(1) \neq 0$, where we use the series expansion about 0 for $(1-t)^{-r}$. In particular, Lemma 4.2.5.1 applies, and the first two limits in the lemma statement exist and equal $p(1)/q(1) \in \mathbb{Q}$. As we clearly have $|\mathrm{sign}(\beta_{c,\lambda}^{\leq n})| \leq \dim L_c(\lambda)^{\leq n}$, we also have $p(1)/q(1) \in [-1,1]$, giving the first claim. The final claim follows similarly, noting that for $n >> 0$ we have that $\mathrm{sign}(\beta_{c,\lambda}[n])$ and $\dim L_c(\lambda)[n]$ are polynomials in n of degree at most r and with coefficients of n^r given by $q(1)/(r-1)!$ and $p(1)/(r-1)!$, respectively. ☐

Definition 4.2.5.3. *For $c \in \mathfrak{p}_{\mathbb{R}}$, the rational number $a_{c,\lambda} \in [-1, 1]$ appearing in Lemma 4.2.5.2 is the* asymptotic signature *of $L_c(\lambda)$. If $a_{c,\lambda} = \pm 1$, we say $L_c(\lambda)$ is* quasi-unitary*.*

4.2.6 Isotypic Signature Characters

As contravariant forms are not only graded but also W-invariant, it is natural to consider the *isotypic signature characters* recording the signatures of contravariant forms both within graded components and also within W-isotypic components:

Definition 4.2.6.1. *Let $c \in \mathfrak{p}_{\mathbb{R}}$ be a real parameter, let β be a contravariant Hermitian form on the module $M \in \mathcal{O}_c(W, \mathfrak{h})$, and for all $z \in \mathbb{C}$ and $\pi \in Irr(W)$ let $M_z^\pi \subset M$ denote the π-isotypic component of the homogeneous degree z part of M and let $\beta_z^\pi = \beta|_{M_z^\pi}$. The* π-isotypic signature character *$sch^\pi(M, \beta)$ of M with respect to β is the formal series*

$$sch^\pi(M, \beta)(t) := \sum_{z \in \mathbb{C}} sign(\beta_z^\pi) t^z.$$

Clearly, we have $\mathrm{sch}(M, \beta) = \sum_{\pi \in \mathrm{Irr}(W)} \mathrm{sch}^\pi(M, \beta)$. When M is lowest weight, we define $\mathrm{sch}_0^\pi(M)(t) \in \mathbb{Z}[[t]]$ completely analogously to $\mathrm{sch}_0(M)$. Similarly, let $\mathrm{ch}^\pi(M)$ denote the graded dimension of M^π, and when M is lowest weight with lowest weight λ let $\mathrm{ch}_0^\pi(M)(t) = t^{-h_c(\lambda)} \mathrm{ch}^\pi(M)(t) \in \mathbb{Z}^{\geq 0}[[t]]$.

The identification of any standard module $\Delta_c(\lambda)$ with $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ used in Section 4.2.1 respects the W-action. In particular, in the setting considered in that section, each isotypic subspace $\Delta_c(\lambda)^\pi$ is equipped with a Jantzen filtration $\{\Delta_c(\lambda)^{\pi, \geq k}\}_{k \geq 0}$, and we have $\Delta_c(\lambda)^{\geq k} = \bigoplus_{\pi \in \mathrm{Irr}(W)} \Delta_c(\lambda)^{\pi, \geq k}$. Furthermore, the wall-crossing formulas in Lemma 4.2.4.2 and the decomposition in Lemma 4.2.4.3 of an arbitrary signature character $\mathrm{sch}(M, \beta)$ in terms of signature characters $\mathrm{sch}(L_i)$ of irreducible representations, and their proofs, have direct analogues for the isotypic signatures characters sch^π.

Let $l = \dim \mathfrak{h}$, let $\{d_i\}_{i=1}^l$ denote the fundamental degrees of W, and let $\mathbb{C}[\mathfrak{h}]^{coW} := \mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}](\mathbb{C}[\mathfrak{h}]^W)^+$ be the coinvariant algebra. Recall that $\mathbb{C}[\mathfrak{h}]^{coW}$ is graded and is isomorphic to $\mathbb{C}W$ as a $\mathbb{C}W$-module and that the graded dimension of $\mathbb{C}[\mathfrak{h}]^W$ is given by $\prod_{i=1}^l (1-t^{d_i})^{-1}$. For any $\lambda, \pi \in \mathrm{Irr}(W)$, let $\theta_\lambda^\pi \in \mathbb{Z}^{\geq 0}[t]$ denote the graded dimension of the π-isotypic subspace of $\lambda \otimes \mathbb{C}[\mathfrak{h}]^{coW}$.

Lemma 4.2.6.2. *For any $\lambda, \pi \in Irr(W)$, we have:*

(1) $\theta_\lambda^\pi(1) = (\dim \lambda)(\dim \pi)^2$

(2) $ch_0^\pi(\Delta_0(\lambda)) = sch_0^\pi(\Delta_0(\lambda)) = \theta_\lambda^\pi(t)/\prod_{i=1}^l (1-t^{d_i})$.

(3)
$$\lim_{n\to\infty} \frac{\dim \Delta_0(\lambda)^{\pi, \leq n}}{\dim \Delta_0(\lambda)^{\leq n}} = \lim_{t \to 1^-} \frac{ch_0^\pi(\Delta_0(\lambda))(t)}{ch_0(\Delta_0(\lambda))(1,t)} = \frac{(\dim \pi)^2}{|W|}.$$

Proof. $\theta_\lambda^\pi(1)$ is the dimension of the π-isotypic subspace of $\lambda \otimes \mathbb{C}[\mathfrak{h}]^{coW}$; as $\mathbb{C}[\mathfrak{h}]^{coW} \cong \mathbb{C}W$ as a $\mathbb{C}W$-module and as $\lambda \otimes \mathbb{C}W \cong \mathbb{C}W^{\oplus \dim \lambda}$, the first statement follows. The second statement follows from the fact that $\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^{coW} \otimes \mathbb{C}[\mathfrak{h}]^W$ as a graded $\mathbb{C}W$-module. The third statement follows from the first two statements and from Lemma 4.2.5.1, using the facts that $\mathrm{ch}_0(\Delta_0(\lambda))(1,t) = (\dim \lambda) P_W(t)/\prod_{i=1}^l (1-t^{d_i})$, where $P_W(t)$ is the Poincaré polynomial of W, and $P_W(1) = |W|$. □

Note that Lemma 4.2.4.2 holds for the isotypic signature characters sch^π as well, by the same proof as for the usual signature characters sch. As the signs ϵ_i appearing in Lemma 4.2.4.3 have no dependence on the irreducible representation π, it follows from Lemma 4.2.6.2 and the proof of Proposition 4.2.4.1 that we have:

Lemma 4.2.6.3. *For any $c \in \mathfrak{p}_\mathbb{R}$, there exists a collection of polynomials $\{n_c^{\lambda,\mu}(t) \in \mathbb{Z}[t] : \lambda, \mu \in Irr(W)\}$ such that for every $\lambda, \pi \in Irr(W)$ we have*

$$sch_0^\pi(L_c(\lambda))(t) = \sum_{\mu \in Irr(W)} n_c^{\lambda,\mu}(t) sch_0^\pi(\Delta_0(\mu))(t)$$

$$= \prod_{i=1}^l (1-t^{d_i})^{-1} \sum_{\mu \in Irr(W)} n_c^{\lambda,\mu}(t) \theta_\mu^\pi(t).$$

In particular, $sch_0^\pi(L_c(\lambda))(t)$ is a rational function of t.

Corollary 4.2.6.4. *In the setting of Lemma 4.2.6.3, the rational function*

$$sch_0^\pi(L_c(\lambda))(t)$$

has a pole of order at most $r := \dim Supp(L_c(\lambda))$ *at* $t = 1$.

Proof. By Lemma 4.2.6.3, we see that $\mathrm{sch}_0^\pi(L_c(\lambda))(t)$ is absolutely convergent for $|t| < 1$, and by a comparison of coefficients we see that the rational function

$$\mathrm{sch}_0^\pi(L_c(\lambda))(t)/\mathrm{ch}_0(L_c(\lambda))(1,t)$$

takes values in $[-1,1]$ on the interval $[0,1)$. As $\mathrm{ch}_0(L_c(\lambda))(1,t)$ has a pole of order r at $t = 1$, the claim follows. $\square$

The following proposition allows the asymptotic signature $a_{c,\lambda}$ of $L_c(\lambda)$ to be computed in any isotypic component when $L_c(\lambda)$ has full support. Considering the case when $L_c(\lambda)$ is finite-dimensional, note that this need not be true when $L_c(\lambda)$ has proper support.

Proposition 4.2.6.5. *Let* $c \in \mathfrak{p}_{\mathbb{R}}$, $\lambda \in Irr(W)$, *and suppose* $L_c(\lambda)$ *has full support. Then for all* $\pi \in Irr(W)$ *the limit*

$$\lim_{n\to\infty} \frac{sign(\beta_{c,\lambda}^{\pi,\leq n})}{\dim L_c(\lambda)^{\pi,\leq n}}$$

exists and equals $a_{c,\lambda}$. *In particular, this limit is independent of* π.

Proof. As we have

$$a_{c,\lambda} = \lim_{n\to\infty} \frac{\mathrm{sign}(\beta_{c,\lambda}^{\leq n})}{\dim L_c(\lambda)^{\leq n}} = \lim_{n\to\infty} \frac{\sum_{\pi\in\mathrm{Irr}(W)} \mathrm{sign}(\beta_{c,\lambda}^{\pi,\leq n})}{\sum_{\pi\in\mathrm{Irr}(W)} \dim L_c(\lambda)^{\pi,\leq n}}$$

it suffices to show that the limit in the proposition statement exists and is independent of π. As $L_c(\lambda)$ has full support, considering the decomposition of $[L_c(\lambda)]$ in $K_0(\mathcal{O}_c(W,\mathfrak{h}))$ in terms of the classes of standard modules $[\Delta_c(\mu)]$ for various $\mu \in \mathrm{Irr}(W)$, we have

$$\lim_{n\to\infty} \frac{\dim L_c(\lambda)^{\pi,\leq n}}{\dim L_c(\lambda)^{\leq n}} = \lim_{t\to 1^-} \frac{\mathrm{ch}_0^\pi(L_c(\lambda))(t)}{\mathrm{ch}_0(L_c(\lambda))(1,t)} = \frac{(\dim \pi)^2}{|W|} \tag{4.2.1}$$

where the first equality follows from Lemma 4.2.5.1 and the second equality follows from Lemma 4.2.6.2(3) and the fact that $L_c(\lambda)$ has full support in $\mathfrak{h}$. By equation (4.2.1) and Lemmas 4.2.5.1 and 4.2.6.2 we have

$$\lim_{n\to\infty} \frac{\mathrm{sign}(\beta_{c,\lambda}^{\pi,\leq n})}{\dim L_c(\lambda)^{\pi,\leq n}}$$

$$= \frac{|W|}{(\dim \pi)^2} \lim_{n\to\infty} \frac{\mathrm{sign}(\beta_{c,\lambda}^{\pi,\leq n})}{\dim L_c(\lambda)^{\leq n}}$$

$$= \frac{|W|}{(\dim \pi)^2} \lim_{t\to 1^-} \frac{\mathrm{sch}_0^\pi(L_c(\lambda))(t)}{\mathrm{ch}_0(L_c(\lambda))(1,t)}$$

$$= \frac{|W|}{(\dim \pi)^2} \lim_{t\to 1^-} \frac{\sum_{\mu\in\mathrm{Irr}(W)} n_c^{\lambda,\mu}(t)\theta_\mu^\pi(t)/\prod_{i=1}^l(1-t^{d_i})}{\mathrm{ch}_0(L_c(\lambda))(1,t)}$$

$$= |W| \lim_{t\to 1^-} \frac{\sum_{\mu\in\mathrm{Irr}(W)} n_c^{\lambda,\mu}(1)\dim\mu}{\mathrm{ch}_0(L_c(\lambda))(1,t)\prod_{i=1}^l(1-t^{d_i})},$$

which is visibly independent of π, as needed. Note that this final limit exists because $L_c(\lambda)$ has full support in $\mathfrak{h}$, so $\mathrm{ch}_0(L_c(\lambda))(1,t)$ as a pole of order l at $t=1$. $\square$

4.3 The Dunkl Weight Function

4.3.1 The Contravariant Pairing and Contravariant Form

As in the previous section, let W be a finite complex reflection group with reflection representation $\mathfrak{h}$. Fix a positive-definite W-invariant inner product on $\mathfrak{h}_{\mathbb{R}}$, and let $(\cdot,\cdot)_{\mathfrak{h}}$ be its unique extension to a W-invariant positive-definite Hermitian form on $\mathfrak{h}$. Let $T:\mathfrak{h}\to\mathfrak{h}^*$, $T(y)(x)=(x,y)_{\mathfrak{h}}$, be the conjugate-linear W-invariant isomorphism introduced in Section 4.2.1, giving rise for any parameter $c\in\mathfrak{p}$ to the isomorphism $\sigma:\overline{H_c(W,\mathfrak{h})^{opp}}\to H_{c^\dagger}(W,\mathfrak{h})$ of $\mathbb{C}$-algebras and conjugate-linear equivalence of categories

$$^h : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_{c^\dagger}(W,\mathfrak{h})^{opp}$$

introduced in Section 4.2.2.

Let $\lambda \in \mathrm{Irr}(W)$ be an irreducible representation of W, and fix a positive-definite W-invariant Hermitian form $(\cdot,\cdot)_\lambda$ on λ. For any $c \in \mathfrak{p}$, the W-invariant isomorphism $\varphi : \lambda \to \overline{\lambda^*}$ given by $\varphi(v)(v') = (v', v)$ between the lowest weight spaces of $\Delta_{c^\dagger}(\lambda)$ and $\Delta_c(\lambda)^h$ extends uniquely to an $H_{c^\dagger}(W, \mathfrak{h})$-homomorphism $\varphi : \Delta_{c^\dagger}(\lambda) \to \Delta_c(\lambda)^h$. When it is convenient, we will use the notation $v_2^\dagger v_1$ to denote the inner product $(v_1, v_2)_\lambda$ for $v_1, v_2 \in \lambda$, borrowing notation from the standard inner product on $\mathbb{C}^N$. Similarly, we will use the notation $A^\dagger$ to denote the adjoint of an operator $A \in \mathrm{End}_{\mathbb{C}}(\lambda)$ with respect to the form $(\cdot,\cdot)_\lambda$.

Definition 4.3.1.1. *For $c \in \mathfrak{p}$ and $\lambda \in Irr(W)$, let $\beta_{c,\lambda}$ be the sesquilinear pairing*

$$\beta_{c,\lambda} : \Delta_c(\lambda) \times \Delta_{c^\dagger}(\lambda) \to \mathbb{C}$$

given by

$$\beta_{c,\lambda}(v, v') = \varphi(v')(v).$$

Remark 4.3.1.2. *$\beta_{c,\lambda}$ depends on the choice of forms $(\cdot,\cdot)_{\mathfrak{h}}$ and $(\cdot,\cdot)_\lambda$ and is therefore determined by $c \in \mathfrak{p}$ and $\lambda \in Irr(W)$ only up to a positive real multiple.*

Remark 4.3.1.3. *When the parameter $c \in \mathfrak{p}$ is real, i.e. $c \in \mathfrak{p}_{\mathbb{R}}$ and $c = c^\dagger$, the pairing $\beta_{c,\lambda}$ is precisely the contravariant Hermitian form on $\Delta_c(\lambda)$ considered in [10] and [33, Definition 4.5]. While this case is the primary case of interest, we will consider the more general setting of complex c to show that the associated Dunkl weight function, introduced later, is defined and holomorphic for all $c \in \mathfrak{p}$.*

The following proposition is standard (for $c \in \mathfrak{p}_{\mathbb{R}}$ it is a reformulation of [33, Proposition 2.2]):

Proposition 4.3.1.4.

(i) $\beta_{c,\lambda}$ is the unique sesquilinear form $\beta_{c,\lambda} : \Delta_c(\lambda) \times \Delta_{c^\dagger}(\lambda) \to \mathbb{C}$ that is:

(a) normalized: $\beta_{c,\lambda}|_\lambda = (\cdot,\cdot)_\lambda$

(b) contravariant: $\beta_{c,\lambda}(hv, v') = \beta_{c,\lambda}(v, \sigma(h)v')$ for all $h \in H_c(W, \mathfrak{h})$, $v \in \Delta_c(\lambda)$, and $v' \in \Delta_{c^\dagger}(\lambda)$,

(ii) Any contravariant form $\beta : \Delta_c(\lambda) \times \Delta_{c^\dagger}(\lambda) \to \mathbb{C}$ is proportional to $\beta_{c,\lambda}$.

(iii) $\beta_{c,\lambda}$ is graded, W-invariant, and satisfies $\beta_{c,\lambda} = \beta^\dagger_{c^\dagger,\lambda}$.

(iv) $\beta_{c,\lambda}$ descends to a nondegenerate pairing $\beta_{c,\lambda} : L_c(\lambda) \times L_{c^\dagger}(\lambda) \to \mathbb{C}$.

4.3.2 The Gaussian Pairing and Gaussian Inner Product

For the remainder of this chapter, let W be a finite real reflection group (equivalently, a finite Coxeter group). Let $\mathfrak{h}_\mathbb{R}$ denote its real reflection representation, let $\mathfrak{h} = \mathbb{C}\otimes_\mathbb{R}\mathfrak{h}_\mathbb{R}$ denote the complexified reflection representation, let $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*_\mathbb{R}$ be a system of positive roots for W, let $\{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}_\mathbb{R}$ be the associated system of positive coroots, and fix an orthonormal basis $y_1, ..., y_l$ of $\mathfrak{h}_\mathbb{R}$ with associated dual basis $x_1, ..., x_l \in \mathfrak{h}^*_\mathbb{R}$. In this setting, $H_c(W, \mathfrak{h})$ contains an internal W-invariant $\mathfrak{sl}_2$-triple $\mathbf{e}, \mathbf{f}, \mathbf{h}$ given by

$$\mathbf{h} = \sum_i x_i y_i + \frac{1}{2}\dim\mathfrak{h} - \sum_{s\in S} c_s s = \sum_i (x_i y_i + y_i x_i)/2, \quad \mathbf{e} := -\frac{1}{2}\sum_i x_i^2 \quad \mathbf{f} := \frac{1}{2}\sum_i y_i^2.$$

The elements $\mathbf{e}, \mathbf{f}, \mathbf{h}$ do not depend on the choice of orthonormal basis $y_1, ..., y_l \in \mathfrak{h}_\mathbb{R}$.

Note that the operator $\mathbf{f}$ acts locally nilpotently on any lowest weight module. In particular, following [10] and [33, Definition 4.5], we make the following definition:

Definition 4.3.2.1. *The* Gaussian pairing $\gamma_{c,\lambda}$ *is the sesquilinear pairing*

$$\gamma_{c,\lambda} : \Delta_c(\lambda) \times \Delta_{c^\dagger}(\lambda) \to \mathbb{C}$$

defined by

$$\gamma_{c,\lambda}(P, Q) = \beta_{c,\lambda}(\exp(\mathbf{f})P, \exp(\mathbf{f})Q).$$

When $c \in \mathfrak{p}_\mathbb{R}$, $\gamma_{c,\lambda}$ is a Hermitian form on $\Delta_c(\lambda)$ and will be called the Gaussian inner product.

Remark 4.3.2.2. *For $c \in \mathfrak{p}_\mathbb{R}$, the study of the signatures of the Hermitian forms $\beta_{c,\lambda}$ and $\gamma_{c,\lambda}$ is equivalent; we have $sign(\beta^{\leq n}_{c,\lambda}) = sign(\gamma^{\leq n}_{c,\lambda})$ for all $n \geq 0$, where $\beta^{\leq n}_{c,\lambda}$ and $\gamma^{\leq n}_{c,\lambda}$ denote the restrictions of $\beta_{c,\lambda}$ and $\gamma_{c,\lambda}$, respectively, to $\Delta_c(\lambda)^{\leq n}$. In particular, the asymptotic signature $a_{c,\lambda}$ can be equally well computed using the form $\gamma_{c,\lambda}$. As* *we will see, the advantage of the form $\gamma_{c,\lambda}$ is that $\mathfrak{h}_{\mathbb{R}}^*$ acts on $\Delta_c(\lambda)$ by self-adjoint operators, making an integral representation of this form possible and allowing for analytic techniques to be used to study the signatures.*

The following proposition generalizes [33, Proposition 4.6] to complex $c \in \mathfrak{p}$ and provides a useful characterization of $\gamma_{c,\lambda}$:

Proposition 4.3.2.3.

(i) The pairing $\gamma_{c,\lambda}$ satisfies:

$$\gamma_{c,\lambda}(xP, Q) = \gamma_{c,\lambda}(P, xQ) \quad \textit{for all } x \in \mathfrak{h}_{\mathbb{R}}^*, P \in \Delta_c(\lambda), Q \in \Delta_{c^\dagger}(\lambda).$$

(ii) Up to scaling, $\gamma_{c,\lambda}$ is the unique W-invariant sesquilinear pairing $\Delta_c(\lambda) \times \Delta_{c^\dagger}(\lambda) \to \mathbb{C}$ satisfying

$$\gamma_{c,\lambda}((-y + Ty)P, Q) = \gamma_{c,\lambda}(P, yQ) \quad \textit{for all } y \in \mathfrak{h}_{\mathbb{R}}, P \in \Delta_c(\lambda), Q \in \Delta_{c^\dagger}(\lambda)$$

and

$$\gamma_{c,\lambda}(yP, Q) = \gamma_{c,\lambda}(P, (-y + Ty)Q) \quad \textit{for all } y \in \mathfrak{h}_{\mathbb{R}}, P \in \Delta_c(\lambda), Q \in \Delta_{c^\dagger}(\lambda).$$

$\gamma_{c,\lambda}$ is determined among such forms by the normalization condition $\gamma_{c,\lambda}|_\lambda = (\cdot, \cdot)_\lambda$.

(iii) Up to scaling, $\gamma_{c,\lambda}$ is the unique W-invariant sesquilinear pairing $\Delta_c(\lambda) \times \Delta_{c^\dagger}(\lambda) \to \mathbb{C}$ satisfying

$$\gamma_{c,\lambda}(xP, Q) = \gamma_{c,\lambda}(P, xQ) \quad \textit{for all } x \in \mathfrak{h}_{\mathbb{R}}^*, P \in \Delta_c(\lambda), Q \in \Delta_{c^\dagger}(\lambda)$$

and

$$\gamma_{c,\lambda}((-y + Ty)P, v) = 0 \quad \textit{for all } y \in \mathfrak{h}_{\mathbb{R}}, P \in \Delta_c(\lambda), v \in \lambda.$$

$\gamma_{c,\lambda}$ is determined among such forms by the normalization condition $\gamma_{c,\lambda}|_\lambda = (\cdot, \cdot)_\lambda$.

(iv) $\gamma_{c,\lambda} = \gamma_{c^\dagger,\lambda}^\dagger$.

Proof. Parts (i) and (ii) are proved using Proposition 4.3.1.4 and exactly the same calculations and arguments appearing in the proof of [33, Proposition 4.6]. To prove (iii), it suffices to show that the properties in (iii) imply the properties in (ii). Let γ be a pairing as in (iii). As $\mathfrak{h}_{\mathbb{R}}^*$ acts by operators self-adjoint with respect to γ, the two equations appearing in (ii) are equivalent because $Ty \in \mathfrak{h}_{\mathbb{R}}^*$ is self-adjoint, so it suffices to prove that

$$\gamma((-y+Ty)P, Q) = \gamma(P, yQ) \tag{4.3.1}$$

for all $y \in \mathfrak{h}_{\mathbb{R}}, P \in \Delta_c(\lambda)$, and $Q \in \Delta_{c^\dagger}(\lambda)$. We will prove equation (4.3.1) by induction on $\deg Q$. As $yv = 0$ for all $y \in \mathfrak{h}_{\mathbb{R}}$ and $v \in \lambda$, equation (4.3.1) holds for all $y \in \mathfrak{h}_{\mathbb{R}}, P \in \Delta_c(\lambda)$, and $Q \in \lambda$ by the second property in (iii), establishing the base case $\deg Q = 0$. Now suppose, for some $N \geq 0$, that equation (4.3.1) holds for all $y \in \mathfrak{h}_{\mathbb{R}}, P \in \Delta_c(\lambda)$, and $Q \in \Delta_{c^\dagger}(\lambda)$ with $\deg Q \leq N$. For any $x \in \mathfrak{h}_{\mathbb{R}}^*, y \in \mathfrak{h}_{\mathbb{R}}, P \in \Delta_c(\lambda)$, and $Q \in \Delta_{c^\dagger}(\lambda)$ we have

$$\begin{aligned}
&\gamma((-y+Ty)P, xQ) - \gamma(P, yxQ) \\
&= \gamma(x(-y+Ty)P, Q) - \gamma\left(P, \left(xy + x(y) - \sum_{s\in S} \overline{c_s}\alpha_s(y)\alpha_s^\vee(x)s\right)Q\right) \\
&= \gamma(x(-y+Ty)P, Q) - \gamma\left(\left((-y+Ty)x + x(y) - \sum_{s\in S} c_s\alpha_s(y)\alpha_s^\vee(x)s\right)P, Q\right) \\
&= 0
\end{aligned}$$

where we use the commutation relation for $[y, x]$ in $H_{c^\dagger}(W, \mathfrak{h})$ and $H_c(W, \mathfrak{h})$ in the first and last equality, respectively, and the fact that $[x, Ty] = 0$, establishing the inductive step.

Statement (iv) follows immediately from $\beta_{c,\lambda} = \beta^\dagger_{c^\dagger,\lambda}$. □

4.3.3 Dunkl Weight Function: Characterization and Properties

The remainder of Section 4.3 is dedicated to proving the following main theorem:

Theorem 4.3.3.1. *For any finite Coxeter group W and irreducible representation $\lambda \in Irr(W)$, there is a unique family $K_{c,\lambda}$, holomorphic in $c \in \mathfrak{p}$, of $End_{\mathbb{C}}(\lambda)$-valued tempered distributions on $\mathfrak{h}_{\mathbb{R}}$ such that the following integral representation of the Gaussian pairing $\gamma_{c,\lambda}$ holds for all $c \in \mathfrak{p}$:*

$$\gamma_{c,\lambda}(P,Q) = \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^{\dagger} K_{c,\lambda}(x) P(x) e^{-|x|^2/2} dx \qquad \text{for all } P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda,$$

where we make the standard identification of $\Delta_c(\lambda)$ and $\Delta_{c^\dagger}(\lambda)$ with $\mathbb{C}[\mathfrak{h}] \otimes \lambda$. Furthermore, $K_{c,\lambda}$ satisfies the additional properties:

(i) For all $M > 0$ there exists an integer $N \geq 0$, which may be taken to be 0 for M sufficiently small, such that for $c \in \mathfrak{p}$ with $|c_s| < M$ for all $s \in S$ the distribution $\delta^N K_{c,\lambda}$, where $\delta := \prod_{s \in S} \alpha_s \in \mathbb{C}[\mathfrak{h}]$ is the discriminant element, is given by integration against an analytic function on $\mathfrak{h}_{\mathbb{R},reg}$ that is locally integrable over $\mathfrak{h}_{\mathbb{R}}$.

(ii) For any $x \in \mathfrak{h}_{\mathbb{R},reg}$ the operator $K_{c,\lambda}(x) \in End_{\mathbb{C}}(\lambda)$ determines a B_W-invariant sesquilinear pairing

$$KZ_x(\Delta_c(\lambda)) \times KZ_x(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}$$

by the formula $(v_1, v_2) = v_2^\dagger K_{c,\lambda}(x) v_1$. When $c \in \mathfrak{p}_{\mathbb{R}}$, i.e. $c = c^\dagger$, this pairing is a Hermitian form.

Remark 4.3.3.2. *The uniqueness of $K_{c,\lambda}$ is immediate from the standard fact that the subspace $\mathbb{C}[\mathfrak{h}]e^{-|x|^2/2}$ is dense in the space of complex-valued Schwartz functions $\mathscr{S}(\mathfrak{h}_{\mathbb{R}})$ on $\mathfrak{h}_{\mathbb{R}}$.*

Remark 4.3.3.3. *The existence of the distribution $K_{c,\lambda}$ was shown in the case that W is a dihedral group and c is real and small (for any λ) by Dunkl [19]. The case in which λ is 1-dimensional is much simpler than the case of general λ and was used by Etingof in [28] to study the support of the irreducible representation $L_c(triv)$. Related results in the trigonometric case in type A appear in [22, 23, 24].*

In preparation for the proof of Theorem 4.3.3.1, in the remainder of Section 4.3.3 we will now establish several necessary properties and an additional characterization of the distribution $K_{c,\lambda}$. The proof of Theorem 4.3.3.1 will then proceed in three steps:

in Section 4.3.4 we will establish the existence of $K_{c,\lambda}$ for small c and show that in this case it is given by integration against a locally L^1 function; in Section 4.3.5 we will produce an analytic continuation for all $c \in \mathfrak{p}$ of this function over $\mathfrak{h}_{\mathbb{R},reg}$; and in Section 4.3.6 we will show that these functions extend naturally to distributions on all of $\mathfrak{h}_{\mathbb{R}}$, completing the proof of Theorem 4.3.3.1.

An element $y \in \mathfrak{h} \subset H_c(W, \mathfrak{h})$ acts in the representation $\Delta_c(\lambda) = \mathbb{C}[\mathfrak{h}] \otimes \lambda$ by the operator

$$D_y = \partial_y - \sum_{s \in S} c_s \frac{\alpha_s(y)}{\alpha_s}(1 - s) \otimes s.$$

This operator D_y acts naturally as a continuous operator on the Schwartz space $\mathscr{S}(\mathfrak{h}_{\mathbb{R}})$, and we will use the notation D_y for this continuous operator as well, with the representation λ and parameter c implicit and suppressed from the notation. We may equivalently view any tempered distribution K on $\mathfrak{h}_{\mathbb{R}}$ with values in $\mathrm{End}_{\mathbb{C}}(\lambda)$ as a continuous operator

$$K : \mathscr{S}(\mathfrak{h}_{\mathbb{R}}) \otimes \lambda \to \lambda.$$

In particular, for any such distribution K, the composition $K \circ D_y$ is defined and is another such distribution. With this in mind, it is now straightforward to translate the conditions in Proposition 4.3.2.3(iii) into a convenient characterization of the Dunkl weight function $K_{c,\lambda}$:

Proposition 4.3.3.4. *Let K be an $End_{\mathbb{C}}(\lambda)$-valued tempered distribution on $\mathfrak{h}_{\mathbb{R}}$, and let γ be the sesquilinear form on $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ defined by*

$$\gamma(P, Q) = \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger K(x) P(x) e^{-|x|^2/2} dx.$$

Then $\gamma = \gamma_{c,\lambda}$ if and only if the following hold:

(i) normalization:

$$\int_{\mathfrak{h}_{\mathbb{R}}} K(x) e^{-|x|^2/2} dx = Id_\lambda$$

(ii) W-invariance: $wK(w^{-1}x)w^{-1} = K(x)$ for all $w \in W$

(iii) annihilation by Dunkl operators: $K \circ D_y = 0$ for all $y \in \mathfrak{h}_{\mathbb{R}}$.

Proof. It is immediate that the normalization condition (i) above and the normalization condition $\gamma|_\lambda = (\cdot,\cdot)_\lambda$ are equivalent and that the W-invariance condition (ii) above is equivalent to the W-invariance of the form γ. It is also immediate that $\gamma(xP,Q) = \gamma(P,xQ)$ for all $x \in \mathfrak{h}_\mathbb{R}$ and $P,Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$, so it remains to show that condition (iii) is equivalent to the condition $\gamma((-y_i+x_i)P,v) = 0$ for all $P \in \mathbb{C}[\mathfrak{h}] \otimes \lambda, v \in \lambda$, and $i = 1,...,l = \dim\mathfrak{h}$. As $e^{-|x|^2/2}$ is W-invariant we have $D_{y_i}(P(x)e^{-|x|^2/2}) = (D_{y_i}P(x) - x_i)e^{-|x|^2/2}$, and a direct calculation then gives

$$\gamma((-y_i+x_i)P,v) = \int_{\mathfrak{h}_\mathbb{R}} v^\dagger K(x)(-y_i+x_i)(P(x))e^{-|x|^2/2}dx$$

$$= \int_{\mathfrak{h}_\mathbb{R}} v^\dagger (K \circ D_{y_i})(P(x)e^{-|x|^2/2})dx.$$

As $\mathbb{C}[\mathfrak{h}]e^{-|x|^2/2}$ is dense in $\mathscr{S}(\mathfrak{h}_\mathbb{R})$, we see that $\gamma((-y_i+x_i)P,v) = 0$ for all $P \in \mathbb{C}[\mathfrak{h}]\otimes\lambda$ and $v \in \lambda$ if and only if $K \circ D_{y_i} = 0$, as needed. □

Proposition 4.3.3.5. *Any family of tempered distributions $K_{c,\lambda}$ as appearing in Theorem 4.3.3.1 satisfies the equation $K_{c,\lambda} = K^\dagger_{c^\dagger,\lambda}$. In particular, $K_{c,\lambda}$ takes values in Hermitian forms on λ when $c \in \mathfrak{p}_\mathbb{R}$.*

Proof. Immediate from the equality $\gamma_{c,\lambda} = \gamma^\dagger_{c^\dagger,\lambda}$. □

Proposition 4.3.3.6. *Any distribution K on $\mathfrak{h}_\mathbb{R}$ with values in $End_\mathbb{C}(\lambda)$ satisfying the conditions (i), (ii), and (iii) appearing in Proposition 4.3.3.4 satisfies the following properties:*

(i) The restriction $K|_{\mathfrak{h}_{\mathbb{R},reg}}$ of K to the real regular locus

$$\mathfrak{h}_{\mathbb{R},reg} := \{x \in \mathfrak{h}_\mathbb{R} : \alpha_s(x) \neq 0 \textit{ for all } s \in S\}$$

satisfies the 2-sided KZ-type system of differential equations:

$$\partial_y K + \sum_{s\in S} c_s \frac{\alpha_s(y)}{\alpha_s}(sK+Ks) = 0 \qquad \textit{for all } y \in \mathfrak{h}_\mathbb{R} \tag{4.3.2}$$

(ii) K is a homogeneous distribution of degree $-2\chi_\lambda(\sum_{s\in S} c_s s)/\dim\lambda$, where χ_λ

is the character of λ. In particular, K is tempered.

Proof. To prove (i), take an arbitrary test function $\varphi \in C_c^\infty(\mathfrak{h}_{\mathbb{R},reg}) \otimes \lambda$. For any $y \in \mathfrak{h}_{\mathbb{R}}$, we have $(K \circ D_y)(\varphi) = 0$. From the definition of D_y and the fact that $(1 \otimes s)(\varphi)/\alpha_s$ and $(s \otimes s)(\varphi)/\alpha_s$ are also well-defined test functions in $C_c^\infty(\mathfrak{h}_{\mathbb{R}}) \otimes \lambda$ for any $s \in S$, we have:

$$0 = \int_{\mathfrak{h}_{\mathbb{R}}} (K \circ D_y)(\varphi) dx$$

$$= \int_{\mathfrak{h}_{\mathbb{R}}} K_c(x) \left(\partial_y \varphi(x) - \sum_{s \in S} c_s \frac{\alpha_s(y)}{\alpha_s(x)} (s\varphi(x) - s\varphi(sx)) \right) dx$$

$$= \int_{\mathfrak{h}_{\mathbb{R}}} \left(-\partial_y K(x) - \sum_{s \in S} c_s \frac{\alpha_s(y)}{\alpha_s(x)} K(x) s \right) \varphi(x) dx$$

$$+ \sum_{s \in S} c_s \int_{\mathfrak{h}_{\mathbb{R}}} \frac{\alpha_s(y)}{\alpha_s(x)} K(x) s\varphi(sx) dx$$

$$= \int_{\mathfrak{h}_{\mathbb{R}}} \left(-\partial_y K(x) - \sum_{s \in S} c_s \frac{\alpha_s(y)}{\alpha_s(x)} K(x) s \right) \varphi(x) dx$$

$$+ \sum_{s \in S} c_s \int_{\mathfrak{h}_{\mathbb{R}}} \frac{\alpha_s(y)}{\alpha_s(sx)} K(sx) s\varphi(x) dx$$

$$= \int_{\mathfrak{h}_{\mathbb{R}}} \left(-\partial_y K(x) - \sum_{s \in S} c_s \frac{\alpha_s(y)}{\alpha_s(x)} K(x) s \right) \varphi(x) dx$$

$$- \sum_{s \in S} c_s \int_{\mathfrak{h}_{\mathbb{R}}} \frac{\alpha_s(y)}{\alpha_s(x)} sK(x) \varphi(x) dx$$

$$= - \int_{\mathfrak{h}_{\mathbb{R}}} \left(\partial_y K + \sum_{s \in S} c_s \frac{\alpha_s(y)}{\alpha_s(x)} (sK + Ks) \right) \varphi(x) dx,$$

where in the fourth equality we change variables $x \mapsto sx$ and in the fifth equality we use the W-invariance of K and the equality $\alpha_s(sx) = -\alpha_s(x)$. As $\varphi \in C_c^\infty(\mathfrak{h}_{\mathbb{R},reg}) \otimes \lambda$ was arbitrary, this proves (i).

Next, note that as $x_1, ..., x_l \in \mathfrak{h}_{\mathbb{R}}^*$ and $y_1, ..., y_l \in \mathfrak{h}_{\mathbb{R}}$ are dual bases we have

$$\sum_{i=1}^{l} f(y_i) w(x_i) = w(f)$$

for all $f \in \mathfrak{h}^*$ and $s \in S$. It follows that we have, as operators on $\mathbb{C}[\mathfrak{h}] \otimes \lambda$,

$$\sum_{i=1}^{l} \frac{\alpha_s(y_i)}{\alpha_s}(1-s)(x_i) = \frac{1}{\alpha_s}\left(\sum_{i=1}^{l} \alpha_s(y_i)(x_i - s(x_i)s)\right) = \frac{1}{\alpha_s}(\alpha_s - s(\alpha_s)s) = 1+s \tag{4.3.3}$$

Using the fact that $K \circ D_y = 0$ for all $y \in \mathfrak{h}_{\mathbb{R}}$, the action of the Euler operator $\sum_{i=1}^{l} x_i \partial_{y_i}$ on the distribution $K(x)$ is therefore given by

$$\left(\sum_{i=1}^{l} x_i \partial_{y_i}\right) K(x) = -K(x) \sum_{i=1}^{l} \partial_{y_i} x_i$$

$$\begin{aligned}
&= -K(x) \sum_{i=1}^{l} \left(\sum_{s \in S} c_s \frac{\alpha_s(y_i)}{\alpha_s}(1-s) \otimes s\right) x_i \\
&= -K(x) \left(\sum_{s \in S} c_s \left(\sum_{i=1}^{l} \frac{\alpha_s(y_i)}{\alpha_s}(1-s)x_i\right) \otimes s\right) \\
&= -K(x) \sum_{s \in S} c_s(1+s) \otimes s \\
&= -\sum_{s \in S} c_s(sK(x) + K(x)s) \\
&= \frac{-2\chi_\lambda(\sum_{s \in S} c_s s)}{\dim \lambda} K(x),
\end{aligned}$$

where the second equality follows from the equation $K \circ D_{y_i} = 0$, the fourth equality uses equation (4.3.3), the fifth equality uses the W-invariance of $K(x)$, and the final equality uses the fact that $\sum_{s \in S} c_s s$ is central in $\mathbb{C}W$ and therefore acts on λ by the scalar $\chi_\lambda(\sum_{s \in S} c_s s)/\dim \lambda$, giving (ii). $\square$

Proposition 4.3.3.7. *For any $c \in \mathfrak{p}$, the support of the distribution $K_{c,\lambda}$ coincides with the set of real points of the support of $L_c(\lambda)$:*

$$Supp(K_{c,\lambda}) = Supp(L_c(\lambda))_{\mathbb{R}}.$$

Proof. This was shown by Etingof [28, Proposition 3.10] in the case $\lambda = \text{triv}$, and the proof generalizes to arbitrary λ without modification. $\square$

4.3.4 Existence of Dunkl Weight Function for Small c

Let $\alpha_1, ..., \alpha_l$ be the simple roots associated to the system of positive roots $\{\alpha_s\}_{s\in S}$, let

$$\mathcal{C} := \{x \in \mathfrak{h}_{\mathbb{R}} : \alpha_i(x) > 0 \text{ for } i = 1, ..., l\}$$

be the open associated fundamental Weyl chamber, and let $\overline{\mathcal{C}}$ denote its closure. Recall that

$$\delta := \prod_{s\in S} \alpha_s \in \mathbb{C}[\mathfrak{h}]$$

is the discriminant element defining the reflection hyperplanes of W.

Lemma 4.3.4.1. *Let $c \in \mathfrak{p}$ and let K be a End(λ)-valued analytic function on $\mathfrak{h}_{\mathbb{R},reg}$ satisfying the differential equation*

$$\partial_y K + \sum_{s\in S} \frac{c_s \alpha_s(y)}{\alpha_s}(sK + Ks) = 0$$

for each $y \in \mathfrak{h}_{\mathbb{R}}$. Then

(i) K is homogeneous of degree $-2\chi_\lambda(\sum_{s\in S} c_s s)/\dim\lambda$;

(ii) there exists an integer $N \geq 0$ such that $\delta^N K$, extended to $\mathfrak{h}_{\mathbb{R}}$ with value 0 along the reflection hyperplanes $\ker(\alpha_s)$, is a continuous function on $\mathfrak{h}_{\mathbb{R}}$;

(iii) if $|c_s|$ is sufficiently small for all $s \in S$ then $K(x)$ is a locally L^1 function on $\mathfrak{h}_{\mathbb{R}}$.

Proof. Statement (i) follows immediately from the differential equation and the fact that the central element $\sum_{s\in S} c_s s \in \mathbb{C}W$ acts on λ by the scalar $\chi_\lambda(\sum_{s\in S} c_s s)/\dim\lambda$.

To establish (ii) and (iii) it suffices to show that K is locally integrable on $\mathcal{C}$ when $|c_s|$ is sufficiently small for all $s \in S$ and that there exists $N \geq 0$ such that $\delta^N K|_{\mathcal{C}}$ extends to a continuous function on $\overline{\mathcal{C}}$ vanishing on the boundary. Let $\omega_1^\vee, ..., \omega_l^\vee \in \overline{\mathcal{C}}$ be the fundamental dominant coweights, so that $\alpha_i(\omega_j^\vee) = \delta_{ij}$, and let R be the region

$$R := \{x \in \mathfrak{h}_{\mathbb{R},reg} : \alpha_i(x) \in (0, 1), i = 1, ..., l\} \subset \mathcal{C}.$$

Given a point $v \in \mathcal{C}$, we have, for those $t > 0$ such that $\alpha_i(v - t\omega_i^\vee) > 0$,

$$\frac{d}{dt}||K(v - t\omega_i^\vee)|| \leq ||\frac{d}{dt}K(v - t\omega_i^\vee)||$$

$$= ||\sum_{s \in S} \frac{c_s \alpha_s(\omega_i^\vee)}{\alpha_s(v - t\omega_i^\vee)}(sK(v - t\omega_i^\vee) + K(v - t\omega_i^\vee)s)||$$

$$\leq \sum_{s \in S} \frac{2|c_s|\alpha_s(\omega_i^\vee)}{\alpha_s(v - t\omega_i^\vee)}||K(v - t\omega_i^\vee)||,$$

where the norm $||\cdot||$ on $\mathrm{End}_{\mathbb{C}}(\lambda)$ is the norm arising from the natural inner product on $\mathrm{End}_{\mathbb{C}}(\lambda)$ associated to the inner product $(\cdot,\cdot)_\lambda$ on λ. Note that this norm is differentiable away from $0 \in \mathrm{End}_{\mathbb{C}}(\lambda)$, the system of differential equations K satisfies implies that if $K(x) = 0$ for any $x \in \mathcal{C}$ then K is identically 0 on $\mathcal{C}$, and any $s \in S$ acts by an isometry with respect to this norm. It follows that

$$||K(v - t\omega_i^\vee)|| \leq ||K(v)|| \prod_{s \in S} \left(\frac{\alpha_s(v)}{\alpha_s(v - t\omega_i^\vee)} \right)^{2|c_s|}.$$

With $\rho^\vee := \sum_i \omega_i^\vee \in \mathcal{C}$, we therefore have

$$||K(x)|| \leq ||K(\rho^\vee)|| \prod_{s \in S} (\alpha_s(\rho^\vee)/\alpha_s(x))^{2|c_s|} \tag{4.3.4}$$

for all $x \in R$, and the claim follows immediately from this estimate and the homogeneity of K. $\square$

We need to impose further conditions, in addition to W-equivariance, on such a locally integrable function K in order for it to represent the form $\gamma_{c,\lambda}$. In [20, Section 5] and [19, Equation 9], Dunkl derived certain conditions on such K near the boundaries of the Weyl chambers. We will now establish similar conditions in terms of the nature of the singularities of K along the reflection hyperplanes, and we will relate these conditions to the invariance of sesquilinear pairings on certain representations of the Hecke algebra.

Definition 4.3.4.2. *For each simple reflection s_i, let*

$$\mathcal{C}_i := \{x \in \mathfrak{h}_{\mathbb{R}} : \alpha_i(x) = 0, \alpha_j(x) > 0 \textit{ for } j \neq i\}$$

be the open codimension-1 face of $\overline{\mathcal{C}}$ determined by α_i.

Let $\lambda \in \mathrm{Irr}(W)$ and let $c \in \mathfrak{p}$ be such that $c_s \notin \frac{1}{2}\mathbb{Z}\backslash\{0\}$ for all $s \in S$. Fix a simple reflection s_i, and choose coordinates $z_1, ..., z_l \in \mathfrak{h}_{\mathbb{R}}^* \subset \mathfrak{h}^*$ for $\mathfrak{h}$ with $z_1 = \alpha_i$ and $z_j(\alpha_i^\vee) = 0$ for $j > 1$. Consider the modified KZ connection

$$\nabla'_{KZ} := d - \sum_{s \in S} c_s \frac{d\alpha_s}{\alpha_s} s$$

on the trivial vector bundle on $\mathfrak{h}_{reg}$ with fiber λ. The connection ∇'_{KZ} is flat and therefore has a unique local extension of any initial value to a local homomorphic flat section. Furthermore, considering the restriction of ∇'_{KZ} in the z_1 direction near a point $x_0 \in \mathcal{C}_i$ lying on the reflection hyperplane $\ker(\alpha_i)$, it follows from the standard theory of ordinary differential equations with regular singularities (see, e.g. [72, Theorem 5.5]) that there is a holomorphic function $P_i(z)$, defined in a complex analytic z_1-disc about x_0 and satisfying $P_i(x_0) = \mathrm{Id}$, such that the function

$$z \mapsto P_i(z) z_1^{c_i s_i}$$

gives a fundamental $\mathrm{End}_{\mathbb{C}}(\lambda)$-valued (multivalued) solution to the restriction of the system ∇'_{KZ} to a small punctured z_1-disc about x_0. We may extend $P_i(z)$ in the $z_2, ..., z_l$ directions by taking solutions of the restricted connection

$$\nabla''_{KZ} := d - \sum_{s \in S \backslash \{s_i\}} c_s \frac{d\alpha_s}{\alpha_s} s$$

along affine hyperplanes parallel to $\ker(\alpha_i)$ to produce a single valued function $P_i(z)$, holomorphic in $z_2, ..., z_l$, defined in a simply connected complex analytic s_i-stable neighborhood of D of $\mathcal{C} \cup \mathcal{C}_i \cup s_i(\mathcal{C})$ (note that the restricted connection above is regular along $\mathcal{C}_i$ itself). The independence of the function $z_1^{c_i s_i}$ on the variables $z_2, ..., z_l$ and the uniqueness of extensions of solutions of ∇'_{KZ} then implies that the such that the function

$$N_i(z) := P_i(z) z_1^{c_i s_i}$$

gives a fundamental $\mathrm{End}_{\mathbb{C}}(\lambda)$-valued (multivalued) solution to the system ∇'_{KZ} on the region $D_{reg} := D \cap \mathfrak{h}_{reg}$. In particular, $P_i(z)$ is holomorphic on D_{reg}. Continuous dependence of solutions to ordinary differential equations on parameters and initial conditions implies that $P_i(z)$ is continuous on all of D. In particular, viewing $P_i(z)$ as a function on D_{reg} holomorphic in z_1, the singularities of $P_i(z)$ along $\ker(\alpha_i)$ are removable. It follows that $P_i(z)$ is holomorphic in z_1 on all of D. As $P_i(z)$ is holomorphic in each z_j for $j > 1$, it follows by Hartogs' theorem that $P_i(z)$ is holomorphic on all of D.

We next derive further properties of $P_i(z)$ arising from W-equivariance of the connection ∇'_{KZ} and the initial condition $P_i(x_0) = \mathrm{Id}$. In particular, we see that $s_i N_i(s_i z) s_i$ is another multivalued fundamental solution of ∇'_{KZ} on D_{reg}. As

$$s_i N_i(s_i z) s_i = s_i P_i(s_i z) s_i (-z_1)^{c_i s_i} = s_i P_i(s_i z) s_i z_1^{c_i s_i} e^{\pi\sqrt{-1} c_i s_i},$$

it follows that $s_i P_i(s_i z) s_i z_1^{c_i s_i}$ is such a solution as well (to reduce ambiguity, here we use the notation $\sqrt{-1}$ for the imaginary unit $i \in \mathbb{C}$ to avoid confusion with the index of the simple reflection s_i). By the proof of [72, Theorem 5.5], in particular its use of [72, Theorem 4.1], and the assumption that $c_i \notin \frac{1}{2}\mathbb{Z}\backslash\{0\}$, any function $\widetilde{P}_i(z)$ holomorphic in a small complex analytic z_1-disc about x_0 such that $\widetilde{P}_i(x_0) = \mathrm{Id}$ and the function $\widetilde{P}_i(z) z_1^{-c_i s_i}$ is a solution to the system ∇'_{KZ} on the punctured z_1-disc must coincide with $P_i(z)$ along the entire disc. It follows that $s_i P_i(s_i z) s_i = P_i(z)$ for z in a small complex analytic z_1-disc about x_0 and therefore for all $z \in D$.

We summarize the conclusions of the above discussion in the following lemma:

Lemma 4.3.4.3. *Let $\lambda \in Irr(W)$, let $c \in \mathfrak{p}$ be such that $c_i \notin \frac{1}{2}\mathbb{Z}\backslash\{0\}$ for a given simple reflection $s_i \in S$, and let $z_1, ..., z_l$ be coordinates for $\mathfrak{h}$ as in the discussion above. There is a s_i-stable simply connected complex analytic neighborhood D, independent of c, of $\mathcal{C} \cup \mathcal{C}_i \cup s_i(\mathcal{C})$ in $\mathfrak{h}$ and a holomorphic $GL(\lambda)$-valued function $P_i(z)$ on D such* *that*

(1) $P_i(z)\alpha_i(z)^{c_i s_i}$ *is a* $End_{\mathbb{C}}(\lambda)$*-valued multivalued holomorphic fundamental solution to the modified* KZ *system*

$$\nabla'_{KZ} := d - \sum_{s\in S} c_s \frac{d\alpha_s}{\alpha_s} s$$

on the domain $D_{reg} := D \cap \mathfrak{h}_{reg}$

(2) $s_i P_i(s_i z) s_i = P_i(z)$ *for all* $z \in D$*. In particular,* P_i *takes values in* $End_{s_i}(\lambda)$ *along* $\ker(\alpha_i) \cap D$*.*

(3) $P_i(z)$ *is a solution of the system*

$$\nabla''_{KZ} := d - \sum_{s\in S\setminus\{s_i\}} c_s \frac{d\alpha_s}{\alpha_s} s$$

along $\ker(\alpha_i) \cap D$*.*

Furthermore, any such P_i *is determined by its value at any point* $x_0 \in \mathcal{C}_i$*, and for any constant invertible* $A \in Aut_{s_i}(\lambda)$ *the function* $P_i(z)A$ *is another such function. In particular, any two such functions* $P_i(z), \widetilde{P}_i(z)$ *are related by an equality* $\widetilde{P}_i(z) = P_i(z)A$ *for a unique such* A*.*

Now, let $K : \mathcal{C} \to \mathrm{End}_{\mathbb{C}}(\lambda)$ be a real analytic function satisfying differential equation (4.3.2) appearing in Proposition 4.3.3.6

$$\partial_y K + \sum_{s\in S} c_s \frac{\alpha_s(y)}{\alpha_s}(sK + Ks) = 0 \qquad \text{for all } y \in \mathfrak{h}_{\mathbb{R}}.$$

Choose a simple reflection s_i, and fix functions $P_{i,c}(z)$ and $P_{i,c^\dagger}$ as in Lemma 4.3.4.3 for parameters c and $c^\dagger$, respectively. By the uniqueness of solutions of differential equation (4.3.2) given initial conditions, there is a matrix $K_i \in \mathrm{End}_{\mathbb{C}}(\lambda)$, uniquely determined given $P_i(z)$, such that K is given by

$$z \mapsto P_{i,c^\dagger}(z)^{\dagger,-1}\alpha_i(z)^{-c_i s_i} K_i \alpha_i(z)^{-c_i s_i} P_{i,c}(z)^{-1}$$

for $z \in \mathcal{C}$. As $P_{i,c}(z)$ and $P_{i,c^\dagger}(z)$ are determined only up to right multiplication by some $A \in \mathrm{Aut}_{s_i}(\lambda)$, here K_i is determined by K only up to the action of $\mathrm{Aut}_{s_i}(\lambda)$ on $\mathrm{End}_{\mathbb{C}}(\lambda)$ by $A.M = AMA^\dagger$. As $s_i^\dagger = s_i$, s_i-invariance of A implies that of $A^\dagger$, and we see that s_i-invariance of K_i is a property of K, not depending on the choice of $P_{i,c}(z)$ and $P_{i,c^\dagger}(z)$. In particular, we make the following definition:

Definition 4.3.4.4. *In the setting of the previous paragraph, we say that the function K is* asymptotically W-invariant *if $K_i \in End_{s_i}(\lambda)$ for all simple reflections s_i.*

We will need the following lemma:

Lemma 4.3.4.5. *Fix $x_0 \in \mathfrak{h}_{\mathbb{R},reg}$. There is a holomorphic function $B : \mathfrak{p} \to End_{\mathbb{C}}(\lambda)$ such that $B(0) = Id$ and, for all $c \in \mathfrak{p}$, $B(c)$ defines a B_W-invariant sesquilinear pairing*

$$KZ_{x_0}(\Delta_c(\lambda)) \times KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}$$

$$(v_1, v_2) \mapsto v_2^\dagger B(c) v_1,$$

where we make the standard identification of $KZ_{x_0}(\Delta_c(\lambda))$ and $KZ_{x_0}(\Delta_{c^\dagger}(\lambda))$ with λ as vector spaces. Any two such functions $B(c)$ and $\widetilde{B}(c)$ are related by $\widetilde{B}(c) = b(c)B(c)$ for a meromorphic function $b(c)$. Replacing $B(c)$ by $(B(c)+B(c^\dagger)^\dagger)/2$, we may assume $B(c)$ takes values in Hermitian forms for all $c \in \mathfrak{p}_{\mathbb{R}}$.

Proof. An operator $A_c \in \mathrm{End}_{\mathbb{C}}(\lambda)$ defines a B_W-invariant sesquilinear pairing

$$KZ_{x_0}(\Delta_c(\lambda)) \times KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}$$

as in the lemma statement if and only if

$$T_{i,c^\dagger}^{\dagger,-1} A_c T_{i,c}^{-1} = A_c, \tag{4.3.5}$$

where $T_{i,c}$ and $T_{i,c^\dagger}$ denote the action of the generator $T_i \in B_W$ in $KZ_{x_0}(\Delta_c(\lambda))$ and $KZ_{x_0}(\Delta_{c^\dagger}(\lambda))$, respectively, for all generators T_i of B_W. Similarly to Lemma 4.2.2.5, such a sequilinear pairing is equivalent to a homomorphism of B_W-representations

$$KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to KZ_{x_0}(\Delta_c(\lambda))^h \tag{4.3.6}$$

where $KZ_{x_0}(\Delta_c(\lambda))^h$ denotes the Hermitian dual of $KZ_{x_0}(\Delta_c(\lambda))$ as a representation of B_W, i.e. the representation of B_W which is $\overline{KZ_{x_0}(\Delta_c(\lambda))^*}$ as a $\mathbb{C}$-vector space and in which the action of $g \in B_W$ is by $g.f = f \circ g^{-1}$. When $|c_s|$ is small for all $s \in S$, the Hecke algebra $\mathsf{H}_q(W)$, $q(s) = e^{-2\pi i c(s)}$, is semisimple and isomorphic to $\mathbb{C}W$, and the KZ functor is an equivalence of categories $\mathcal{O}_c(W, \mathfrak{h}) \cong \mathsf{H}_q(W)\text{-mod}_{f.d.}$. For such c, the irreducible representations of $\mathsf{H}_q(W)$ therefore are precisely the images of the standard modules under the KZ functor, and in particular they can be distinguished by their character as B_W-representations, as this is so for $c = 0$ and these characters are continuous (in fact, holomorphic) functions of c. In particular, for such c, as $KZ_{x_0}(\Delta_{c^\dagger}(\lambda))$ and $KZ_{x_0}(\Delta_c(\lambda))^h$ are irreducible, isomorphic for $c = 0$, and have characters as B_W-representations that are continuous functions of c, it follows that they are isomorphic. It follows that, when $|c_s|$ is small for all $s \in S$, there is a 1-dimensional space of B_W-isomorphisms as in (4.3.6), and hence a unique solution $A_c \in \operatorname{End}_{\mathbb{C}}(\lambda)$ to the system of equations

$$\begin{cases} T_{i,c^\dagger}^{\dagger,-1} A_c T_{i,c}^{-1} = A_c, \text{ for all generators } T_i \in B_W \\ \operatorname{Tr}(A_c) = \dim(\lambda). \end{cases} \tag{4.3.7}$$

The operators $T_{i,c}^{-1}$ and $T_{i,c^\dagger}^{\dagger,-1}$ are holomorphic in $c \in \mathfrak{p}$, so we may view system (4.3.7) as a system of linear system of equations in the variable $A_c \in \operatorname{End}_{\mathbb{C}}(\lambda)$ with coefficients that are holomorphic in $c \in \mathfrak{p}$. It follows that there is a solution A_c that is meromorphic in $c \in \mathfrak{p}$ with singularities where the system is degenerate. Clearing denominators by multiplying by an appropriate determinant $d(c)$, holomorphic in c, the resulting holomorphic function $d(c)A_c$, up to scaling by a global constant, satisfies the properties of $B(c)$ stated in the lemma. The uniqueness statement follows as well. Finally, if $B(c)$ satisfies the properties in the Lemma then $\widetilde{B}(c) := (B(c) + B(c^\dagger)^\dagger)/2$

is also holomorphic in c and satisfies equation (4.3.5) for all $c \in \mathfrak{p}_{\mathbb{R}}$, and therefore for all $c \in \mathfrak{p}$ as well by analyticity. $\square$

Remark 4.3.4.6. *Lemma 4.3.4.5 holds for complex reflection groups as well, with the same proof, although this level of generality will not be needed.*

Theorem 4.3.4.7 (Existence of Dunkl Weight Function for Small c)**.** *Let $\lambda \in Irr(W)$ and let $c \in \mathfrak{p}$ with $|c_s|$ sufficiently small for all $s \in S$. Let $K : \mathfrak{h}_{\mathbb{R},reg} \to End_{\mathbb{C}}(\lambda)$ be a nonzero W-equivariant function satisfying system (4.3.2). Then the following are equivalent:*

(a) K represents the Gaussian pairing $\gamma_{c,\lambda}$ up to rescaling by a nonzero complex number.

(b) For any point $x \in \mathfrak{h}_{\mathbb{R},reg}$, $K(x)$ determines a B_W-invariant sesquilinear pairing

$$KZ_x(\Delta_c(\lambda)) \times KZ_x(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}.$$

(c) $K|_{\mathcal{C}}$ is asymptotically W-invariant in the sense of Definition 4.3.4.4.

Furthermore, the space of such K satisfying (a)-(c) forms a one-dimensional complex vector space.

Proof. We will first show that (a) and (c) are equivalent. As the function K is locally integrable and homogeneous by Lemma 4.3.4.1 when $|c_s|$ is small for all $s \in S$, it determines a tempered distribution with values in $\mathrm{End}_{\mathbb{C}}(\lambda)$. Therefore, we may define a sesquilinear pairing $\gamma_K : \Delta_c(\lambda) \times \Delta_{c^\dagger}(\lambda) \to \mathbb{C}$ by the formula

$$\gamma_K(P, Q) = \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger K(x) P(x) e^{-|x|^2/2} dx$$

for all $P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$. As K is nonzero and W-equivariant, the same is true for γ_K by the density of $\mathbb{C}[\mathfrak{h}]e^{-|x|^2/2} \subset \mathscr{S}(\mathfrak{h}_{\mathbb{R}})$. By Proposition 4.3.3.4, γ_K is proportional to $\gamma_{c,\lambda}$ if and only if

$$\int_{\mathfrak{h}_{\mathbb{R}}} K(x) D_y \varphi(x) dx = 0$$

for all $\varphi \in \mathscr{S}(\mathfrak{h}_\mathbb{R})$ and $y \in \mathfrak{h}_\mathbb{R}$. Following Dunkl, for any $\epsilon > 0$ define the region

$$\Omega_\epsilon := \{x \in \mathfrak{h}_\mathbb{R} : |\alpha_s(x)| > \epsilon \text{ for all } s \in S\} \subset \mathfrak{h}_{\mathbb{R},reg}.$$

For any $\varphi \in \mathscr{S}(\mathfrak{h}_\mathbb{R}) \otimes \lambda$ the function $K(D_y\varphi)$ is integrable, so we have

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} K(x) D_y \varphi(x) dx = \int_{\mathfrak{h}_\mathbb{R}} K(x) D_y \varphi(x) dx$$

by the dominated convergence theorem (see, e.g., [64, Section 4.4]). The region Ω_ϵ avoids the singularities of $1/\alpha_s$ for all $s \in S$, and so, following the calculations appearing after Equation (14) in [20, Section 5], a direct calculation using the W-invariance of K, the W-invariance of Ω_ϵ, and the system of differential equations that K satisfies shows that

$$\int_{\Omega_\epsilon} K(x) D_y \varphi(x) dx = \int_{\Omega_\epsilon} \partial_y (K(x)\varphi(x)) dx.$$

By Proposition 4.3.3.4, we see that γ_K is proportional to $\gamma_{c,\lambda}$ if and only if

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \partial_y (K(x)\varphi(x)) dx = 0 \tag{4.3.8}$$

for a spanning set of $y \in \mathfrak{h}_\mathbb{R}$ and dense set of $\varphi \in \mathscr{S}(\mathfrak{h}_\mathbb{R})$.

Fix a simple reflection s_i. Note that the boundary wall of the cone $\Omega_\epsilon \cap \mathcal{C}$ parallel to $\mathcal{C}_i$ is the translate of $\mathcal{C}_i$ by $\epsilon \rho^\vee$ and that the boundary wall of the cone $\Omega_\epsilon \cap s_i\mathcal{C}$ parallel to $\mathcal{C}_i$ is the translate of $\mathcal{C}_i$ by $\epsilon s_i \rho^\vee = \epsilon(\rho^\vee - \alpha_i^\vee)$. Computing the integral above as an iterated integral with inner integrals chosen as in [20, Section 5] and choosing test functions φ to be bump functions supported near a point $x_0 \in \mathcal{C}_i$ and s_i-invariant, we see that the vanishing of the limit

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \partial_{\alpha_i^\vee} (K\varphi) dx$$

for all φ implies that

$$\lim_{\epsilon\to 0}\int_{C_i}(K - s_iKs_i)(x+\epsilon\rho^\vee)\psi(x)dx = \lim_{\epsilon\to 0}\int_{C_i}(K(x+\epsilon\rho^\vee) - K(x+\epsilon s_i\rho^\vee))\psi(x)dx = 0$$

for all $\psi \in C_c^\infty(\mathcal{C}_i)$, where the first equality follows from the W-invariance of K. Choosing coordinates $z_1, ..., z_l$ adapted to the wall $\mathcal{C}_i$ as in the discussion preceding Definition 4.3.4.4, we have that K is given on $\mathcal{C}$ by the formula

$$z \mapsto P_{i,c^\dagger}(z)^{\dagger,-1}\alpha_i(z)^{-c_is_i}K_i\alpha_i(z)^{-c_is_i}P_{i,c}(z)^{-1}$$

with $P_{i,c}(z)$, $P_{i,c^\dagger}$, and K_i as above. As $P_{i,c}$ and $P_{i,c^\dagger}$ take values in $\mathrm{Aut}_{s_i}(\lambda)$ on $\mathcal{C}_i$, it follows that the commutators $[s_i, P_{i,c^\dagger}(z)^{\dagger,-1}]$ and $[P_{i,c}(z)^{-1}, s_i]$ are antiholomorphic and holomorphic, respectively, in z and vanish along $\mathcal{C}_i$. In particular, there is a holomorphic function $Q_{i,c}(z)$ and an antiholomorphic function $Q_{i,c^\dagger}(z)$ on D with values in $\mathrm{End}_{\mathbb{C}}(\lambda)$ such that $[s_i, P_{i,c^\dagger}(z)^{\dagger,-1}] = \overline{z_1}Q_{i,c^\dagger}(z)$ and $[P_{i,c}(z)^{-1}, s_i] = z_1Q_{i,c}(z)$. For $\sigma_1, \sigma_2 \in \{\pm 1\}$ let $K_i^{\sigma_1,\sigma_2} \in \mathrm{End}_{\mathbb{C}}(\lambda)$ be the projection of K_i to the (σ_1, σ_2) simultaneous eigenspace of the commuting operators λ_{s_i} and ρ_{s_i} on $\mathrm{End}_{\mathbb{C}}(\lambda)$ of left and right, respectively, multiplication by s_i. We have $K_i = K_i^{1,1} + K_i^{1,-1} + K_i^{-1,1} + K_i^{-1,-1}$, and a straightforward computation shows

$$(K - s_iKs_i)(z) = 2P_{i,c^\dagger}(z)^{\dagger,-1}(K_i^{1,-1} + K_i^{-1,1})P_{i,c}(z)^{-1} + z_1^{1-2|c_i|}R_i(z) \qquad (4.3.9)$$

for $z \in \mathcal{C}$, where $R_i(z) \in \mathrm{End}_{\mathbb{C}}(\lambda)$ is analytic on $\mathcal{C}$ and extends continuously to $\mathcal{C} \cup \mathcal{C}_i$. It follows that we have

$$\lim_{\epsilon\to 0}\int_{\mathcal{C}_i}(K - s_iKs_i)(x+\epsilon\rho^\vee)\psi(x)dx$$

$$= \int_{\mathcal{C}_i} 2P_{i,c^\dagger}(x+\epsilon\rho^\vee)^{\dagger,-1}(K_i^{1,-1} + K_i^{-1,1})P_{i,c}(x+\epsilon\rho^\vee)^{-1}\psi(x)dx$$

for all $\psi \in C_c^\infty(\mathcal{C}_i)$. For this integral to vanish for all such ψ, we must have that $K_i^{1,-1} + K_i^{-1,1} = 0$, and hence that $K_i = K_i^{1,1} + K_i^{-1,-1}$ commutes with s_i, so that

$K|_{\mathcal{C}}$ is asymptotically W-invariant. In particular, we see that (a) implies (c).

Conversely, if (c) holds, equation (4.3.9) implies that $K - s_i K s_i$ extends continuously to all of $\mathcal{C} \cup \mathcal{C}_i \cup s_i(\mathcal{C})$ with value 0 along $\mathcal{C}_i$. By the W-equivariance of K and the inner integrals used by Dunkl recalled in the previous paragraph, to show the vanishing of limit (4.3.8), for $y \in \mathfrak{h}_{\mathbb{R},reg}$ and $\varphi \in C_c^\infty(\mathfrak{h}_\mathbb{R})$, it suffices to show that the limit

$$\lim_{\epsilon \to 0} \int_{\mathcal{C}_i} (K\varphi)(x + \epsilon\rho^\vee) - (K\varphi)(s_i(x + \epsilon\rho^\vee))dx \tag{4.3.10}$$

vanishes. It suffices to treat the cases in which $\varphi \circ s_i = \pm\varphi$. In the case $\varphi \circ s_i = \varphi$, as K is W-invariant the integrand of (4.3.10) is $(K - s_i K s_i)(x + \epsilon\rho^\vee)\varphi(x + \epsilon\rho^\vee)$. In the case $\varphi \circ s_i = -\varphi$, there exists a test function $\psi \in C_c^\infty(\mathfrak{h}_\mathbb{R})$ such that $\varphi = \alpha_i\psi$, and the integrand of (4.3.10) takes the form $(\alpha_i \cdot (K + s_i K s_i))(x + \epsilon\rho^\vee)\psi(x + \epsilon\rho^\vee)$. In particular, in either case the integrand is of the form $f(x + \epsilon\rho^\vee)\psi(x + \epsilon\rho^\vee)$ where $\psi \in C_c^\infty(\mathfrak{h}_\mathbb{R})$ is a test function and f is either $K - s_i K s_i$ or $\alpha_i \cdot (K + s_i K s_i)$. In either case, from estimate (4.3.4) in Lemma 4.3.4.1 and equation (4.3.9), there is a constant $\mu > 0$, independent of c, such that

$$\delta_i(x)^{\mu \sum_{s \in S} |c_s|} \alpha_i(x)^{2|c_i|-1} f(x)$$

extends to a continuous function on $\overline{\mathcal{C}}$. In particular, let $M \geq 0$ be

$$M := \max_{x \in \mathrm{Supp}(\psi) \cap \overline{\mathcal{C}}} ||\delta_i(x)^{\mu \sum_{s \in S} |c_s|} \alpha_i(x)^{2|c_i|-1} f(x)||$$

and let $M' := \max_{x \in \mathfrak{h}_\mathbb{R}} |\psi(x)|$. We then have

$$||f(x)|| \leq M\delta_i(x)^{-\mu \sum_{s \in S} |c_s|} \alpha_i(x)^{1-2|c_i|}$$

for $x \in \mathrm{Supp}(\psi) \cap \overline{\mathcal{C}}$, and hence

$$||f(x + \epsilon\rho^\vee)\psi(x + \epsilon\rho^\vee)||$$

$$\leq MM'\delta_i(x + \epsilon\rho^\vee)^{-\mu \sum_{s \in S} |c_s|} \alpha_i(x + \epsilon\rho^\vee)^{1-2|c_i|}$$

$$\leq MM'\delta_i(x)^{-\mu \sum_{s\in S}|c_s|}\epsilon^{1-2|c_i|}$$

for all $x \in \mathcal{C}_i \cap \mathrm{Supp}(\psi)$. Taking $|c_s|$ sufficiently small for all $s \in S$ so that $\delta_i(x)^{-\mu\sum_{s\in S}|c_s|}$ is locally integrable on $\mathcal{C}_i$ and $1-2|c_i| > 0$, the vanishing of limit (4.3.10) now follows from the dominated convergence theorem. This completes the proof that (c) implies (a).

Now we will show the equivalence of (b) and (c). Fix a point $x_0 \in \mathfrak{h}_{\mathbb{R},reg}$ and identify $K(x_0)$ with a sesquilinear pairing $KZ_{x_0}(\Delta_c(\lambda)) \times KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}$. Here we make the usual identification of $KZ_{x_0}(\Delta_c(\lambda))$ and $KZ_{x_0}(\Delta_{c^\dagger}(\lambda))$ with λ as $\mathbb{C}$-vector spaces, and $K(x_0) \in \mathrm{End}_{\mathbb{C}}(\lambda)$ defines a sesquilinear form by $(v_1, v_2)_{K(x_0)} = v_2^\dagger K(x_0)v_1$. Without loss of generality, we take $x \in \mathcal{C}$. The braid group $B_W = \pi_1(\mathfrak{h}_{reg}/W, x_0)$ is generated by the elements T_i given by positively oriented half loops around the hyperplanes $\ker(\alpha_i)$ for simple roots α_i. For each simple root α_i, let $T_{i,c}$ and $T_{i,c^\dagger}$ denote the action of T_i on $KZ_{x_0}(\Delta_c(\lambda))$ and on $KZ_{x_0}(\Delta_{c^\dagger}(\lambda))$, respectively, so that the action of T_i on the space of sesquilinear pairings $KZ_{x_0}(\Delta_c(\lambda)) \times KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}$, identified with $\mathrm{End}_{\mathbb{C}}(\lambda)$, is given by

$$X \mapsto T_{i,c^\dagger}^{\dagger,-1} X T_{i,c}^{-1}. \tag{4.3.11}$$

Consider the W-equivariant $\mathrm{End}_{\mathbb{C}}(\lambda)$-valued multivalued function

$$\widetilde{K}(z) := (f_{c^\dagger}(z)M_{c^\dagger}(z))^{\dagger,-1}K(x_0)(f_c(z)M_c(z))^{-1}$$

on $\mathfrak{h}_{reg}$, where $M_c(z)$ is the monodromy of the KZ connection

$$\nabla_{KZ} := d + \sum_{s\in S} c_s \frac{d\alpha_s}{\alpha_s}(1-s)$$

on the trivial bundle $\lambda \times \mathfrak{h}_{reg} \to \mathfrak{h}_{reg}$ from x_0 to z, $f_c(z)$ is the (scalar) monodromy of the scalar-valued W-equivariant flat connection

$$d - \sum_{s\in S} c_s \frac{d\alpha_s}{\alpha_s}$$

on the trivial bundle $\mathbb{C}\times\mathfrak{h}_{reg} \to \mathfrak{h}_{reg}$ from x_0 to z, and $M_{c^\dagger}$ and $f_{c^\dagger}$ are defined similarly with the parameter $c^\dagger$ in place of the parameter c. A straightforward computation shows that $\widetilde{K}(z)$ satisfies system (4.3.2) of differential equations on $\mathcal{C}$, so it follows from the equality $\widetilde{K}(x_0) = K(x_0)$ and the uniqueness of solutions to (4.3.2) given initial conditions that $\widetilde{K}(x) = K(x)$ for all $x \in \mathcal{C}$.

View $\widetilde{K}(z)$ as a multivalued section of the vector bundle

$$(\mathrm{End}_{\mathbb{C}}(\lambda) \times \mathfrak{h}_{reg})/W \to \mathfrak{h}_{reg}/W.$$

It follows by considering residues that the monodromy of $f_c(z)$ is inverse to the monodromy of $\overline{f_{c^\dagger}(z)}$, and in particular the function $f_c(z)\overline{f_{c^\dagger}(z)}$ is single valued on $\mathfrak{h}_{reg}/W$. As the monodromy of $M_c(z)$ based at x_0 is given, by definition, by the B_W-action on $KZ_{x_0}(\Delta_c(\lambda))$ and similarly for $M_{c^\dagger}(z)$, it follows that the monodromy in $\mathfrak{h}_{reg}/W$ of $\widetilde{K}(z)$ based at x_0 is precisely given by the B_W-action on sesquilinear pairings appearing in (4.3.11) above. In particular, $K(x_0)$ is B_W-invariant as a sesquilinear pairing $KZ_{x_0}(\Delta_c(\lambda)) \times KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}$ if and only if $\widetilde{K}(z)$ is single-valued on $\mathfrak{h}_{reg}/W$.

It remains to show that $\widetilde{K}(z)$ is single-valued if and only if $K(x)$ is asymptotically W-invariant in the sense of Definition 4.3.4.4. Fix a simple reflection α_i, and let $P_{i,c}(z)$ and $P_{i,c^\dagger}(z)$ be holomorphic functions on a domain D as in Lemma 4.3.4.3. It follows from that same lemma and the remarks following it that there is a unique $K_i \in \mathrm{End}_{\mathbb{C}}(\lambda)$ such that

$$\widetilde{K}(z) = P_{i,c^\dagger}(z)^{\dagger,-1}\overline{\alpha_i(z)}^{-c_i s_i} K_i \alpha_i(z)^{-c_i s_i} P_{i,c}(z)^{-1}$$

for all $z \in D \cap \mathfrak{h}_{reg}$. Decomposing K_i into simultaneous left and right eigenspaces for s_i as $K_i = K_i^{1,1} + K_i^{1,-1} + K_i^{-1,1} + K_i^{-1,-1}$ as before, we have that $P_{i,c^\dagger}(z)^\dagger \widetilde{K}(z) P_{i,c}(z)$, for $z \in D \cap \mathfrak{h}_{reg}$, is given by:

$$|\alpha_i(z)|^{-2c_i} K_i^{1,1} + \left(\frac{\overline{\alpha_i(z)}}{\alpha_i(z)}\right)^{-c_i} K_i^{1,-1} + \left(\frac{\overline{\alpha_i(z)}}{\alpha_i(z)}\right)^{c_i} K_i^{-1,1} + |\alpha_i(z)|^{2c_i} K_i^{-1,-1}.$$

Clearly, this function, and hence $\widetilde{K}(z)$ itself, is single-valued in $\mathfrak{h}_{reg}$ near the hyperplane $\ker(\alpha_i)$ if and only if $K_i^{1,-1} = K_i^{-1,1} = 0$, i.e. if $K_i = K_i^{1,1} + K_i^{-1,-1}$ is s_i-invariant. In that case, by the s_i-equivariance of $P_{i,c}$ and $P_{i,c^\dagger}$, it then follows that $\widetilde{K}(z)$ is also single-valued in $\mathfrak{h}_{reg}/W$ near the hyperplane $\ker(\alpha_i)$. So, $\widetilde{K}(z)$ is invariant under the monodromy action of $T_i \in B_W$ if and only if $K_i \in \mathrm{Aut}_{s_i}(\lambda)$. In particular, $\widetilde{K}(z)$ is single-valued if and only if K is asymptotically W-invariant, completing the proof of the equivalence of (b) and (c).

The final statement of the theorem follows from Lemma 4.3.4.5. $\square$

4.3.5 Analytic Continuation on the Regular Locus

Theorem 4.3.5.1 (Existence of Dunkl Weight Function on $\mathfrak{h}_{\mathbb{R},reg}$)**.** *There exists a unique family of analytic functions $K_c : \mathfrak{h}_{\mathbb{R},reg} \to End_{\mathbb{C}}(\lambda)$ with holomorphic dependence on c such that the following holds: for all $M > 0$ there exists an integer $N \geq 0$, which may be taken to be 0 for M sufficiently small, such that for $c \in \mathfrak{p}$ with $|c_s| < M$ for all $s \in S$ the function $\delta^{2N} K_c$ is locally integrable on $\mathfrak{h}_{\mathbb{R}}$ and represents the Gaussian pairing γ_c on $\delta^N \Delta_c(\lambda) \times \delta^N \Delta_{c^\dagger}(\lambda)$ in the following sense:*

$$\gamma_c(\delta^N P, \delta^N Q) = \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger \delta^{2N}(x) K_c(x) P(x) e^{-|x|^2/2} dx \qquad \textit{for all } P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda.$$

For each $c \in \mathfrak{p}$ and $x \in \mathfrak{h}_{\mathbb{R},reg}$, $K_c(x)$ defines a B_W-invariant sesquilinear pairing

$$KZ_x(\Delta_c(\lambda)) \times KZ_x(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}.$$

For $c \in \mathfrak{p}_{\mathbb{R}}$, $K_c(x)$ defines a B_W-invariant Hermitian form on $KZ_x(\Delta_c(\lambda))$.

Proof. Uniqueness of K_c follows from the density of $\mathbb{C}[\mathfrak{h}]e^{-|x|^2/2}$ in the Schwartz space $\mathscr{S}(\mathfrak{h}_{\mathbb{R}})$, so it suffices to establish existence of K_c. Let $B : \mathfrak{p} \to \mathrm{End}_{\mathbb{C}}(\lambda)$ be a holomorphic function as in Lemma 4.3.4.5 defined relative to the point $x_0 \in \mathcal{C}$. It follows from the holomorphic dependence on the parameter c and on initial conditions of solutions of the system of differential equations (4.3.2) appearing in Proposition 4.3.3.6 that there is a unique function $K' : \mathfrak{p} \times \mathfrak{h}_{\mathbb{R},reg} \to \mathrm{End}_{\mathbb{C}}(\lambda)$, $K'(c,x) = K'_c(x)$, holomorphic in $c \in \mathfrak{p}$ and analytic in $x \in \mathfrak{h}_{\mathbb{R},reg}$, such that

(1) $K'_c(x_0) = B(c)$ for all $c \in \mathfrak{p}$

(2) $wK'_c(w^{-1}x)w^{-1} = K'_c(x)$ for all $c \in \mathfrak{p}$, $x \in \mathfrak{h}_{\mathbb{R},reg}$, and $w \in W$

(3) For all fixed $c \in \mathfrak{p}$, $K'_c(x)$, as a function of $x \in \mathcal{C}$ satisfies the system of differential equations (4.3.2).

Furthermore, as $B(c)$ defines a B_W-invariant sesquilinear pairing

$$KZ_{x_0}(\Delta_c(\lambda)) \times KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C},$$

it follows from the proof of the equivalence of statements (b) and (c) in Theorem 4.3.4.7 that $K'_c(x)$ defines a B_W-invariant sesquilinear pairing

$$KZ_x(\Delta_c(\lambda)) \times KZ_x(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C}$$

for all $c \in \mathfrak{p}$ and $x \in \mathfrak{h}_{\mathbb{R},reg}$. Note that $K'_0(x) = \mathrm{Id}$ for all $x \in \mathfrak{h}_{\mathbb{R},reg}$.

Let $M > 0$. By Lemma 4.3.4.1, there exists an integer $N \geq 0$, which may be taken to be 0 if M is sufficiently small, such that for all c with $|c_s| < M$ for all $s \in S$ the function $\delta^{2N}K'_c$ is locally integrable on $\mathfrak{h}_{\mathbb{R}}$. As $\delta^{2N}K'_c$ is homogeneous by the same lemma, it follows that integration against $\delta^{2N}K'_c$ determines a tempered distribution on $\mathfrak{h}_{\mathbb{R}}$. In particular, we may define a sesquilinear pairing $\widetilde{\gamma}_c$ on $\delta^N\Delta_c(\lambda) \times \delta^N\Delta_{c^\dagger}(\lambda)$ as follows:

$$\widetilde{\gamma}_c(\delta^N P, \delta^N Q) := \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger \delta^{2N}(x) K'_c(x) P(x) e^{-|x|^2/2} dx \qquad \text{for all } P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda.$$

Let $U_M \subset \mathfrak{p}$ be the open subset $U_M := \{c \in \mathfrak{p} : |c_s| < M \text{ for all } s \in S\}$. For any $P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$, we see that $\widetilde{\gamma}_c(\delta^N P, \delta^N Q)$ is a holomorphic function of $c \in U_M$, and the same is true for $\gamma_c(\delta^N P, \delta^N Q)$. In particular, for any such P and Q such that $\widetilde{\gamma}_c(\delta^N P, \delta^N Q)$ is not identically zero for $c \in U_M$, we may define the meromorphic function $f_{P,Q} : U_M \to \mathbb{C}$ by

$$f_{P,Q}(c) := \gamma_c(\delta^N P, \delta^N Q)/\widetilde{\gamma}_c(\delta^N P, \delta^N Q).$$

By Theorem 4.3.4.7 and the properties of K'_c listed in the previous paragraph, we see that, for those $c \in \mathfrak{p}$ with $|c_s|$ sufficiently small for all $s \in S$, the pairings γ_c and $\widetilde{\gamma}_c$ agree up to a nonzero complex scalar multiple. It follows that all of the functions $f_{P,Q}$ for $P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$ coincide in a neighborhood of 0 in $\mathfrak{p}$ and hence coincide with a single meromorphic function $f : U_M \to \mathbb{C}$. Let $K_c : \mathfrak{h}_{\mathbb{R},reg} \to \mathrm{End}_{\mathbb{C}}(\lambda)$ be defined by $K_c(x) = f(c)K'_c(x)$. Then $K_c(x)$ is meromorphic in $c \in U_M$, holomorphic in c in a neighborhood of 0 in $\mathfrak{p}$, satisfies the remaining properties of $K'_c(x)$ from the previous paragraph, and

$$\gamma_c(\delta^N P, \delta^N Q) = \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger \delta^{2N}(x) K_c(x) P(x) e^{-|x|^2/2} dx \qquad \text{for all } P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$$

for all $c \in U_M$ for which $K_c(x)$ is holomorphic.

It remains to show that $K_c(x)$ is in fact holomorphic for all $c \in U_M$. To this end, it suffices to show that $K_c(x)$ has no singularities for c in $U_M \cap \mathfrak{l}$ for any complex line $\mathfrak{l}$ in $\mathfrak{p}$. By the Weierstrass theorem on the existence of meromorphic functions with prescribed zeros and poles, there exists a meromorphic function g on $U_M \cap \mathfrak{l}$ such that $K_c(x_0) = g(c)K''_c(x_0)$, where $K''_c(x_0) \in \mathrm{End}_{\mathbb{C}}(\lambda)$ is nonzero for all $c \in U_M \cap \mathfrak{l}$. As above, we may extend $K''_c(x_0)$, via W-equivariance and the system of differential equations (4.3.2), to a non-vanishing function $K'' : (U_M \cap \mathfrak{l}) \times \mathfrak{h}_{\mathbb{R},reg} \to \mathrm{End}_{\mathbb{C}}(\lambda)$, holomorphic in $c \in U_M \times \mathfrak{l}$ and analytic in $x \in \mathfrak{h}_{\mathbb{R},reg}$, so that $K_c(x) = g(c)K''_c(x)$ for all $c \in U_M \cap \mathfrak{l}$ and $x \in \mathfrak{h}_{\mathbb{R},reg}$. As above, we have

$$\gamma_c(\delta^N P, \delta^N Q) = g(c) \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger \delta^{2N}(x) K''_c(x) P(x) e^{-|x|^2/2} dx \qquad \text{for all } P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda \tag{4.3.12}$$

for all $c \in U_M \cap \mathfrak{l}$ for which $g(c)$ is holomorphic. But by the density of $\mathbb{C}[\mathfrak{h}]e^{-|x|^2/2}$ in the Schwartz space $\mathscr{S}(\mathfrak{h}_{\mathbb{R}})$ and the non-vanishing of $\delta^{2N}(x)K''_c(x)$ for $x \in \mathfrak{h}_{\mathbb{R},reg}$, it follows that for any $c \in U_M \cap \mathfrak{l}$ there are $P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$ such that the integral on the righthand side of equation (4.3.12) is nonzero. It follows that $g(c)$ is holomorphic for all $c \in U_M \cap \mathfrak{l}$, as needed. □

Corollary 4.3.5.2. *a) Let $q : S \to \mathbb{C}^\times$ be a W-invariant function satisfying $|q_s| = 1$*

for all $s \in S$. Then every irreducible representation of the Hecke algebra $\mathsf{H}_q(W)$ admits a nondegenerate B_W-invariant Hermitian form, unique up to $\mathbb{R}^\times$-scaling.

b) Let $c \in \mathfrak{p}_\mathbb{R}$ and let $x_0 \in \mathfrak{h}_{\mathbb{R},reg}$. If $\lambda \in Irr(W)$ is such that $KZ_{x_0}(L_c(\lambda))$ is nonzero and unitary, i.e. it admits a positive-definite B_W-invariant Hermitian form, then $L_c(\lambda)$ is quasi-unitary in the sense of Definition 4.2.5.3.

Remark 4.3.5.3. *Chlouveraki-Gordon-Griffeth [12, Section 4] have considered certain invariant symmetric forms on representations of the Hecke algebra $\mathsf{H}_q(W)$ for arbitrary q.*

Proof of Corollary 4.3.5.2. Any q as in (a) is of the form $q_s = e^{-2\pi i c_s}$ for some $c \in \mathfrak{p}_\mathbb{R}$. Fix such $c \in \mathfrak{p}_\mathbb{R}$ and choose a point $x_0 \in \mathfrak{h}_{\mathbb{R},reg}$. Any irreducible representation of $\mathsf{H}_q(W)$ is isomorphic to some $KZ_{x_0}(L_c(\lambda))$ for some $\lambda \in \mathrm{Irr}(W)$ such that $\mathrm{Supp}(L_c(\lambda)) = \mathfrak{h}$. Let $N_c(\lambda)$ be the maximal proper submodule of $\Delta_c(\lambda)$, so that $L_c(\lambda) = \Delta_c(\lambda)/N_c(\lambda)$. Recall from Proposition 4.3.1.4(iv) that $N_c(\lambda)$ is the radical of γ_c.

Let $K_c : \mathfrak{h}_{\mathbb{R},reg} \to \mathrm{End}_\mathbb{C}(\lambda)$ be as in Theorem 4.3.5.1, so that in particular $K_c(x_0)$ defines a B_W-invariant Hermitian form on $\lambda =_{v.s.} KZ_{x_0}(\Delta_c(\lambda))$. It follows from the system of differential equations (4.3.2) that if $K_c(x_0) = 0$ then $K_c(x) = 0$ for all $x \in \mathfrak{h}_{\mathbb{R},reg}$. In this case, by Theorem 4.3.5.1, for sufficiently large $N > 0$, we have that the restriction of γ_c to $\delta^N \Delta_c(\lambda)$ is zero. As $\delta \in \mathbb{C}[\mathfrak{h}]$ is self-adjoint with respect to the form γ_c, it follows that $\delta^{2N}\Delta_c(\lambda) \subset N_c(\lambda)$, so $\delta^{2N} L_c(\lambda) = 0$, contradicting $\mathrm{Supp}(L_c(\lambda)) = \mathfrak{h}$. In particular, $K_c(x_0)$ defines a nonzero B_W-invariant Hermitian form on $KZ_{x_0}(\Delta_c(\lambda))$.

As KZ_{x_0} is exact, we have $KZ_{x_0}(L_c(\lambda)) \cong KZ_{x_0}(\Delta_c(\lambda))/KZ_{x_0}(N_c(\lambda))$. Furthermore, as $KZ_{x_0}(L_c(\lambda))$ is irreducible, to show that it admits a nondegenerate B_W-invariant Hermitian form, unique up to $\mathbb{R}^\times$ scaling, it suffices to show that $KZ_{x_0}(N_c(\lambda))$ lies in the radical of $K_c(x_0)$. To this end, take a vector $v \in KZ_{x_0}(N_c(\lambda))$. Viewing $KZ_{x_0}(N_c(\lambda))$ as a submodule of $KZ_{x_0}(\Delta_c(\lambda))$ and identifying the latter with λ as a vector space, let $\widetilde{v} \in N_c(\lambda) \subset \Delta_c(\lambda) = \mathbb{C}[\mathfrak{h}] \otimes \lambda$ be such that its value at x_0 is

v. In particular, $\delta^N\widetilde{v}$ is in the radical of γ_c, and it follows from Theorem 4.3.5.1 that

$$0 = \gamma_c(\delta^N\widetilde{v}, \delta^N Q) = \int_{\mathfrak{h}_\mathbb{R}} Q(x)^\dagger \delta^{2N}(x) K_c(x)\widetilde{v}e^{-|x|^2/2}dx \qquad \text{for all } Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda.$$

As $\delta^{2N}K_c\widetilde{v}$ defines a tempered distribution on $\mathfrak{h}_\mathbb{R}$ with values in λ, the density of $\mathbb{C}[\mathfrak{h}]e^{-|x|^2/2}$ in the Schwartz space $\mathscr{S}(\mathfrak{h}_\mathbb{R})$ implies that $\delta^{2N}K_c\widetilde{v} = 0$ pointwise on $\mathfrak{h}_{\mathbb{R},reg}$. In particular, $K_c(x_0)v = 0$. As $v \in KZ_{x_0}(N_c(\lambda))$ was arbitrary, it follows that $KZ_{x_0}(N_c(\lambda)) \subset \mathrm{rad}(K_c(x_0))$, where $\mathrm{rad}(K_c(x_0))$ denotes the radical of the form $K_c(x_0)$. In particular, $K_c(x_0)$ descends to a nondegenerate B_W-invariant Hermitian form on $KZ_{x_0}(\Delta_c(\lambda))$, proving (a). Note that this also shows that in fact $KZ_{x_0}(N_c(\lambda)) = \mathrm{rad}(K_c(x_0))$.

For (b), if $KZ_{x_0}(L_c(\lambda))$ is unitary, it follows from the argument above that $K_c(x_0)$, appropriately scaled, is positive-semidefinite and descends to a positive-definite Hermitian form on $KZ_{x_0}(L_c(\lambda))$. The same is then true for all points $x \in \mathfrak{h}_{\mathbb{R},reg}$, and the integral formula in Theorem 4.3.5.1 then implies that the restriction of γ_c to $\delta^N L_c(\lambda)$ is positive-definite. As $L_c(\lambda)$ has full support in $\mathfrak{h}$ and as $L_c(\lambda)/\delta^N L_c(\lambda)$ is supported on the reflection hyperplanes, it follows that the Hilbert polynomial of $L_c(\lambda)$ is of strictly higher degree than the Hilbert polynomial of $L_c(\lambda)/\delta^N L_c(\lambda)$. In particular, the Hilbert polynomial of $\delta^N L_c(\lambda)$ has the same leading term as the Hilbert polynomial of $L_c(\lambda)$ itself, so we have

$$\lim_{n\to\infty} \frac{\dim(\delta^N L_c(\lambda))^{\leq n}}{\dim L_c(\lambda)^{\leq n}} = 1.$$

As γ_c is positive-definite on $\delta^N L_c(\lambda)$, we see that $L_c(\lambda)$ is quasi-unitary, as needed. $\square$

We gave the proof of Corollary 4.3.5.2(a) as above to demonstrate a use of the Dunkl weight function and because the arguments appearing in that proof would be repeated later for the proof of Theorem 4.4.0.1. However, there is an alternate proof of Corollary 4.3.5.2(a) that is also valid for finite *complex* reflection groups, due to Raphaël Rouquier and communicated to me by Pavel Etingof, as follows.

Proposition 4.3.5.4. *Corollary 4.3.5.2(a) is valid for finite complex reflection groups*

W.

Proof. Let $c \in \mathfrak{p}_\mathbb{R}$ be a real parameter corresponding to q via the KZ functor. Let L be an irreducible representation of $\mathsf{H}_q(W)$, and let λ be an irreducible representation of W such that $L_c(\lambda)$ has full support and $L \cong KZ(L_c(\lambda))$. To show that L admits a nondegenerate B_W-invariant Hermitian form, necessarily uniquely determined up to $\mathbb{R}^\times$-multiple, it suffices to show that L is isomorphic to its Hermitian dual L^h as a B_W-representation. By Remark 4.3.4.6, there is a holomorphic function $B : \mathfrak{p} \to \mathrm{End}_\mathbb{C}(\lambda)$ satisfying the conditions of Lemma 4.3.4.5. Restricting to the complex line in $\mathfrak{p}$ spanned by c and scaling B by an appropriate meromorphic function, we may assume that $B(c)$ is nonzero and therefore determines a nonzero B_W-invariant Hermitian form on $KZ(\Delta_c(\lambda))$, which we will regard as a nonzero map $\beta : KZ(\Delta_c(\lambda)) \to KZ(\Delta_c(\lambda))^h$. As β defines a Hermitian form, it factors through an isomorphism $\overline{\beta} : KZ(\Delta_c(\lambda))/\ker\beta \cong (KZ(\Delta_c(\lambda))/\ker\beta)^h$. The head of the module $KZ(\Delta_c(\lambda))$ is $KZ(L_c(\lambda))$; this follows from the fact that $L_c(\lambda)$ has full support and is the head of $\Delta_c(\lambda)$ and the fact that the KZ functor admits a right adjoint $\pi^* : \mathsf{H}_q(W)\text{-mod}_{f.d.} \to \mathcal{O}_c(W, \mathfrak{h})$ such that $KZ \circ \pi^* \cong \mathrm{Id}_{\mathsf{H}_q(W)\text{-mod}_{f.d.}}$. In particular, $KZ(L_c(\lambda))$ is also the head of $KZ(\Delta_c(\lambda))/\ker\beta$, and it appears with multiplicity 1 as a composition factor because the same is true for $L_c(\lambda)$ in $\Delta_c(\lambda)$. If $KZ(\Delta_c(\lambda))/\ker\beta$ is irreducible, then it isomorphic to $KZ(L_c(\lambda))$, and the proof is complete. Otherwise, let $S \subset KZ(\Delta_c(\lambda))/\ker\beta$ be a simple submodule, necessarily not isomorphic to $KZ(L_c(\lambda))$ because the latter appears with multiplicity 1. Taking the Hermitian dual, we obtain a surjection $(KZ(\Delta_c(\lambda))/\ker\beta)^h \to S^h$, and composing with $\overline{\beta}$ we conclude that S^h appears in the head of $KZ(\Delta_c(\lambda))/\ker\beta$, so $S^h \cong KZ(L_c(\lambda))$. But S must be isomorphic to $KZ(L_c(\mu))$ for some $\mu \in \mathrm{Irr}(W)$, $\mu \neq \lambda$, such that $L_c(\mu)$ has full support and appears as a composition factor in $\Delta_c(\lambda)$. By induction on the highest weight order $\leq_c$ of $\mathcal{O}_c(W, \mathfrak{h})$, we may assume that $S \cong KZ(L_c(\mu))$ admits a nondegenerate B_W-invariant Hermitian form, so that $S \cong S^h$. But then $S \cong KZ(L_c(\lambda))$, a contradiction. It follows that $KZ(\Delta_c(\lambda))/\ker\beta$ is irreducible and isomorphic to $KZ(L_c(\lambda))$, and the claim follows. $\square$

Remark 4.3.5.5. *In Section 4.4, using techniques from semiclassical analysis, we will substantially generalize part (b) of Corollary 4.3.5.2. In particular, we will see that, whenever $L_c(\lambda)$ has full support, the asymptotic signature $a_{c,\lambda}$ of $L_c(\lambda)$ equals, up to sign, the signature of an invariant Hermitian form on $KZ_{x_0}(L_c(\lambda))$ normalized by its dimension. In particular, unitarity of $KZ_{x_0}(L_c(\lambda))$ does not only imply quasi-unitarity of $L_c(\lambda)$, as in part (b) of Corollary 4.3.5.2, but is equivalent to it.*

4.3.6 Extension to Tempered Distribution via the Wonderful Model

In this section we will complete the proof of the existence of the Dunkl weight function, i.e. of Theorem 4.3.3.1. As we will see, essentially what remains is to extend the function $K_c : \mathfrak{h}_{\mathbb{R},reg} \to \mathrm{End}_{\mathbb{C}}(\lambda)$ from Theorem 4.3.5.1 to a tempered distribution on $\mathfrak{h}_{\mathbb{R}}$ in a natural way. To achieve this we will use a known approach (see, e.g., [4]) in which the extension is carried out on the De Concini-Procesi wonderful model [14]; the desired extension is then obtained by pushing forward to $\mathfrak{h}_{\mathbb{R}}$. The advantage of working in the wonderful model rather than in $\mathfrak{h}_{\mathbb{R}}$ is that the hyperplane arrangement is replaced by a normal crossings divisor, allowing the application of standard fact that for any $\lambda \in \mathbb{C}$ the function $|x|^{z\lambda}$, locally integrable for $|z|$ small, has a natural distributional meromorphic continuation to $z \in \mathbb{C}$.

For $R > 0$, let $B_R(0)$ denote the open disk $B_R(0) := \{z \in \mathbb{C} : |z| < R\}$ of radius R in the complex plane centered at 0. For any integer $N > 0$ let $Mat_{N\times N}(\mathbb{C})$ denote the space of $N \times N$ complex matrices, and for any $a \in Mat_{N\times N}(\mathbb{C})$ let $\mathrm{Spec}(A)$ denote the set of eigenvalues of a.

Lemma 4.3.6.1. *Let $R > 0$, let $a : \mathfrak{p} \to Mat_{N\times N}(\mathbb{C})$ be a linear function, and let $A(z; c)$ be a holomorphic function of $(c, z) \in \mathfrak{p} \times B_R(0)$, taking values in the space $Mat_{N\times N}(\mathbb{C})$ of $N \times N$ complex matrices, such that $A(0; c) = a(c)$ for all $c \in \mathfrak{p}$. Then for any bounded domain $U \subset \mathfrak{p}$ there is a $Mat_{N\times N}(\mathbb{C})$-valued holomorphic function*

$Q(z;c)$ of $(c,z) \in U \times B_R(0)$ such that

$$F(z;c) = Q(z;c)z^{a(c)}$$

is a solution of the differential equation

$$z\frac{dF(z;c)}{dz} = A(z;c)F(z;c) \tag{4.3.13}$$

for all $c \in \mathfrak{p}$ and a fundamental solution whenever $a(c)$ has no two eigenvalues differing by a nonzero integer.

Proof. The lemma follows from a straightforward adaptation of the arguments and results appearing in [72, Section 5] as follows. Expand the function $A(z;c)$ as $A(z;c) = \sum_{n=0}^{\infty} a_{c,n}z^n$, where the $a_{c,n}$ are entire functions of $c \in \mathfrak{p}$ and $a_{c,0} = a(c)$. First, consider a formal series $P(z;c) = \sum_{n=0}^{\infty} p_{c,n}z^n$ with values in $Mat_{N\times N}(\mathbb{C})$. A direct calculation shows that the formal series $P(z;c)z^{a(c)}$ satisfies differential equation (4.3.13) if and only if

$$p_{c,n}(n + a(c)) - a(c)p_{c,n} = \sum_{k=1}^{n} a_{c,k}p_{c,n-k} \qquad \text{for all } n \geq 1. \tag{4.3.14}$$

Let $\rho_{n+a(c)}$ denote the operator of right multiplication by $n + a(c)$ and let $\lambda_{a(c)}$ denote the operator of left multiplication by $a(c)$. Note that $(\rho_{n+a(c)} - \lambda_{a(c)})^{-1}$ is a meromorphic $GL(Mat_{N\times N}(\mathbb{C}))$-valued function of $c \in \mathfrak{p}$ that is holomorphic after multiplication by the polynomial $\det(\rho_{n+a(c)} - \lambda_{a(c)})$. In particular, we may define meromorphic functions $p_{c,n}$, $n \geq 0$, of $c \in \mathfrak{p}$ by setting $p_{c,0} = \mathrm{Id}$ and

$$p_{c,n} := (\rho_{n+c_i a} - \lambda_{c_i a})^{-1} \sum_{k=1}^{n} a_{c,k}p_{c,n-k} \qquad \text{for all } n \geq 1. \tag{4.3.15}$$

Let $q_{c,n} := k(c_i)p_{c,n}$ for all $n \geq 0$.

It is a standard fact that the operator $\rho_{n+a(c)} - \lambda_{a(c)}$ is singular if and only if $n+a(c)$ and $a(c)$ have an eigenvalue in common, i.e. if $n = \lambda - \lambda'$ for some eigenvalues $\lambda, \lambda' \in \mathrm{Spec}(a(c))$. As $a(c)$ depends linearly on $c \in \mathfrak{p}$ and $U \subset \mathfrak{p}$ is bounded, it follows that there exists $m > 0$ such that $\rho_{n+a(c)} - \lambda_{a(c)}$ is invertible for all $n > m$ and $c \in \overline{U}$, where $\overline{U}$ denotes the closure of U. Let $k(c) = \prod_{n=1}^{m} \det(\rho_{n+a(c)} - \lambda_{a(c)})$, and let $q_{c,n} := k(c)p_{c,n}$ for all $n \geq 0$. Then $q_{c,n}$ is holomorphic function of c in a neighborhood of the closure $\overline{U}$ for all $n \geq 0$ and the formal series $Q(z;c) := \sum_{n=0}^{\infty} q_{c,n}z^n$ satisfies differential equation (4.3.13).

It now suffices to show that there exists $r > 0$ such that the series $Q(z;c) = \sum_{n=0}^{\infty} q_{c,n}z^n$ converges absolutely and uniformly for $(c,z) \in U \times B_r(0)$. In particular, it then follows that $Q(z;c)$ is holomorphic for $(c,z) \in U \times B_r(0)$, and therefore also for $(c,z) \in U \times B_R(0)$ by standard results on holomorphic dependence of solutions on parameters and initial conditions (see, e.g., [49, Theorem 1.1]). That $Q(z;c)$ is invertible, and hence that $Q(z;c)z^{a(c)}$ is a fundamental solution to (4.3.13), for those $c \in U$ such that $a(c)$ has no two eigenvalues differing by a nonzero integer follows from the equality $Q(0;c) = k(c)\mathrm{Id}$.

The desired uniform convergence of $Q(z;c) = \sum_{n=0}^{\infty} q_{c,n}z^n$ can be shown by a modification of the proof of [72, Theorem 5.3] as follows. As $A(z;c) = \sum_{n=0}^{\infty} a_{c,n}z^n$ is holomorphic on $\mathfrak{p} \times B_R(0)$, it follows by considering the Taylor expansions of the $a_{c,n}$ and the boundedness of U that there is a scalar-valued formal series $b(z) = \sum_{n=1}^{\infty} b_n z^n$ defining a holomorphic function on $B_R(0)$ and such that $||a_{c,n}|| \leq b_n$ for all $n \geq 1$ and $c \in U$. By the boundedness of U again, it follows that there exist $N', C, D > 0$ such that

$$||(\rho_{n+a(c)} - \lambda_{a(c)})^{-1}|| \leq C \qquad \text{for all } n > N', c \in U$$

and

$$||q_{c,n}|| \leq D \qquad \text{for all } n \leq N', c \in U.$$

Now, consider the formal scalar-valued series $m(z) = \sum_{n=0}^{\infty} m_n z^n$ defined by

$$m_r = D \qquad \text{for all } r \leq N'$$

and

$$m_r = C \sum_{s=1}^{r} b_s m_{r-s} \qquad \text{for all } r > N'.$$

As $m(z)$ satisfies the equation

$$m(z) = Cb(z)m(z) + D + \sum_{s=1}^{N} (D - C\sum_{t=1}^{s} b_t D)z^s$$

and $b(z)$ is holomorphic near 0 and $b(0) = 0$, it follows that there exists $r > 0$ such that $m(z)$ is holomorphic in $B_r(0)$. By construction we have $||q_{c,n}|| \leq m_n$ for all $n \geq 0$ and $c \in U$, so for all $c \in U$ and $z \in B_r(0)$ the series $Q(z;c) = \sum_{n=0}^{\infty} q_{c,n} z^n$ is majorized by the series $m(z)$. In particular, $Q(z;c) = \sum_{n=0}^{\infty} q_{c,n} z^n$ is absolutely and uniformly convergent for all $c \in U$ and $z \in B_r(0)$, as needed. $\square$

Lemma 4.3.6.2. *Let $n, N > 0$ be positive integers, let $R > 0$ be a positive real number, let $a_1, ..., a_n : \mathfrak{p} \to Mat_{N\times N}(\mathbb{C})$ be linear functions, and let*

$$A_c = \sum_{j=1}^{n} a(c) z_j^{-1} dz_j + \Omega_c$$

be a meromorphic 1-form on $B_R(0)^n \subset \mathbb{C}^n$ with values in $Mat_{N\times N}(\mathbb{C})$ such that

(1) the form Ω_c is holomorphic on $B_R(0)^n$ with holomorphic dependence on $c \in \mathfrak{p}$

(2) $d + A_c$ defines a flat connection on $(B_R(0)\backslash\{0\})^n$ for all $c \in \mathfrak{p}$.

Then for any bounded domain $U \subset \mathfrak{p}$ there exists, for all $1 \leq j \leq n$, a function $Q_j(z_j, ..., z_n; c)$ with values in $Mat_{N\times N}(\mathbb{C})$, holomorphic in $(c, z_j, ..., z_n) \in U \times B_R(0)^{n-j+1}$, such that

$$F(z;c) := Q_1(z_1, ..., z_n; c) z_1^{a_1(c)} \cdots Q_{n-1}(z_{n-1}, z_n; c) z_{n-1}^{a_{n-1}(c)} Q_n(z_n; c) z_n^{a_n(c)}$$

is a solution of the differential equation

$$(d + A_c)F(z;c) = 0$$

for all $c \in U$ and a fundamental solution whenever no two eigenvalues of any $a_i(c)$ differ by a nonzero integer.

Proof. The claim follows from iterated applications of Lemma 4.3.6.1 and standard results (e.g. [49, Theorem 1.1]) on holomorphic dependence on parameters and initial conditions of solutions of differential equations. $\square$

We can now complete the proof of the existence of the Dunkl weight function as a tempered distribution with values in $\mathrm{End}_{\mathbb{C}}(\lambda)$ with holomorphic dependence on $c \in \mathfrak{p}$:

Proof of Theorem 4.3.3.1. Let $K_c : \mathfrak{h}_{\mathbb{R},reg} \to \mathrm{End}_{\mathbb{C}}(\lambda)$ be the family of analytic functions with holomorphic dependence on $c \in \mathfrak{p}$ in the statement of Theorem 4.3.5.1. Note that for $|c_s|$ sufficiently small K_c already defines a holomorphic family of tempered distributions on $\mathfrak{h}_{\mathbb{R}}$ with the desired properties. It therefore suffices to show that this family of distributions for $|c_s|$ small has a holomorphic continuation to all $c \in \mathfrak{p}$. To see this, note that by Proposition 4.3.3.4 the distribution K_c for $|c_s|$ small satisfies properties (i)-(iii) of that proposition, and therefore by Proposition 4.3.3.6 is homogeneous of degree $-2\chi_\lambda(\sum_{s\in S} c_s s)/\dim\lambda$. It follows that the holomorphic continuation is homogeneous of degree $-2\chi_\lambda(\sum_{s\in S} c_s s)/\dim\lambda$, and hence tempered, for all $c \in \mathfrak{p}$. The characterizing properties (i)-(iii) appearing in Proposition 4.3.3.4 then must hold for all parameters $c \in \mathfrak{p}$ for the holomorphic continuation as they hold for $|c_s|$ sufficiently small. It follows from Proposition 4.3.3.4 that such a holomorphic continuation is a family of tempered distributions as in Theorem 4.3.3.1. So, it remains to construct such a holomorphic continuation.

Let $\pi : Y \to \mathfrak{h}$ be the De Concini-Procesi wonderful model [14] for the hyperplane arrangement $\cup_{s\in S}\ker(\alpha_s) \subset \mathfrak{h}$. Recall that Y is a smooth $\mathbb{C}$-variety with W-action, π is proper and W-equivariant, π restricts to an isomorphism $\pi^{-1}(\mathfrak{h}_{reg}) \to \mathfrak{h}_{reg}$, and $\pi^{-1}(\cup_{s\in S}\ker(\alpha_s))$ is a normal crossings divisor. Furthermore, let $\pi_{\mathbb{R}} : Y_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}}$ denote the real locus of π, which shares the same properties as π but for the corresponding real hyperplane arrangement in $\mathfrak{h}_{\mathbb{R}}$. Let $Y_{reg} = \pi^{-1}(\mathfrak{h}_{reg})$ and let $Y_{\mathbb{R},reg} = Y_{reg} \cap Y_{\mathbb{R}} = \pi_{\mathbb{R}}^{-1}(\mathfrak{h}_{\mathbb{R},reg})$.

Note that it follows from the W-equivariance of π and the fact that $\pi|_{Y_{reg}} : Y_{reg} \to \mathfrak{h}_{reg}$ is an isomorphism that π preserves stabilizers in W, i.e. that for all $d \in Y$ we have $\mathrm{Stab}_W(d) = \mathrm{Stab}_W(\pi(d))$. The inclusion $\mathrm{Stab}_W(d) \subset \mathrm{Stab}_W(\pi(d))$ is trivial, so consider $w \in \mathrm{Stab}_W(\pi(d))$. Let $U \subset Y$ be an open neighborhood of d in Y that is stable under $\mathrm{Stab}_W(d)$ and such that the $W/\mathrm{Stab}_W(d)$-orbits of U are disjoint. Let $U_{reg} = U \cap Y_{reg}$. As $w(\pi(d)) = \pi(d)$ it follows that $w\pi(U) \cap \pi(U) \neq \emptyset$, and hence $w\pi(U_{reg}) \cap \pi(U_{reg}) \neq \emptyset$ as well. It then follows that $wU_{reg} \cap U_{reg} = w\pi^{-1}\pi(U_{reg}) \cap \pi^{-1}\pi(U_{reg}) \neq \emptyset$ as well. In particular, $w \in \mathrm{Stab}_W(d)$, so $\mathrm{Stab}_W(d) = \mathrm{Stab}_W(\pi(d))$ for all $d \in Y$, as claimed.

Consider the modified KZ connection

$$\nabla'_{KZ} := d - \sum_{s\in S} c_s \frac{d\alpha_s}{\alpha_s} s = d - \sum_{s\in S} c_s d(\log \alpha_s) s \tag{4.3.16}$$

on $\mathfrak{h}_{reg}$. Fix a point $x_0 \in \mathcal{C}$, and let $F_c(z)$ be a $\mathrm{End}_{\mathbb{C}}(\lambda)$-valued fundamental solution of ∇'_{KZ} with $F_c(x_0) = \mathrm{Id}$. By standard results on holomorphic dependence of solutions on parameters, it follows that $F_c(z)$ is holomorphic in $c \in \mathfrak{p}$. Regard $F_c(z)$ as a multivalued function on $\mathfrak{h}_{reg}$. As the function $K_c : \mathcal{C} \to \mathrm{End}_{\mathbb{C}}(\lambda)$ satisfies differential equation (4.3.2) it follows that $K_c(x) = F_{c^\dagger}(x)^{\dagger,-1} K_c(x_0) F_c(x)^{-1}$ for all $c \in \mathfrak{p}$ and $x \in \mathcal{C}$. In fact, by the proof of the equivalence of statements (b) and (c) in Theorem 4.3.4.7, we have that the function $\widetilde{K}_c : \mathfrak{h}_{reg} \to \mathrm{End}_{\mathbb{C}}(\lambda)$, $\widetilde{K}_c(z) := F_{c^\dagger}(z)^{\dagger,-1} K_c(x_0) F_c(z)^{-1}$, is single-valued and satisfies $\widetilde{K}_c(x) = K_c(x)$ for all $x \in \mathfrak{h}_{\mathbb{R},reg}$.

The pullback $\pi^*\nabla'_{KZ}$ is a meromorphic 1-form on Y with values in $\mathrm{End}_{\mathbb{C}}(\lambda)$ and singularities along the components of the divisor $Y\backslash Y_{reg} = \pi^{-1}(\cup_{s\in S} \ker(\alpha_s))$, with holomorphic dependence on $c \in \mathfrak{p}$. Following the discussion in the introduction to [15], consider the residue of $\pi^*\nabla'_{KZ}$ along the components of $Y\backslash Y_{reg}$ as follows. As $Y\backslash Y_{reg}$ is a normal crossings divisor, for any point $d \in Y_{\mathbb{R}}$ there is a local complex coordinate system $z_1, ..., z_l$ such that $z_1(d) = \cdots z_l(d) = 0$ and the divisor $Y\backslash Y_{reg}$ is given near d by the equation $z_1 \cdots z_m = 0$ for some m, $0 \leq m \leq l$. It follows that for each reflection $s \in S$ the pullback $\pi^*\alpha_s$ is of the form $\pi^*\alpha_s = \prod_{j=1}^m z_j^{k_{j,s}} p_s(z)$ for some integers $k_{j,s} \in \mathbb{Z}^{\geq 0}$ and regular functions $p_s(z)$ with $p_s(0) \neq 0$. In these local coordinates, we have, by equation (4.3.16)

$$\pi^*\nabla'_{KZ} = d - \sum_{s\in S} c_s d(\log(\prod_{j=1}^m z_j^{k_{j,s}} p_s))s = d - \sum_{j=1}^m \left(\sum_{s\in S} c_s k_{j,s} s \right) d\log z_j + \Omega_c$$

where Ω_c is a 1-form with values in $\mathrm{End}_{\mathbb{C}}(\lambda)$, holomorphic in z and c.

Note that this connection is of the form considered in Lemma 4.3.6.2. For some $1 \leq j \leq m$, consider the residue $a_j(c) := \sum_{s \in S} c_s k_{j,s} s$ along the hyperplane $z_j = 0$. Recall that the map π is W-equivariant. Let d' be a generic point on the hyperplane $z_j = 0$, and consider the stabilizer $W' = \mathrm{Stab}_W(\pi(d')) = \mathrm{Stab}_W(d')$, a parabolic subgroup of W. For $s \in S \backslash W'$, we have $s(\pi(d')) \neq \pi(d')$ and hence $\alpha_s(\pi(d')) \neq 0$. In particular, in this case $\pi^*\alpha_s(d') \neq 0$, so $k_{j,s} = 0$. Now consider instead $s \in S \cap W'$ and $w \in W'$. The element $w \in W' = \mathrm{Stab}_W(d')$ stabilizes the hyperplane $z_1 = 0$ and has finite order, from which it follows that $w^* z_1 = z_1 p(z)$ for some holomorphic function $p(z)$ with $p(0) \neq 0$. The W-equivariance of π then implies that $k_{j,s} = k_{j,wsw^{-1}}$. We see that the residue $a_j(c)$ is a $\mathfrak{p}$-linear combination of sums of conjugacy classes of reflections in W'. It follows that the residues $\{a_j(c)\}_{c \in \mathfrak{p}}$ are simultaneously diagonalizable with eigenvalues that are linear functions of $c \in \mathfrak{p}$.

By Lemma 4.3.6.2, for every bounded open set $U \subset \mathfrak{p}$ there is a meromorphic function $G(c)$ of $c \in U$ such that, in the local coordinates $z_1, ..., z_l$ on Y near d as above, we have

$$\pi^* F_c(z) = Q_1(z_1, ..., z_l; c) z_1^{a_1(c)} \cdots Q_{l-1}(z_{l-1}, z_l; c) z_{l-1}^{a_{l-1}(c)} Q_l(z_l; c) z_l^{a_l(c)} G(c)$$

where the functions $Q_j(z_j, ..., z_l; c)$ are holomorphic both in z in a neighborhood of d in Y and also in $c \in U$. As the residues $\{a_j(c)\}_{c \in \mathfrak{p}}$ are simultaneously diagonalizable and as $\widetilde{K}_c(z) = F_{c^\dagger}(z)^{\dagger,-1} K(x_0) F_c(z)^{-1}$ is single valued on $\mathfrak{h}_{reg}$ and holomorphic in c, it follows from the form of $\pi^* F_c(z)$ above that the matrix entries of $\pi^* \widetilde{K}_c(z)$ are linear combinations of functions of the form $f_c(z)|z_1|^{g_1(c)} \cdots |z_l|^{g_l(c)}$ for some linear functions $g_1, ..., g_l : \mathfrak{p} \to \mathbb{C}$ and function $f_c(z)$ holomorphic in both $c \in U$ and also z in a neighborhood of d in Y. It is a standard fact that for $\lambda \in \mathbb{C}$ the function $|x|^\lambda$ is locally integrable on $\mathbb{R}$ when the real part of λ satisfies $\mathrm{Re}(\lambda) > -1$ and therefore defines a homogeneous distribution for such λ, and this distribution $|x|^\lambda$ has a meromorphic continuation to all $\lambda \in \mathbb{C}$ with (simple) poles at the negative odd integers. As $f_c(z)$ is holomorphic in $c \in U$ and holomorphic, in particular smooth, in z, it follows that the function $\pi^* K_c(x) = \pi^* \widetilde{K}_c(x)$ on $Y_{\mathbb{R},reg}$ has an extension, meromorphic in $c \in U$, to a distribution on an open neighborhood of d in $Y_{\mathbb{R}}$. As the point $d \in Y_{\mathbb{R}}$ and the bounded open set $U \subset \mathfrak{p}$ were arbitrary, it follows that the function $\pi^* K_c(x)$ on $Y_{\mathbb{R},reg}$ has an extension to a distribution on $Y_{\mathbb{R},reg}$ meromorphic in $c \in \mathfrak{p}$. Taking the pushforward of this distribution along π, one obtains an extension of the function K_c to a distribution on $\mathfrak{h}_{\mathbb{R}}$ that is meromorphic in c and coincides with K_c as a distribution when $|c_s|$ is small for all $s \in S$. Denote this family of distributions also by $\mathscr{K}_c$

It only remains to show that $\mathscr{K}_c$ is in fact holomorphic, not merely meromorphic, in $c \in \mathfrak{p}$. This follows immediately from the finiteness of the Hermite coefficients of K_c, which are given by the Gaussian pairing $\gamma_{c,\lambda}$ by analyticity, as this is so for $|c_s|$ small:

$$\int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger \mathscr{K}_c(z) P(x) e^{-|x|^2/2} dx = \gamma_{c,\lambda}(P,Q) \in \mathbb{C} \qquad \text{for all } P,Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda \text{ and } c \in \mathfrak{p}.$$

$\square$

4.4 Signatures and the KZ Functor

In this section we will apply the Dunkl weight function and methods from semiclassical analysis to prove the following comparison theorem for signatures of irreducible representations of rational Cherednik algebras and Hecke algebras, generalizing Corollary 4.3.5.2:

Theorem 4.4.0.1. *Let $\lambda \in Irr(W)$ be an irreducible representation of the finite Coxeter group W, and let $c \in \mathfrak{p}_{\mathbb{R}}$ be a real parameter. If $KZ(L_c(\lambda))$ is nonzero then the asymptotic signature $a_{c,\lambda}$ of $L_c(\lambda)$ is given by the formula*

$$a_{c,\lambda} = \frac{p-q}{\dim KZ(L_c(\lambda))} \tag{4.4.1}$$

where $p-q$ is the signature, up to sign, of a B_W-invariant Hermitian form on $KZ(L_c(\lambda))$.

In the special case that W is a symmetric group, Theorem 4.4.0.1 was proved by Venkateswaran [70, Theorem 1.4] by providing exact formulas for the left and right sides of equation (4.4.1). The generalization above to arbitrary finite Coxeter groups was expected by Venkateswaran [70] and Etingof.

We will begin with a reminder on semiclassical analysis in Section 4.4.1, prove a key analytic lemma in Section 4.4.2, and finally prove Theorem 4.4.0.1 in Section 4.4.3.

4.4.1 Reminder on Semiclassical Analysis

We briefly review notation and results from semiclassical analysis, referring the reader to [75] for details.

Denote by $\mathrm{Op}_h^{\mathrm{w}}$ the Weyl quantization on $\mathbb{R}^n$, mapping a symbol

$$a(x,\xi) \in C_c^\infty(\mathbb{R}^{2n})$$

to an h-dependent family of operators $\mathrm{Op}_h^{\mathrm{w}}(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined as follows:

$$\mathrm{Op}_h^{\mathrm{w}}(a)f(x) = \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\Big(\frac{x+y}{2},\xi\Big) f(y)\, dy d\xi. \tag{4.4.2}$$

More generally one can quantize h-dependent symbols $a(x,\xi;h)$ which satisfy the following derivative bounds for all multiindices α on $\mathbb{R}^{2n}$:

$$|\partial^\alpha_{(x,\xi)} a(x,\xi;h)| \le C_\alpha m(x,\xi), \quad (x,\xi)\in\mathbb{R}^{2n}, \quad 0<h\le 1, \tag{4.4.3}$$

where C_α is an h-independent constant depending only on the multiindex α and where $m(x,\xi)$ is an *order function* as defined in [75, §4.4.1]. The order functions we will use here are $m \equiv 1$ and

$$m_1(x,\xi) := 1 + |x|^2 + |\xi|^2. \tag{4.4.4}$$

As in [75, Definition 4.4.2], for any order function m we denote by $S(m)$ the class of (possibly h-dependent) symbols $a(x,\xi;h)$ such that $a(x,\xi;h) \in C^\infty(\mathbb{R}^{2n})$ for all

$h \in (0,1]$ and that satisfy the derivative bounds (4.4.3). The Weyl quantization $\mathrm{Op}_h^{\mathrm{w}}$ is defined for symbols $a \in S(m)$ and has the following properties:

1. for $a \in S(m)$, the operator $\mathrm{Op}_h^{\mathrm{w}}(a)$ acts continuously on the space of Schwartz functions $\mathscr{S}(\mathbb{R}^n)$ and also on the space of tempered distributions and $\mathscr{S}'(\mathbb{R}^n)$ [75, Theorem 4.16];

2. if $a \in S(1)$, then by [75, Theorem 4.23]
$$\sup_{0<h\leq 1} \|\mathrm{Op}_h^{\mathrm{w}}(a)\|_{L^2\to L^2} < \infty; \tag{4.4.5}$$

3. if $a \in S(m)$, $b \in S(\widetilde{m})$, then there exists $a\#b$ such that by [75, Theorem 4.18]
$$\mathrm{Op}_h^{\mathrm{w}}(a)\mathrm{Op}_h^{\mathrm{w}}(b) = \mathrm{Op}_h^{\mathrm{w}}(a\#b), \quad a\#b \in S(m\widetilde{m}), \quad a\#b = ab + \mathcal{O}(h)_{S(m\widetilde{m})}. \tag{4.4.6}$$

4. if $a \in S(m)$, then by [75, (4.1.13)]
$$\mathrm{Op}_h^{\mathrm{w}}(a)^* = \mathrm{Op}_h^{\mathrm{w}}(\overline{a}); \tag{4.4.7}$$

5. if $a \in S(1)$ is real-valued and satisfies $a \geq c > 0$ for some constant c and all sufficiently small h, then there exists $h_0 > 0$ such that by [75, Theorem 4.30]
$$\langle \mathrm{Op}_h^{\mathrm{w}}(a)f, f\rangle_{L^2} \geq \frac{c}{2}\|f\|_{L^2}^2 \quad \text{for all } f \in L^2(\mathbb{R}^n), \quad 0 < h \leq h_0. \tag{4.4.8}$$

Remark 4.4.1.1. *In fact, as will soon be important for the proof of Theorem 4.4.0.1, one may also consider, for any sufficiently large positive integer k, h-dependent symbols that are only C^k. Specifically, for any integer $k \geq 0$ and order function m one may also consider the symbol class $S^{(k)}(m)$ of symbols $a(x,\xi;h)$ such that $a(x,\xi;h) \in C^k(\mathbb{R}^{2n})$ and for which the derivative bounds (4.4.3) hold for all muti-indices α with $|\alpha| \leq k$. The construction of the Weyl quantization makes sense for such symbols (in fact for distributions as well, see [75, Theorem 4.2]), and the proofs* *of the properties recalled above are based on only a fixed finite number of terms in the semiclassical expansion, and therefore on only a fixed finite number of derivatives of the symbol. In particular, for k sufficiently large, the properties of the Weyl quantization listed above are valid for symbols from the class $S^{(k)}(m)$.*

The quantum harmonic oscillator

Consider the quantum harmonic oscillator in $\mathbb{R}^n$ [75, §6.1],

$$P := -h^2\Delta + |x|^2.$$

It is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$. We have $P = \mathrm{Op}_h^{\mathrm{w}}(p)$ where (recalling (4.4.4))

$$p(x,\xi) = |\xi|^2 + |x|^2 \in S(m_1).$$

We will also use the shifted operators determined by $(y,\eta) \in \mathbb{R}^{2n}$

$$\begin{aligned} P_{(y,\eta)} &= \sum_{j=1}^{n}(-ih\partial_{x_j} - \eta_j)^2 + |x-y|^2; \\ P_{(y,\eta)} &= \mathrm{Op}_h^{\mathrm{w}}(p_{(y,\eta)}), \quad p_{(y,\eta)}(x,\xi) = |\xi-\eta|^2 + |x-y|^2 \in S(m_1). \end{aligned} \tag{4.4.9}$$

Note that $P_{(y,\eta)}$ is conjugate to P:

$$P_{(y,\eta)} = T_{(y,\eta)} P T_{(y,\eta)}^{-1}, \quad T_{(y,\eta)} f(x) = e^{\frac{i}{h}\langle x,\eta\rangle} f(x-y). \tag{4.4.10}$$

We use the following consequence of functional calculus of pseudodifferential operators, see [75, Theorem 14.9] and [18, §8]:

Lemma 4.4.1.2. *Assume that $\varphi \in C_c^\infty(\mathbb{R})$ is h-independent and fix $(y,\eta) \in \mathbb{R}^{2n}$. Then $\varphi(P_{(y,\eta)}) = Op_h^{\mathrm{w}}(a)$ for some a satisfying for all N*

$$a \in S(m_1^{-N}), \quad a = \varphi \circ p_{(y,\eta)} + \mathcal{O}(h)_{S(m_1^{-N})}.$$

4.4.2 A Key Lemma

Before proving the main lemma, we will need the following elementary measure theoretic result:

Lemma 4.4.2.1. *For any Lebesgue measurable set $X \subset \mathbb{R}^n$ let vol(X) denote its volume. For any compact set $K \subset \mathbb{R}^n$ and every $\epsilon, r > 0$, there exist finitely many points $x_1, ..., x_N \in \mathbb{R}^n$ and positive real numbers $r_i \in (0, r)$ such that*

(a) $K \subset \bigcup_{i=1}^N B_{r_i}(x_i)$

(b) $\sum_{i=1}^N vol(B_{r_i}(x_i)) < vol(K) + \epsilon$.

Proof. Fix the compact set $K \subset \mathbb{R}^n$ and positive numbers $\epsilon, r > 0$. Recall that the outer measures defined by coverings by rectangles or balls coincide, and that each gives rise to Lebesgue measure. In particular, there are countably many open rectangles R_i such that $K \subset \bigcup_{i=1}^\infty R_i$ and $\sum_{i=1}^\infty \mathrm{vol}(R_i) < \mathrm{vol}(K) + \epsilon/2$. Subdividing and slightly enlarging the rectangles if necessary, we may further assume that $\mathrm{vol}(R_i) < \mathrm{vol}(B_{r/2}(0))$. Similarly, there are countably many open balls $\{B_{r_{ij}}(x_{ij})\}_{j=1}^\infty$ such that $R_i \subset \cup_{j=1}^\infty B_{r_{ij}}(x_{ij})$ and $\sum_{j=1}^\infty \mathrm{vol}(B_{r_{ij}}(x_{ij})) < \mathrm{vol}(R_{ij}) + \min\{\epsilon/2^{i+1}, \mathrm{vol}(B_{r/2}(0))\}$. In particular, for each i, j we have $\mathrm{vol}(B_{r_{ij}}(x_{ij})) < 2\mathrm{vol}(B_{r/2}(0)) \le \mathrm{vol}(B_r(0))$ so $r_{ij} < r$. Also, the countably many open balls $\{B_{r_{ij}}(x_{ij})\}_{i,j=1}^\infty$ cover the compact set K, and we may extract a finite subcover $\{B_{r_i}(x_i)\}_{i=1}^N$ of K. We have $\sum_{i=1}^N \mathrm{vol}(B_{r_i}(x_i)) < \sum_{i,j=1}^\infty \mathrm{vol}(B_{r_{ij}}(x_{ij})) < \mathrm{vol}(K) + \sum_{i=1}^\infty \epsilon/2^i = \mathrm{vol}(K) + \epsilon$, as needed. □

For $\beta \ge 0$, denote by $V_h(\beta)$ the range of the spectral projection $\mathbf{1}_{[0,\beta]}(P)$:

$$V_h(\beta) = \mathrm{span}\{u \in L^2(\mathbb{R}^n) \mid Pu = \lambda u \text{ for some } \lambda \in [0, \beta]\} \subset L^2(\mathbb{R}^n).$$

By Weyl's Law [75, Theorem 6.3] we have for any fixed β

$$\dim V_h(\beta) = c_n \beta^n h^{-n} + o(h^{-n}) \quad \text{as } h \to 0, \quad c_n := \frac{1}{2^n \cdot n!}. \tag{4.4.11}$$

Denote by

$$\Pi_h : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

the orthogonal projector onto $V_h(1)$. It is a direct consequence of [75, Theorem 6.2] on the eigenfunctions of P that we have

$$V_h(1) = \mathbb{C}[\mathbb{R}^n]^{\leq \frac{1}{2}(\frac{1}{h}-n)} e^{-|x|^2/2h}, \tag{4.4.12}$$

where $\mathbb{C}[\mathbb{R}^n]^{\leq \frac{1}{2}(\frac{1}{h}-n)}$ denotes the space of complex valued polynomials on $\mathbb{R}^n$ with degree at most $\frac{1}{2}(\frac{1}{h} - n)$.

We will denote by $\mathrm{Mat}_{m\times m}(\mathbb{C})$ and $\mathrm{Herm}_{m\times m}(\mathbb{C})$ the spaces of $m \times m$ complex matrices and $m \times m$ complex Hermitian matrices, respectively.

Lemma 4.4.2.2. *Let m, n be positive integers, let $k >> n$, and let $q : \mathbb{R}^n \to Herm_{m\times m}(\mathbb{C})$ be a C^k function taking values in Hermitian $m \times m$ matrices and with matrix entries $q_{ij}(x)$ satisfying, for some fixed nonnegative integer N_0, the derivative bounds:*

$$|\partial_\alpha q_{ij}(x)| \leq C_\alpha (1 + |x|)^{N_0} \tag{4.4.13}$$

for all multiindices α with $|\alpha| \leq k$. Consider the family of operators, for $h > 0$,

$$Q = Q(h) : V_h(1) \otimes \mathbb{C}^m \to V_h(1) \otimes \mathbb{C}^m, \quad Q(f) = (\Pi_h \otimes Id)(qf).$$

Let $Spec(q(x))$ denote the set of eigenvalues of $q(x)$ with multiplicity, and put (here $B_r(x, \xi)$ denotes the open ball with radius r and center (x, ξ) in $\mathbb{R}^{2n}$)

$$\gamma := \frac{\int_{B_1(0)} \#\{\lambda \in Spec(q(x)) : \lambda \leq 0\} dx d\xi}{vol(B_1(0))}.$$

Then, for every $\epsilon > 0$ there exists $h_0 > 0$ such that for $0 < h < h_0$ the number of non-positive eigenvalues of $Q(h)$, counted with multiplicity, is bounded above by $(\gamma + \epsilon) c_n h^{-n}$.

Proof. We follow the idea of the proof of [75, Theorem 6.8].

First, note that the operator $Q(h)$ is well-defined; it follows from (4.4.12) and the polynomial growth (4.4.13) of the matrix entries q_{ij} that multiplication by q defines a map

$$q : V_h(1) \otimes \mathbb{C}^m \to L^2(\mathbb{R}^n) \otimes \mathbb{C}^m,$$

so $Q(h)(f) = (\Pi_h \otimes \mathrm{Id})(qf)$ does define a linear operator on the finite-dimensional space $V_h(1)$.

Given q and $\epsilon > 0$, we may choose finitely many open balls

$$B_j := B_{r_j}((y_j, \eta_j)) \subset \mathbb{R}^{2n}, \quad (y_j, \eta_j) \in \mathbb{R}^{2n}, \quad r_j > 0; \quad j = 1, \ldots, N,$$

vectors $v_1, ..., v_N \in \mathbb{C}^m$, and constants $c_0, C_1, ..., C_N > 0$ all such that

$$q(x) + \sum_{j=1}^{N} C_j \mathbb{1}_{B_{r_j}((y_j,\eta_j))}(x, \xi) v_j v_j^T \geq 3c_0 \text{ for all } (x, \xi) \in B_1(0) \tag{4.4.14}$$

$$\sum_j \mathrm{vol}(B_j) \leq \Big(\gamma + \frac{\varepsilon}{3}\Big) \mathrm{vol}(B(0, 1)), \tag{4.4.15}$$

where the inequality in the first condition indicates that all eigenvalues are at least $3c_0$ and where $\mathbb{1}_{B_{r_j}((y_j,\eta_j))}$ denotes the indicator function of $B_{r_j}((y_j, \eta_j))$. To see that this is possible, we use Lemma 4.4.2.1 and descending induction on $M := \max_{(x,\xi)\in B_1(0)} \#\{\lambda \in \mathrm{Spec}(q(x)) : \lambda \leq 0\}$. If $M = 0$, there is nothing to prove. Otherwise, consider the compact set

$$K_M := \{(x, \xi) \in B_1(0) : \#\{\lambda \in \mathrm{Spec}(q(x)) : \lambda \leq 0\} = M\}.$$

For each point $(x, \xi) \in K_M$, let $v_{(x,\xi)} \in \mathbb{C}^m$ be an eigenvector of $q(x)$ with eigenvalue $\lambda_{(x,\xi)} \leq 0$, and let $C_{(x,\xi)} = 1 - \lambda_{(x,\xi)}$. Then for all $(x, \xi) \in K_M$ the matrix $q(x) + C_{(x,\xi)} v_{(x,\xi)} v_{(x,\xi)}^T$ has at most $M - 1$ nonpositive eigenvalues counted with multiplicity, and it follows that there exists a number $r_{(x,\xi)} > 0$ such that for all $(y, \eta) \in B_{r_{(x,\xi)}}((x, \xi))$ the matrix $q(y) + C_{(x,\xi)} v_{(x,\xi)} v_{(x,\xi)}^T$ has at most $M - 1$ nonpositive eigenvalues counted with multiplicity. As K_M is compact, the open covering $\{B_{r_{(x,\xi)}}((x, \xi))\}_{(x,\xi)\in K_M}$ of K_M admits a Lebesgue number $r > 0$. Applying Lemma 4.4.2.1 to K_M for this r and replacing q_1 by a function defined similarly to the expression appearing in (4.4.14), we may reduce to the case that $\max_{(x,\xi)\in B_1(0)} \#\{\lambda \in \mathrm{Spec}(q_1(x,\xi)) : \lambda \leq 0\} < M$ and continue in this manner by induction, defining K_{M-1} to be the closure of $\{(x,\xi) \in B_1(0) : \#\{\lambda \in \mathrm{Spec}(q(x)) : \lambda \leq 0\} = M-1\}$, etc. The ordered eigenvalues of the resulting function in (4.4.14) are then positive and clearly lower-semicontinuous so are bounded away from 0 on the compact set $B_1(0)$, so the eigenvalues are bounded below by $3c_0$ for some $c_0 > 0$.

As above, for $(y,\eta) \in \mathbb{R}^{2n}$ let $p_{(y,\eta)}$ be defined by $p_{(y,\eta)}(x,\xi) = |x-y|^2 + |\xi-\eta|^2$, and let $p_j = p_{(y_j,\eta_j)}$. It is clear that there exist real numbers $\tilde{r}_i > r_i$ and functions $\chi, \chi_1, ..., \chi_N \in C_c^\infty(\mathbb{R}^{2n})$ such that

$$\widetilde{q} := (\chi \circ p)^2 q + (1 - \chi \circ p)\mathrm{Id} + \sum_{j=1}^{N} (\chi_j \circ p_j) v_j v_j^T \geq 2c_0, \tag{4.4.16}$$

$$[0,1] \cap \mathrm{Supp}(1-\chi) = \emptyset, \quad \mathrm{Supp}(\chi_j) \subset (-\infty, \tilde{r}_j^2], \tag{4.4.17}$$

$$\sum_j \tilde{r}_j^{2n} \leq \gamma + \frac{2\varepsilon}{3}, \tag{4.4.18}$$

where $\mathrm{Id} \in \mathrm{Mat}_{m\times m}(\mathbb{C})$ denotes the identity matrix.

By Remark 4.4.1.1, we may take k sufficiently large so that the properties of the Weyl quantization recalled in Section 4.4.1 hold for the symbol classes $S^{(k)}(m)$. Furthermore, let $\mathrm{Op}_h^{\mathrm{w}}$ act element-wise on matrices of such symbols. Define the operator

$$Q_1 := \chi(P)\mathrm{Op}_h^{\mathrm{w}}(q)\chi(P) + (I - \chi(P))\mathrm{Id} + \sum_{j=1}^{N} \chi_j(P_j) v_j v_j^T.$$

Here $\mathrm{Op}_h^{\mathrm{w}}(q)$ is the multiplication operator by q; by (4.4.13) we have

$$q \in S^{(k)}(m_1^M) \otimes \mathrm{Herm}_{m\times m}(\mathbb{C})$$

for some M. Using Lemma 4.4.1.2 in each matrix entry, by the product formula (4.4.6) we see that

$$Q_1 = \mathrm{Op}_h^{\mathrm{w}}(a) \quad \text{for some } a \in S^{(k)}(1) \otimes \mathrm{Herm}_{m\times m}(\mathbb{C}),$$

$$a = \widetilde{q} + \mathcal{O}(h)_{S^{(k)}(1)\otimes \mathrm{Herm}_{m\times m}(\mathbb{C})}.$$

That we may take $a \in S^{(k)}(1) \otimes \mathrm{Herm}_{m\times m}(\mathbb{C})$ rather than simply $S^{(k)}(1) \otimes \mathrm{Mat}_{m\times m}(\mathbb{C})$ follows from [75, (4.1.12)]. In particular, we see that Q_1 is a bounded self-adjoint operator on $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$ by [75, Theorem 4.23].

As $\widetilde{q} \geq 2c_0$ it follows that there exists $h_0 > 0$ such that $a \geq \frac{3}{2}c_0$ for $0 < h < h_0$. Recall that the square root of a positive definite matrix depends smoothly (in fact, analytically) on the matrix entries. In particular, the square root $b := \sqrt{a - c_0}$ is defined for $0 < h < h_0$ and is an element of the symbol class $S^{(k)}(1) \otimes \mathrm{Herm}_{m\times m}(\mathbb{C})$. As the operator $\mathrm{Op}_h^{\mathrm{w}}(b)$ is self-adjoint, it follows from product formula (4.4.6) that we have

$$\mathrm{Op}_h^{\mathrm{w}}(a - c_0) = \mathrm{Op}_h^{\mathrm{w}}(b)^* \mathrm{Op}_h^{\mathrm{w}}(b) + \mathcal{O}(h)_{L^2\otimes\mathbb{C}^m}.$$

As $\mathrm{Op}_h^{\mathrm{w}}(b)^* \mathrm{Op}_h^{\mathrm{w}}(b)$ is manifestly a nonnegative operator, shrinking h_0 if necessary there is a constant $C > 0$ such that $\mathrm{Op}_h^{\mathrm{w}}(a - c_0) \geq -Ch$ for $0 < h < h_0$. In particular, $\mathrm{Op}_h^{\mathrm{w}}(a) \geq c_0 - Ch$ for $0 < h < h_0$. Taking h_0 smaller again if necessary, we have

$$\mathrm{Op}_h^{\mathrm{w}}(a) \geq \frac{c_0}{2} \qquad \text{for } 0 < h < h_0, \tag{4.4.19}$$

i.e.

$$\langle Q_1 f, f\rangle = \langle \mathrm{Op}_h^{\mathrm{w}}(a) f, f\rangle \geq \frac{c_0}{2} ||f||^2_{L^2\otimes\mathbb{C}^m} \text{ for all } f \in L^2(\mathbb{R}^n) \otimes \mathbb{C}^m, \quad 0 < h < h_0. \tag{4.4.20}$$

In particular this applies to all $f \in V_h(1) \otimes \mathbb{C}^m$. However, for such f we have $f = \chi(P)f$, therefore

$$\langle Qf, f\rangle + \langle Q_2 f, f\rangle = \langle Q_1 f, f\rangle \geq \frac{c_0}{2}\|f\|^2 \quad \text{for all } f \in V_h(1) \otimes \mathbb{C}^m \tag{4.4.21}$$

where

$$Q_2 := \sum_{j=1}^{N} \chi_j(P_j) v_j v_j^T. \tag{4.4.22}$$

and where the inner products and norms above are in $L^2(\mathbb{R}^n) \otimes \mathbb{C}^m$. By (4.4.10), (4.4.11), and (4.4.18) we estimate the rank of Q_2 for small h by

$$\mathrm{rank} Q_2 \leq \sum_{j=1}^{N} \dim V_h(\tilde{r}_j^2) = c_n h^{-n} \sum_{j=1}^{N} \tilde{r}_j^{2n} + o(h^{-n}) \leq (\gamma+\varepsilon) c_n h^{-n}. \qquad (4.4.23)$$

Together (4.4.21) and (4.4.23) give the required estimate. $\square$

4.4.3 Proof of the Signature Comparison Theorem

Proof of Theorem 4.4.0.1. Let k be a positive integer sufficiently large so that Lemma 4.4.2.2 holds for C^k $\mathrm{Herm}(\lambda)$-valued functions on $\mathfrak{h}_{\mathbb{R}}$ satisfying the required derivative bounds appearing in that lemma. Let $c \in \mathfrak{p}_{\mathbb{R}}$ be a real parameter, and suppose $\mathrm{Supp} L_c(\lambda) = \mathfrak{h}$. Let

$$K_c : \mathfrak{h}_{\mathbb{R},reg} \to \mathrm{Herm}(\lambda)$$

be the function appearing in Theorem 4.3.5.1, i.e. K_c is the restriction of the Dunkl weight function at parameter c to $\mathfrak{h}_{\mathbb{R},reg}$. By Theorem 4.3.5.1 there is a positive integer $N > 0$ such that

$$\gamma_c(\delta^N P_1, \delta^N P_2) = \int_{\mathfrak{h}_{\mathbb{R}}} P_2(x)^\dagger \delta^{2N}(x) K_c(x) P_1(x) e^{-|x|^2/2} dx \qquad \text{for all } P_1, P_2 \in \mathbb{C}[\mathfrak{h}] \otimes \lambda.$$

By Lemma 4.3.4.1 K_c is a homogeneous function and we may take N large enough so that $\delta^{2N} K_c$ extends continuously to all of $\mathfrak{h}_{\mathbb{R}}$ with value 0 on the union $\mathfrak{h}_{\mathbb{R}} \backslash \mathfrak{h}_{\mathbb{R},reg}$ of the reflection hyperplanes. By the homogeneity of K_c, it follows that we may take N large enough so that $\delta^{2N} K_c : \mathfrak{h}_{\mathbb{R}} \to \mathrm{Herm}(\lambda)$ is in fact C^k and satisfies the hypotheses of the function q appearing in Lemma 4.4.2.2.

Let d be the degree of homogeneity of the function $\delta^{2N} K_c$, let $h > 0$ be arbitrary, and let $Q(h) : V_h(1) \otimes \lambda \to V_h(1) \otimes \lambda$, $V_h(1) \subset L^2(\mathfrak{h}_{\mathbb{R}})$, be the operator

$$Q(h)(f) := (\Pi_h \otimes \mathrm{Id})((\delta^{2N} K_c) f)$$

considered in Lemma 4.4.2.2. Recall from (4.4.12) that

$$V_h(1) = \mathbb{C}[\mathfrak{h}]^{\leq \frac{1}{2}(\frac{1}{h}-l)} e^{-|x|^2/2h}$$

where $l = \dim \mathfrak{h}$. For any $P_1, P_2 \in \mathbb{C}[\mathfrak{h}]^{\leq \frac{1}{2}(\frac{1}{h}-l)} \otimes \lambda$ we have

$$\begin{aligned}
\gamma_{c,\lambda}(\delta^N P_1, \delta^N P_2) &= \int_{\mathfrak{h}_\mathbb{R}} P_2(x)^\dagger (\delta^{2N} K_c)(x) P_1(x) e^{-|x|^2/2} dx \\
&= \int_{\mathfrak{h}_\mathbb{R}} (P_2(x) e^{-|x|^2/4})^\dagger (\delta^{2N} K_c)(x) (P_1(x) e^{-|x|^2/4}) dx \\
&= (2/h)^{l/2} \int_{\mathfrak{h}_\mathbb{R}} (P_2(\sqrt{2/h}x) e^{-|x|^2/2h})^\dagger (\delta^{2N} K_c)(\sqrt{2/h}x) (P_1(\sqrt{2/h}x) e^{-|x|^2/2h}) dx \\
&= (2/h)^{(l+d)/2} \int_{\mathfrak{h}_\mathbb{R}} (P_2(\sqrt{2/h}x) e^{-|x|^2/2h})^\dagger (\delta^{2N} K_c)(x) (P_1(\sqrt{2/h}x) e^{-|x|^2/2h}) dx \\
&= (2/h)^{(l+d)/2} \left\langle Q(h) \left(P_1(\sqrt{2/h}x) e^{-|x|^2/2h} \right), P_2(\sqrt{2/h}x) e^{-|x|^2/2h} \right\rangle_{L^2(\mathfrak{h}_\mathbb{R}) \otimes \lambda}
\end{aligned}$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathfrak{h}_\mathbb{R}) \otimes \lambda}$ denotes the inner product on $L^2(\mathfrak{h}_\mathbb{R}) \otimes \lambda$ induced from the standard inner product on $L^2(\mathfrak{h}_\mathbb{R})$ and the inner product $(\cdot, \cdot)_\lambda$ on λ. In particular, for all $h > 0$,

$$\begin{aligned}
\max\{\dim U : U \subset \mathbb{C}[\mathfrak{h}]^{\leq \frac{1}{2}(\frac{1}{h}-l)} \otimes \lambda, \gamma|_{\delta^N U} \text{ is positive definite}\} \\
= \#\{\mu \in \mathrm{Spec}(Q(h)) : \mu > 0\},
\end{aligned} \tag{4.4.24}$$

$$\begin{aligned}
\max\{\dim U : U \subset \mathbb{C}[\mathfrak{h}]^{\leq \frac{1}{2}(\frac{1}{h}-l)} \otimes \lambda, \gamma|_{\delta^N U} \text{ is negative definite}\} \\
= \#\{\mu \in \mathrm{Spec}(Q(h)) : \mu < 0\},
\end{aligned} \tag{4.4.25}$$

and

$$\dim \mathrm{rad} \left(\gamma|_{\delta^N \mathbb{C}[\mathfrak{h}]^{\leq \frac{1}{2}(\frac{1}{h}-l)} \otimes \lambda} \right) = \dim \ker Q(h). \tag{4.4.26}$$

For a point $x \in \mathfrak{h}_{\mathbb{R},reg}$, let p be the dimension of a maximal positive-definite subspace of λ with respect to the Hermitian form $K_c(x)$, let q be the dimension of a maximal negative-definite subspace of λ with respect to $K_c(x)$, and let $r = \dim \mathrm{rad}(K_c(x)) = \dim \lambda - p - q$ be the dimension of the radical of $K_c(x)$. Recall that the Hermitian forms $K_c(x)$ and $K_c(x')$ are equivalent for all $x, x' \in \mathfrak{h}_{\mathbb{R},reg}$, so the integers p, q, r do not depend on the choice of the point $x \in \mathfrak{h}_{\mathbb{R},reg}$. As $K_c(x)$ descends to a B_W-invariant non-degenerate Hermitian form on $KZ_x(L_c(\lambda))$, the quantity $p-q$ is as in the statement of Theorem 4.4.0.1.

As usual, let $N_c(\lambda)$ denote the maximal proper submodule of $\Delta_c(\lambda)$, and recall that $N_c(\lambda) = \mathrm{rad}(\gamma_{c,\lambda})$. Recall also from the proof of Corollary 4.3.5.2 that for any $x \in \mathfrak{h}_{\mathbb{R},reg}$ we have $KZ_x(N_c(\lambda)) = \mathrm{rad}(K_c(x))$, where we make the usual identification of $KZ_x(\Delta_c(\lambda))$ with λ as a vector space and the identification of $KZ_x(N_c(\lambda))$ as a subspace of $KZ_x(\Delta_c(\lambda))$. In particular, $N_c(\lambda)$ is a graded $\mathbb{C}[\mathfrak{h}]$-module whose restriction to $\mathfrak{h}_{reg}$ is an algebraic vector bundle of rank $r = \dim \mathrm{rad}(K_c(x))$. It follows from standard results on Hilbert series and equation (4.4.26) that we have

$$\lim_{h \to 0} \frac{\dim \ker Q(h)}{\dim V_h(1)} = r. \tag{4.4.27}$$

In the case of the function $\delta^{2N} K_c$, the constant γ appearing in Lemma 4.4.2.2 is given by $\gamma = q + r$. In particular, it follows from that lemma and equation (4.4.27) that we have

$$\limsup_{h \to 0} \frac{\#\{\mu \in \mathrm{Spec}(Q(h)) : \mu < 0\}}{\dim V_h(1)} = q. \tag{4.4.28}$$

Similarly, applying Lemma 4.4.2.2 to the function $-\delta^{2N} K_c$ we have

$$\limsup_{h \to 0} \frac{\#\{\mu \in \mathrm{Spec}(Q(h)) : \mu > 0\}}{\dim V_h(1)} = p. \tag{4.4.29}$$

As

$$\frac{\#\{\mu \in \mathrm{Spec}(Q(h)) : \mu > 0\}}{\dim V_h(1)} + \frac{\#\{\mu \in \mathrm{Spec}(Q(h)) : \mu < 0\}}{\dim V_h(1)} + \frac{\dim \ker Q(h)}{\dim V_h(1)} = \dim \lambda$$

we see that equations (4.4.28) and (4.4.29) hold with limits replacing lim sups. We therefore have, by equations (4.4.24) and (4.4.25),

$$\lim_{n \to \infty} \frac{sign(\gamma_{c,\lambda}|_{(\delta^N L_c(\lambda))^{\leq n}})}{\dim(\delta^N L_c(\lambda))^{\leq n}}$$

$$= \lim_{h \to 0} \frac{\#\{\mu \in \mathrm{Spec}(Q(h)) : \mu > 0\} - \#\{\mu \in \mathrm{Spec}(Q(h)) : \mu < 0\}}{\#\{\mu \in \mathrm{Spec}(Q(h)) : \mu > 0\} + \#\{\mu \in \mathrm{Spec}(Q(h)) : \mu < 0\}}$$

$$= \frac{p-q}{p+q}$$

$$= \frac{p-q}{\dim KZ(L_c(\lambda))}.$$

As $\mathrm{Supp} L_c(\lambda) = \mathfrak{h}$ we have, as explained in the proof of Corollary 4.3.5.2,

$$\lim_{n \to \infty} \frac{\dim(\delta^N L_c(\lambda))^{\leq n}}{\dim L_c(\lambda)^{\leq n}} = 1.$$

In particular, the asymptotic signature of $L_c(\lambda)$ is given by

$$a_{c,\lambda} = \lim_{n \to \infty} \frac{sign(\gamma_{c,\lambda}|_{L_c(\lambda)^{\leq n}})}{\dim L_c(\lambda)^{\leq n}} = \lim_{n \to \infty} \frac{sign(\gamma_{c,\lambda}|_{(\delta^N L_c(\lambda))^{\leq n}})}{\dim(\delta^N L_c(\lambda))^{\leq n}} = \frac{p-q}{\dim KZ(L_c(\lambda))},$$

as claimed. □

4.5 Conjectures and Further Directions

In this section we will discuss various possible extensions of the results of this chapter.

4.5.1 Complex Reflection Groups

The Dunkl weight function $K_{c,\lambda}$ constructed in this chapter is, at present, a phenomenon for finite real reflection groups. There are at least two relevant objects that are absent for finite complex reflection groups. First, the weight function $K_{c,\lambda}$ is a distribution on the real reflection representation $\mathfrak{h}_{\mathbb{R}}$, but the complex reflection representation of a finite complex reflection group in general does not arise as the complexification of a real reflection representation. Second, the Gaussian inner product $\gamma_{c,\lambda}$, for which $K_{c,\lambda}$ gives an integral formula, is itself absent for complex reflection groups. While the contravariant form $\beta_{c,\lambda}$ is defined for an arbitrary finite complex reflection group W and an irreducible representation λ, the definition of $\gamma_{c,\lambda}$ relies on the element $\mathbf{f}$ from the canonical $\mathfrak{sl}_2$-triple $\mathbf{e}, \mathbf{f}, \mathbf{h} \in H_c(W, \mathfrak{h})$ - but for complex W

there is no such $\mathfrak{sl}_2$-triple.

Nevertheless, from the proof of Theorem 4.3.4.7, when $|c_s|$ is small for all $s \in S$ the Dunkl weight function $K_{c,\lambda}$ is given by integration against an analytic function $K : \mathfrak{h}_{\mathbb{R},reg} \to \mathrm{End}_{\mathbb{C}}(\lambda)$ that has a single-valued extension $\widetilde{K}(z)$ to $\mathfrak{h}_{reg}$ of the form

$$\widetilde{K}(z) = F_{c^\dagger}(z)^{\dagger,-1}\widetilde{K}(x_0)F_c(z)^{-1}, \tag{4.5.1}$$

where $x_0 \in \mathfrak{h}_{\mathbb{R},reg}$, $\widetilde{K}(x_0) \in \mathrm{End}_{\mathbb{C}}(\lambda)$ defines a B_W-invariant sesquilinear pairing

$$KZ_{x_0}(\Delta_c(\lambda)) \times KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C},$$

and $F_c(z)$ is the monodromy of the modified KZ connection

$$\nabla'_{KZ} = d - \sum_{s \in S} c_s \frac{d\alpha_s}{\alpha_s} s$$

from x_0 to z. All of these ingredients are available for finite complex reflection groups as well. By Remark 4.3.4.6, when W is a finite complex reflection group, for any $x_0 \in \mathfrak{h}_{reg}$ there is an operator $A_c \in \mathrm{End}_{\mathbb{C}}(\lambda)$ defining a B_W-invariant sesquilinear pairing

$$KZ_{x_0}(\Delta_c(\lambda)) \times KZ_{x_0}(\Delta_{c^\dagger}(\lambda)) \to \mathbb{C},$$

unique up to scalar multiple for generic c. The modified KZ connection is naturally generalized as

$$\nabla'_{KZ} = d - \sum_{s \in S} 2c_s \frac{d\alpha_s}{(1-\lambda_s)\alpha_s} s,$$

where λ_s is the nontrivial eigenvalue of the complex reflection $s \in S$. Taking $\widetilde{K}(x_0) = A_c$ and defining $\widetilde{K}(z)$ for $z \in \mathfrak{h}_{reg}$ as in equation (4.5.1) defines a single-valued function $\widetilde{K} : \mathfrak{h}_{reg} \to \mathrm{End}_{\mathbb{C}}(\lambda)$. For generic c, such a function is uniquely determined up to a global scalar multiple. It would be interesting to understand the meaning of these functions for finite complex reflection groups and to investigate their relationship to the contravariant form $\beta_{c,\lambda}$.

4.5.2 Preservation of Jantzen Filtrations, Epsilon Factors, and Signatures

In Section 4.2.1, we introduced Jantzen filtrations for standard modules $\Delta_c(\lambda)$, depending on a base parameter $c_0 \in \mathfrak{p}_{\mathbb{R}}$ and a deformation direction $c_1 \in \mathfrak{p}_{\mathbb{R}}$ and arising from the analytic (in fact, polynomial) family of forms $\beta_{c,\lambda}$. Given a point $x_0 \in \mathfrak{h}_{\mathbb{R},reg}$, the Dunkl weight function provides a natural family $K_{c,\lambda}(x_0)$ of B_W-invariant Hermitian forms on $KZ_{x_0}(\Delta_c(\lambda))$, holomorphic in $c \in \mathfrak{p}$. Given a base parameter $c_0 \in \mathfrak{p}_{\mathbb{R}}$ and deformation direction $c_1 \in \mathfrak{p}$, one therefore obtains a Jantzen filtration on $KZ_{x_0}(\Delta_c(\lambda))$ by $\mathsf{H}_q(W)$-submodules. The proof of Corollary 4.3.5.2 shows that $KZ_{x_0}(\mathrm{rad}(\beta_{c,\lambda})) = \mathrm{rad}(K_c(x_0))$, i.e. KZ_{x_0} intertwines the first two terms of the Jantzen filtration on $\Delta_c(\lambda)$ with the first two terms of the Jantzen filtration on $KZ_{x_0}(\Delta_c(\lambda))$. It is natural to expect that this is true for all terms in the Jantzen filtration:

Conjecture 4.5.2.1. *For any base parameter $c_0 \in \mathfrak{p}_{\mathbb{R}}$ and deformation direction $c_1 \in \mathfrak{p}_{\mathbb{R}}$, the functor KZ_{x_0} sends the Jantzen filtration on $\Delta_c(\lambda)$ to the Jantzen filtration on $KZ_{x_0}(\Delta_c(\lambda))$.*

In fact, it may be possible to adapt the argument that $KZ_{x_0}(\mathrm{rad}(\beta c, \lambda)) = \mathrm{rad}(K_c(x_0))$ appearing in the proof of Corollary 4.3.5.2 directly to prove Conjecture 4.5.2.1. Alternatively, Ivan Losev has suggested that Conjecture 4.5.2.1 may be able to be proved by a purely categorical argument by considering appropriate $\mathbb{C}[[t]]$-linear categories deforming $\mathcal{O}_c(W, \mathfrak{h})$ and $\mathsf{H}_q(W)\text{-mod}_{f.d.}$ and interpreting the relevant Hermitian forms as maps from standard objects to (complex conjugates of) costandard objects.

Additionally, by Theorem 4.4.0.1, the asymptotic signature of the graded Hermitian form on the first subquotient of the Jantzen filtration of $\Delta_c(\lambda)$ equals the normalized signature of the Hermitian form on the first subquotient of the Jantzen filtration of $KZ_{x_0}(\Delta_c(\lambda))$. It is natural to expect that this too holds for all of the subquotients, although this should be significantly more difficult to show than Conjecture 4.5.2.1. In fact, this will follow from Lemmas 4.2.4.2 and 4.2.4.3, and their analogues at the level of Hecke algebra representations, if the *epsilon factors* $\epsilon_i \in \{\pm 1\}$ appearing in Lemma 4.2.4.3 are also compatible with KZ_{x_0}:

Conjecture 4.5.2.2. *For any base parameter $c_0 \in \mathfrak{p}_\mathbb{R}$ and deformation direction $c_1 \in \mathfrak{p}_\mathbb{R}$, Conjecture 4.5.2.1 holds, and for each $k \geq 0$, the epsilon factors attached to each full-support irreducible constituent of the k^{th} subquotient of the Jantzen filtration on $\Delta_c(\lambda)$ determine epsilon factors for the irreducible constituents of the k^{th} subquotient of the Jantzen filtration on $KZ_{x_0}(\Delta_c(\lambda))$. In particular, the asymptotic signature of the induced form on the k^{th} subquotient of the Jantzen filtration of $\Delta_c(\lambda)$ equals the normalized signature, in the sense of Theorem 4.4.0.1, of the induced form on the k^{th} subquotient of the Jantzen filtration on $KZ_{x_0}(\Delta_c(\lambda))$.*

Furthermore, after making an appropriate definition of Jantzen filtrations on $KZ_{x_0}(\Delta_c(\lambda))$ in the complex reflection group case, perhaps following the ideas of Section 4.5.1, one may similarly formulate Conjecture 4.5.2.2 in the complex reflection group case. One possible approach to prove such a conjecture could be to develop an analogue for rational Cherednik algebras of the signed Kazhdan-Lusztig polynomials introduced by Yee [74]. This would give an approach for providing an entirely algebraic proof of Theorem 4.4.0.1 that is also valid for complex reflection groups.

4.5.3 Local Description of $K_{c,\lambda}$ and Modules of Proper Support

Let $c \in \mathfrak{p}_\mathbb{R}$ be a real parameter. Theorem 4.4.0.1 shows that, when $L_c(\lambda)$ has full support, its asymptotic signature $a_{c,\lambda}$ can be described in a simple way in terms of the local nature of the distribution $K_{c,\lambda}$ near a generic point of its support in $\mathfrak{h}_\mathbb{R}$. It is natural to expect a similar statement to hold for arbitrary $L_c(\lambda)$.

From the proof of Theorem 4.3.4.7, the distribution $K_{c,\lambda}$, locally near any $b \in \mathfrak{h}_{\mathbb{R},reg}$, is given by integration against an analytic function of the form

$$B_c(x)^{\dagger,-1} K_{c,\lambda}(b) B_c(x)^{-1},$$

where $K_{c,\lambda}(b) \in \mathrm{Herm}(\lambda)$ and $B_c(x)$ is an analytic $GL(\lambda)$-valued function of x defined near x_0 and satisfying $B_c(b) = \mathrm{Id}$. Rearranging, we have

$$B_c(x)^\dagger K_{c,\lambda}(x) B_c(x) = K_{c,\lambda}(b) \tag{4.5.2}$$

for $x \in \mathfrak{h}_{\mathbb{R},reg}$ near b.

Now consider an arbitrary point $b \in \mathfrak{h}_\mathbb{R}$ be an arbitrary point. Let $W' = \mathrm{Stab}_W(b)$, let $\mathfrak{h}_{W'}$ be the unique W'-stable complement to $\mathfrak{h}^{W'}$ in $\mathfrak{h}$, and let $\mathfrak{h}_{\mathbb{R},W'} = \mathfrak{h}_\mathbb{R} \cap \mathfrak{h}_{W'}$. The action of W' on $\mathfrak{h}_{\mathbb{R},W'}$ realizes W' as a finite real reflection group, generated by the reflections $S' := S \cap W'$. In particular, we have the rational Cherednik algebra $H_{c'}(W', \mathfrak{h}_{W'})$ and its category $\mathcal{O}_{c'}(W', \mathfrak{h}_{W'})$ with parameter $c' := c|_{S'}$. In a sense, the local structure of a module $M \in \mathcal{O}_c(W, \mathfrak{h})$ near the point $b \in \mathfrak{h}$ respects the decomposition $\mathfrak{h} = \mathfrak{h}^{W'} \oplus \mathfrak{h}_{W'}$, with the structure in the $\mathfrak{h}_{W'}$ component described by the image $\mathrm{Res}_b M \in \mathcal{O}_{c'}(W', \mathfrak{h}_{W'})$ of M under the Bezrukavnikov-Etingof parabolic restriction functor Res_b [5, Section 3.5] and with the structure in the $\mathfrak{h}^{W'}$ component described by an associated local system [5, Section 3.7, Proposition 3.20]. The following conjecture describing the structure of $K_{c,\lambda}$ near b is a natural analogue:

Conjecture 4.5.3.1. *Let $c \in \mathfrak{p}_\mathbb{R}$ be a real parameter, let $b \in \mathfrak{h}_\mathbb{R}$ be an arbitrary point, let $W' = Stab_W(b)$, and let $x = (x', x'')$ denote the decomposition of a vector $x \in \mathfrak{h}_\mathbb{R}$ with respect to the direct sum decomposition $\mathfrak{h}_\mathbb{R} = \mathfrak{h}_{\mathbb{R},W'} \oplus \mathfrak{h}_\mathbb{R}^{W'}$. Then there exists a W'-equivariant $End_\mathbb{C}(\lambda)$-valued analytic function $B(x)$ defined in a neighborhood of b such that $B(b) = Id$ and*

$$B(x)^\dagger K_{c,\lambda}(x) B(x) = \sum_{\mu \in Irr(W')} K_{c,\mu}(x') \otimes h_\mu, \tag{4.5.3}$$

where, for each $\mu \in Irr(W')$, h_μ is a Hermitian form on $Hom_{W'}(\mu, \lambda)$.

A similar statement should hold for all $c \in \mathfrak{p}$. For arbitrary parameters $c \in \mathfrak{p}$, the lefthand side of equation (4.5.3) should be replaced by $B_{c^\dagger}(x)^\dagger K_{c,\lambda}(x) B_c(x)$, and the h_μ should be elements of $\mathrm{End}_\mathbb{C}(\mathrm{Hom}_{W'}(\mu, \lambda))$, not necessarily Hermitian. Generalizing the case $b \in \mathfrak{h}_{\mathbb{R},reg}$ in which h_λ gives a B_W-invariant Hermitian form on the $\mathsf{H}_q(W)$-representation $\lambda \cong_\mathbb{C} KZ_b(\Delta_c(\lambda))$, it is reasonable to expect that for all $\mu \in \mathrm{Irr}(W')$

the space $\mathrm{Hom}_{W'}(\mu,\lambda)$ is a representation of a generalized Hecke algebra, in the sense of [52, Definition 3.25], and that the operator h_μ is invariant under the action of the fundamental group $\pi_1(\mathfrak{h}_{reg}^{W'}/I)$ appearing in [52, Definition 3.25].

We can already see that Conjecture 4.5.3.1 holds in several cases. When $b \in \mathfrak{h}_{\mathbb{R},reg}$, we have $W' = 1$, and Conjecture 4.5.3.1 holds by equation (4.5.2). When $b = 0$, we have $W' = W$, and Conjecture 4.5.3.1 is trivial - simply take $B(x) = \mathrm{Id}$. When b is a generic point on a reflection hyperplane, i.e. $W' = \langle s \rangle$ for some $s \in S$, that Conjecture 4.5.3.1 holds follows from the proof of Theorem 4.3.4.7 - the function $P_{i,c}(x)$ gives the required function $B(x)$ and the operators $K_i^{1,1}, K_i^{-1,-1} \in \mathrm{End}_{\mathbb{C}}(\lambda)$, viewed as Hermitian forms on $\mathrm{Hom}_{W'}(\mathrm{triv},\lambda)$ and on $\mathrm{Hom}_{W'}(\mathrm{sgn},\lambda)$, respectively, give the required forms h_{triv} and h_{sgn}, where triv and sgn denote the trivial and sign representations of W', respectively.

In view of Conjecture 4.5.3.1, a natural extension of Theorem 4.4.0.1 to irreducible representations $L_c(\lambda)$ of arbitrary support is the following:

Conjecture 4.5.3.2. *Let $c \in \mathfrak{p}_{\mathbb{R}}$ be a real parameter, let $\lambda \in Irr(W)$ be an irreducible representation of W, let $b \in \mathfrak{h}_{\mathbb{R}}$ be a generic point in $Supp(L_c(\lambda)) \cap \mathfrak{h}_{\mathbb{R}} = Supp K_{c,\lambda}$. Then, using the notation from Corollary 4.5.3.1, the asymptotic signature $a_{c,\lambda}$ of $L_{c,\lambda}$ is given by*

$$a_{c,\lambda} = \frac{\sum_{\mu \in Irr(W'), \dim L_c(\mu) < \infty} \dim L_c(\mu) a_{c,\mu} sign(h_\mu)}{\dim Res_b L_c(\lambda)}. \tag{4.5.4}$$

Recall that $\mathrm{Res}_b L_c(\lambda)$ is nonzero and finite-dimensional if and only if b is a generic point in $\mathrm{Supp}(L_c(\lambda))$ [5, Proposition 3.23], so the righthand side of equation (4.5.4) is well-defined. When $b = 0$, we have that $W' = W$, $h_\mu = 0$ unless $\mu = \lambda$, h_λ is the standard form on $\mathrm{Hom}_{W'}(\lambda,\lambda) = \mathbb{C}$, and $\mathrm{Res}_b L_c(\lambda) = L_c(\lambda)$, so Conjecture 4.5.3.2 holds trivially. When $b \in \mathfrak{h}_{\mathbb{R},reg}$, Conjecture 4.5.3.2 is precisely Theorem 4.4.0.1, as in that case we have $W' = 1$, $L_c(\mu) = L_c(\mathrm{triv}) = \mathbb{C}$ (there are no other irreducible representations in the sum), $a_{c,\mu} = 1$, and $\mathrm{Res}_b L_c(\lambda) = KZ(L_c(\lambda))$ as vector spaces [5, Remark 3.16].

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