"One Hecke of an Application to $GL_n(\mathbb{F}_q)$":

Dimensions of Unipotent Principal Series Reps

Let $G = GL_n(\mathbb{F}_q)$ and B = its Borel subgroup of upper triangular matrices. Recall we have the associated Hecke algebra $H = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G \mathbb{C})$, a q-deformation of $\mathbb{C}S_n$, isomorphic to $\mathbb{C}S_n$ by Tits'.

Define the <u>unipotent principal series representations</u> of G to be those irreducible representations occurring inside $\mathrm{Ind}_B^G \mathbb{C}$. To start to compute things about these it is convenient to work with idempotents in $\mathbb{C}G$.

nention by F.R. $\langle \mathrm{Ind}_B^G \mathbb{C}, \pi \rangle$ $= \langle \mathbb{C}, \mathrm{Res}_B^G \pi \rangle$, so we're looking at G-irreps w/ B-fixed vectors.

Let $e = \frac{1}{|B|}\sum_{b \in B} b$. Then we have $e^2 = e$ and $\mathrm{Ind}_B^G \mathbb{C} \cong \mathbb{C}Ge$ as $\mathbb{C}G$-reps. We also have

<u>Proposition</u> If M is a $\mathbb{C}G$-module, then we have

$$\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}Ge, M) \cong eM, \qquad f \longmapsto f(e).$$

<u>Proof</u> This is obviously a linear map. To see it lands in eM, note $f(e) = f(e^2) = ef(e) \in eM$. For injectivity, note $\mathbb{C}Ge$ is cyclic/$\mathbb{C}G$, generated by e. For surjectivity, pick $m \in eM$. Then the map $\mathbb{C}Ge \to M$, $ae \mapsto am$ is well-defined since if $ae = 0$ then $am = a(em) = (ae)m = 0$, and this map is visibly $\mathbb{C}G$-linear. $\blacksquare$

<u>Corollary</u> $H := \mathrm{End}_{\mathbb{C}G}(\mathbb{C}Ge) \cong (e\mathbb{C}Ge)^{op}$

$$(a \mapsto ax) \longleftarrow x \quad . \text{ (as algebras)} \ \blacksquare$$

<u>Proposition</u> The map $V \longmapsto eV$ gives a bijection

$$\left\{ \begin{array}{l} \text{principal series unipotent} \\ \text{irreps of } GL_n(\mathbb{F}_q) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{irreducible representations} \\ \text{of } e\mathbb{C}Ge \end{array} \right\}$$

such that

$$(\text{mult of } V \text{ in } \mathbb{C}Ge) = \dim_{\mathbb{C}} eV.$$

Proof The final statement is just the previous proposition: $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}Ge, V) \cong eV$.

That V irred$/\mathbb{C}G \Rightarrow eV$ irred$/e\mathbb{C}Ge$ is just the observation that for $x \in eV$ nonzero we have $e\mathbb{C}Gex = e\mathbb{C}Gx = eV$.

$n_i > 0\ \forall i$ That the map is surjective is just the observation that if $\mathbb{C}Ge = \bigoplus n_i V_i$ with $V_i \not\cong V_j$ for $i \neq j$, we get by left-multiplication by e that

$$e\mathbb{C}Ge = \bigoplus_i n_i eV_i \quad \text{as } e\mathbb{C}Ge\text{-modules}$$

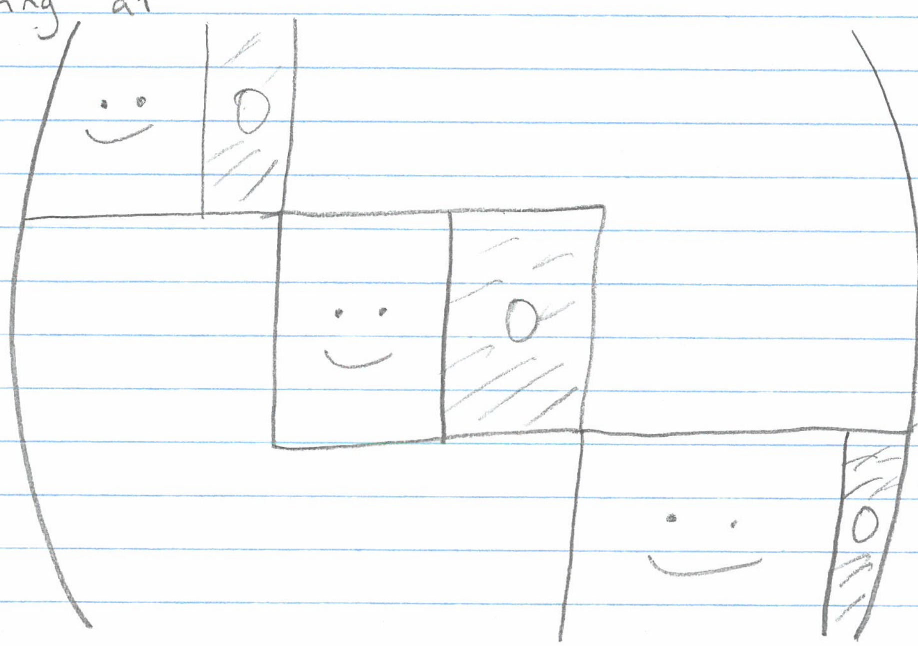
Every irrep of $e\mathbb{C}Ge$ occurs in its regular representation, so the eV_i must exhaust all $e\mathbb{C}Ge$-irreps.

For injectivity, note if ever $eV_i \cong eV_j$ for $i \neq j$ then $\langle e\mathbb{C}Ge, eV_i\rangle \geq n_i + n_j > n_i = \langle \mathbb{C}Ge, V_i\rangle = \dim eV_i$, contradictory to the Wedderburn decomposition of the semisimple algebra $e\mathbb{C}Ge$. ∎

But now I claim this dimension-multiplicity duality can be turned around. In particular, since $e\mathbb{C}Ge$ is semisimple, its irreducible representations are in natural bijection with the irreducible representations of $H = (e\mathbb{C}Ge)^{op}$. In particular, if A is a semisimple finite-dimensional $\mathbb{C}$-algebra, then the isotypic pieces of A as the regular A-bimodule ($= A \otimes A^{op}$-module) are irreducible and are the "matrix blocks," one for each irrep of A, and looking at A as a left A-module the multiplicity space of a irred left A-module is naturally the corresponding irred right A-module ($=$ irred left A^{op}-module). Denote the irred left $H = (e\mathbb{C}Ge)^{op}$-module associated to the irred left $e\mathbb{C}Ge$-module eV by eV^{op}. In particular $\dim eV = \dim eV^{op}$.

<u>Proposition</u> The isotypic pieces of $\mathbb{C}Ge$ as a left $\mathbb{C}G$-module and as a $H = (e\mathbb{C}Ge)^{op} = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G \mathbb{C})$-module coincide, with V-isotypic piece $= eV^{op}$-isotypic piece.

<u>Proof</u> The fact that the isotypic pieces coincide is a general thing, a consequence of Schur's Lemma. Seen a slightly different way, one can just look at the Wedderburn decomposition of $\mathbb{C}G$ in such a way that e takes the form in each matrix block of $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ for I of some size, staring at

$\mathbb{C}Ge =$ 

The smiley faces are visibly the isotypic pieces for both the left and right actions of $\mathbb{C}G$ or $e\mathbb{C}Ge$, respectively.

Now if $U_i = n_i V_i \subset \mathbb{C}Ge$ is the V_i-isotypic piece, then $e : U \to eU = n_i eV_i \subset e\mathbb{C}Ge$ is a homomorphism of right $e\mathbb{C}Ge$-modules. But $n_i eV_i$ is (for the right action) the eV_i^{op}-isotypic piece of $e\mathbb{C}Ge$ acting on itself on the right, and we conclude the V-isotypic piece of $\mathbb{C}Ge$ for $\mathbb{C}G$ is the eV^{op}-isotypic piece for $H = (e\mathbb{C}Ge)^{op}$. ∎

Corollary (The whole point)

The principal series unipotent representations of $GL_n(\mathbb{F}_q)$ are in natural bijection with the irred reps of $H_n(q)$ such that dimension on either side corresponds to multiplicity in $\mathbb{C}Ge$ on the other side.

Pf We have dim(isotypic piece) = (mult)$\cdot$(dim irred). The isotypic pieces agree, and so we get

$$\langle V, \mathbb{C}Ge\rangle_{\mathbb{C}G} \dim V = \langle eV^{op}, \mathbb{C}Ge\rangle_H \dim eV^{op}.$$

But we have $\langle V, \mathbb{C}Ge\rangle = \dim eV^{op}$ already, so

$$\dim V = \langle eV^{op}, \mathbb{C}Ge\rangle_H .$$ ∎

So we've reduced the problem to computing some multiplicities.

Remark We have already found out how to compute the character values χ_V for V a principal series unipotent representation at particular values, i.e. we can compute χ_V on $e\mathbb{C}Ge$, by the formula $\chi_V|_{e\mathbb{C}Ge} = \chi_{eV}$. To see why this formula holds

just note that for $x \in e\mathbb{C}Ge$, write $V = eV \oplus V'$ for some vector space complement V', and then $x \circlearrowright V$ has matrix

$$\begin{array}{c} eV \\ V' \end{array} \left(\begin{array}{c|c} x \circlearrowright eV & \text{stuff} \\ \hline 0 & 0 \end{array}\right)$$

Symmetric Algebra Basics

To go further we'll need to know a little about "orthogonality" for characters of symmetric algebras.

Recall: A symmetric algebra over $\mathbb{C}$ is a finite dimensional associative $\mathbb{C}$-algebra H along with an inner product $(\cdot,\cdot): H \otimes_{\mathbb{C}} H \longrightarrow H$ (symmetric, non-degenerate) such that $(xy, z) = (x, yz)$. Recall Hecke algebras associated to finite Coxeter groups $(q \in \mathbb{C}^\times)$ are symmetric algebras with dual basis to $\{T_w\}$ given by $T_w^\vee = q^{-l(w)} T_{w^{-1}}$.

Construction: Let V, W be finite dimensional H-reps. There is a natural map

$$\operatorname{Hom}_{\mathbb{C}}(V,W) \longrightarrow \operatorname{Hom}_H(V,W)$$
$$\varphi \longmapsto I(\varphi)$$

with

$$I(\varphi)(v) = \sum_{b \in \mathcal{B}} b^\vee \varphi(bv),$$

where $\mathcal{B}$ is a basis of H.

Proposition $I(\varphi)$ is independent of the choice of basis $\mathcal{B}$ and is an H-module homomorphism.

Proof Let C be another basis. Write

$$c_j = \sum_i h_{ij} b_i, \quad c_j^\vee = \sum_i k_{ij} b_i^\vee, \quad h_{ij}, k_{ij} \in \mathbb{C}.$$

Then we have

$$\delta_{j,j'} = (c_j, c_{j'}^\vee) = \sum_{i,i'} h_{ij} k_{i'j'} (b_i, b_{i'}^\vee) = \sum_i h_{ij} k_{ij'}.$$

Thus $(h_{ij})^T (k_{ij}) = I$.

Thus $\sum_j c_j \varphi(c_j^\vee v) = \sum_j \sum_{i,i'} h_{ij} k_{i'j} b_i \varphi(b_{i'}^\vee v)$

$$= \sum_{i,i'} \underbrace{\Big(\sum_j h_{ij} k_{i'j}\Big)}_{=\delta_{i,i'}} b_i \varphi(b_{i'}^\vee v) = \sum_i b_i \varphi(b_i^\vee v),$$ as needed.

To see it is a map of H-modules, let $h \in H$.

Write $bh = \sum_{b'} a_{b',b} b'$

Then $a_{b',b} = (bh, b'^\vee) = (b, hb'^\vee)$ so

$$hb'^\vee = \sum_b a_{b',b} b^\vee.$$

Thus

$$I(\varphi)(hv) = \sum_b b^\vee \varphi(bhv) = \sum_{b'} \Big(\sum_b a_{b',b} b^\vee\Big)\varphi(b'v) = \sum_{b'} hb'^\vee \varphi(b'v)$$

$$= h\, I(\varphi)(v)$$

Now we can prove:

mention that this implies orthogonality of distinct irred characters

<u>Orthogonality of (some) matrix coefficients</u>:

Let $\rho : H \to M_n(\mathbb{C})$, $\rho' : H \to M_m(\mathbb{C})$ be <u>non-isomorphic</u> irreducible representations. Let $\rho_{ij}, \rho'_{k\ell}$ denote the matrix coefficients. Then for any basis $\mathcal{B}$ of H,

$$\sum_{b \in \mathcal{B}} \rho_{ij}(b) \rho'_{k\ell}(b^\vee) = 0.$$

<u>Proof</u> Let $1 \le i \le n$, $1 \le \ell \le m$, and consider the linear map $(v_i \mapsto v'_\ell)$, $v_{i'} \mapsto 0$ for $i' \ne i$, where $v_1, \dots, v_n$ and $v'_1, \dots, v'_m$ are the standard bases of $\mathbb{C}^n$, $\mathbb{C}^m$. Then $I(v_i \mapsto v'_\ell) = 0$ since it is in $\mathrm{Hom}_H(V, W)$, so for any $1 \le j \le n$ we get

$$0 = I(v_i \mapsto v'_\ell)(v_j) = \sum_{b \in \mathcal{B}} b^\vee (v_i \mapsto v'_\ell)(bv_j)$$

$$= \sum_{b \in \mathcal{B}} b^\vee (v_i \mapsto v'_\ell)\Big(\sum_{j'=1}^{n} \rho_{j'j}(b) v_{j'}\Big) = \sum_{b \in \mathcal{B}} b^\vee \rho_{ij}(b) v'_\ell$$

$$= \sum_{b \in \mathcal{B}} \sum_{k=1}^{m} \rho_{ij}(b) \rho'_{k\ell}(b^\vee) v'_k.$$

Taking coefficients of the v'_k $\Longrightarrow$ ■

Back to Dimensions...

Now we apply our knowledge of symmetric algebras. But first a calculation:

Prop In the action of $H = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G \mathbb{C})$ on $\mathrm{Ind}_B^G \mathbb{C}$, T_w acts with trace 0 for $w \neq 1$ (and trace $[G:B]$ for $w=1$).

Pf The statement about T_1 is clear. Let $w \neq 1$. $\mathrm{Ind}_B^G \mathbb{C}$ has a basis of indicator functions of cosets Bg, call this indicator χ_g. Then we have

$$(T_w \chi_g)(g) = \frac{1}{|B|} \sum_{x \in G} T_w(x) \chi_g(x^{-1}g).$$

But $T_w(x) \neq 0 \Rightarrow x \in C(w) := BwB$, and
$\chi_g(x^{-1}g) \neq 0 \Rightarrow x^{-1}g \in Bg \Rightarrow x^{-1} \in B \Rightarrow x \in B = C(1)$,
so $C(w) \cap C(1) \neq \emptyset \Rightarrow w = 1$. Thus
$(T_w \chi_g)(g) = 0$, so the coefficient of χ_g in $T_w \chi_g$ is 0,
so $\mathrm{Tr}(T_w) = 0$ for $w \neq 1$.

Now let χ be an irreducible character of H, and let V_χ be the corresponding irreducible principal series unipotent representation of G. Then we have

Theorem
$$\dim V_\chi = \frac{\left(\sum_{w \in W} q^{\ell(w)}\right) \dim \chi}{\sum_{w \in W} \chi(T_w)\chi(q^{-\ell(w)} T_{w^{-1}})}.$$

Proof Let $\{\chi_j\}$ be the set of irreducible characters of H, and let $\{m_j\}$ denote their multiplicities in $\mathrm{Ind}_B^G \mathbb{C}$. From previous results we have $\dim V_{\chi_j} = m_j$.

We have $\sum_j m_j \chi_j(T_w) = \mathrm{Tr}(T_w; \mathrm{Ind}_B^G \mathbb{C}) = \begin{cases} 0 & w \neq 1 \\ [G:B] & w = 1 \end{cases}$.

Thus we have
$$\sum_{w\in W} \Big(\sum_j m_j \chi_j(T_w) \Big) \chi_i(q^{-\ell(w)} T_{w^{-1}}) = [G:B]\chi_i(T_1)$$

Exchanging order of summation and using orthogonality, recalling $q^{-\ell(w)} T_w = T_w^\vee$, we get this is also
$$\sum_j m_j \Big(\sum_{w\in W} \chi_j(T_w)\chi_i(q^{-\ell(w)} T_{w^{-1}}) \Big)$$
$$= m_i \sum_{w\in W} \chi_i(T_w)\chi_i(q^{-\ell(w)} T_{w^{-1}}).$$

Thus
$$\dim V_i = m_i = \frac{[G:B]\chi_i(T_1)}{\sum_{w\in W} \chi_i(T_w)\chi_i(q^{-\ell(w)} T_{w^{-1}})}$$
(b/c H semisimple $\Rightarrow$ is nonzero)

Finally, recall $\frac{|BwB|}{|B|} = q^{\ell(w)}$

so the Bruhat decomposition $G = \coprod_{w\in W} BwB$ gives
$$[G:B] = \sum_{w\in W} \frac{|BwB|}{|B|} = \sum_{w\in W} q^{\ell(w)}$$
the Poincaré polynomial, as needed.

Immediate Computable Examples

- <u>Deformed Trivial $\longleftrightarrow$ Trivial</u>

H has a "deformed trivial" representation δ given by $\delta: T_w \longmapsto q^{\ell(w)}$. This has dimension 1, so V_δ occurs in $\mathrm{Ind}_B^G \mathbb{C}$ with multiplicity 1. We have
$$\dim V_\delta = \frac{\left(\sum_{w\in W} q^{\ell(w)}\right)(1)}{\sum_{w\in W} \delta(T_w)\delta(q^{-\ell(w)} T_{w^{-1}})} = \frac{\sum_{w\in W} q^{\ell(w)}}{\sum_{w\in W} q^{\ell(w)}} = 1$$

This is actually the trivial representation. This is because $e_B e_G = e_G e_B = e_G$ supports the trivial $\mathbb{C}G$ and $e_B \mathbb{C}G e_B$ rep, where $e_B = \frac{1}{|B|}\sum_{b\in B} b$, $e_G = \frac{1}{|G|}\sum_{g\in G} g$.

Sign $\leftrightarrow$ Steinberg

We also have the linear character $\varepsilon: H \to \mathbb{C}$, $T_w \mapsto (-1)^{\ell(w)}$ deforming the sign representation of W. This has dimension 1, so V_ε also has multiplicity 1 in $\mathrm{Ind}_B^G \mathbb{C}$. We have, using that the Poincare polynomial is palindromic,

$$\dim V_\varepsilon = \frac{\left(\sum_{w\in W} q^{\ell(w)}\right)(1)}{\sum_{w\in W} \varepsilon(T_w)\varepsilon(q^{-\ell(w)} T_{w^{-1}})} = \frac{\sum_{w\in W} q^{\ell(w)}}{\sum_{w\in W} q^{-\ell(w)}}$$

$$= \left(\sum_{w\in W} q^{\ell(w)}\right)\left(\sum_{w\in W} q^{-\ell(w_0)} q^{\ell(w_0)-\ell(w)}\right)^{-1}$$

$$= q^{\ell(w_0)} \left(\sum_{w\in W} q^{\ell(w)}\right)\left(\sum_{w\in W} q^{\ell(w_0 w)}\right)^{-1}$$

$$= q^{\ell(w_0)}$$

where w_0 is the longest element.

Let $P_W = \sum_{w\in W} q^{\ell(w)}$ be the Poincare polynomial

and let $c_\chi = (\deg \chi)^{-1} \sum_{w\in W} \chi(T_w)\chi(T_w^\vee)$

where χ is an irreducible character of H. c_χ is called the Schur element of χ. We have the following general fact about symmetric algebras:

Prop If H is a symmetric algebra, then c_χ is the unique element of $\mathbb{C}$ such that if V is the module affording χ, then for all $\varphi \in \mathrm{End}_{\mathbb{C}}(V)$ we have
$$I(\varphi) = c_\chi \mathrm{Tr}(\varphi)\,\mathrm{id}_V.$$
H is semisimple $\Leftrightarrow$ $c_\chi \neq 0$ $\forall \chi$, and in that case if τ is the symmetrizing trace for H (so $\tau(x) = (x,1)$) we have
$$\tau = \sum_\chi c_\chi^{-1} \chi.$$

Proof The proof is easy and the talk is getting long, so I'll omit this!

Now consider again H to be the Hecke algebra associated to W (finite), now over $\mathbb{C}[q,q^{-1}]$. This is still a symmetric algebra.

Prop The Schur elements $c_\chi \in \mathbb{C}[q,q^{-1}]$ are uniquely determined by the system of equations

$$\sum_{\chi \in Irr(H)} c_\chi^{-1} \chi(T_w) = \begin{cases} 1 & w = 1 \\ 0 & w \neq 1 \end{cases}.$$

It suffices to take only one w from each conjugacy class.

Pf That the formulas hold is just the fact that $\sum_{\chi \in Irr(H)} c_\chi^{-1} \chi = \tau$ and $\tau(T_w) = \begin{cases} 1 & w=1 \\ 0 & w \neq 1 \end{cases}$.

The ~~[illegible]~~ statement follows because if we specialize at $q=1$ the matrix $(\chi(T_w))_{\chi,w}$ becomes the character table of W which is invertible, so it is invertible before specialization. $\blacksquare$

Define $D_\chi = \dim V_\chi$. We have seen $D_\chi = P_W c_\chi^{-1}$.

Let $W' \subset W$ be a parabolic subgroup, and let $H' \subset H$ be the associated parabolic Hecke subalgebra. For $\psi \in Irr(H')$, $\chi \in Irr(H)$, let $m(\chi,\psi)$ denote the multiplicity of χ in $\mathrm{Ind}_{H'}^{H}(\psi)$. Then

Prop $c_\psi^{-1} = \sum_{\chi \in Irr(H)} m(\chi,\psi) c_\chi^{-1} \quad \forall \psi \in Irr(H')$.

Proof By Frobenius reciprocity $m(\chi,\psi)$ is also the multiplicity of ψ in $\chi|H'$, so we have

$$\chi(T_w) = \sum_{\psi \in \mathrm{Irr}(H')} m(\chi,\psi)\psi(T_w) \quad \forall w \in W'.$$

By the previous proposition,

$$\sum_{\psi}\Big(\sum_{\chi} m(\chi,\psi)c_\chi^{-1}\Big)\psi(T_w) = \sum_{\chi} c_\chi^{-1}\chi(T_w) = \begin{cases} 1 & \text{if } w=1 \\ 0 & \end{cases}.$$

By the previous prop again we win. ∎

Now lets use these facts to get a handle on the D_χ in type A.

In type A, $P_{S_n} = P_{A_{n-1}} = [n]_q!$, where

$[n]_q! = [n]_q \cdots [1]_q$ where $[k]_q := 1+q+\cdots+q^{k-1}$.

This is seen by an easy induction on n, using minimal length coset reps for $S_{n-1} \subset S_n$.

Recall the <u>index</u> representation $\mathrm{ind}: H \longrightarrow \mathbb{C}$, $\mathrm{ind}(T_w) = q^{\ell(w)}$. This specializes to the trivial representation of S_n at $q=1$, and we have $c_{\mathrm{ind}} = 1^{-1}\sum_{w\in W} \mathrm{ind}(T_w)\,\mathrm{ind}(q^{-\ell(w)}T_{w^{-1}}) = P_W$.

If μ,λ are partitions of n, let $S_\mu = S_{\mu_1}\times\cdots$ be the associated parabolic subgroup, and define the <u>Kostka #</u> $K_{\lambda\mu}$ to be the multiplicity of χ_λ in $\mathrm{Ind}_{S_\mu}^{S_n}(\mathrm{triv})$.
As tensor product is right exact, induction commutes with specialization, and we obtain

$$(*) \quad c^{-1}_{\mathrm{ind\ on\ } H(S_\mu)} = \sum_{\lambda \vdash n} K_{\lambda\mu} c_\lambda^{-1}.$$

Facts: Kostka numbers have a combinatorial description: $K_{\lambda\mu}$ is the number of semistandard Young tableaux of shape λ and weight μ. The Kostka matrix $(K_{\lambda\mu})$ is invertible, as it is the change of basis matrix between complete homogeneous symmetric functions and Schur functions.

Since we have $D_\lambda = P_W / c_\lambda = [n]_q! / c_\lambda$, we have $c_\lambda^{-1} = [n]_q!^{-1} D_\lambda$. Also it is easy to see that if P_μ is the Poincare polynomial of S_μ, then we have $c_{\mathrm{ind}_\mu} = P_\mu = P_{\mu_1} \times \cdots \times P_{\mu_r} = [\mu_1]_q! \cdots [\mu_r]_q!$

where $\mu = (\mu_1, \ldots, \mu_r)$. Thus $(\ast)$ becomes

$$(\ast\ast) \quad \sum_{\lambda \vdash n} K_{\lambda\mu} D_\lambda = \frac{[n]_q!}{[\mu_1]_q! \cdots [\mu_r]_q!}$$

This gives a method for computing the D_λ, since the Kostka matrix is invertible. OR:

For a partition $\lambda: \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\lambda \vdash n$ (so maybe some $\lambda_i = 0$) and $w \in S_n$, let $\lambda_w = (\lambda_1 + 1 - w(1), \ldots, \lambda_n - n + w(n))$. We can define $K_{\mu\lambda_w}$ as before when λ_w is a composition of n, and to be 0 otherwise. We have:

FACT: $\displaystyle\sum_{w \in S_n} \varepsilon(w) K_{\mu,\lambda_w} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \end{cases}$.

We then get

$$D_\lambda = \sum_{\nu \vdash n} \Big(\sum_{w \in S_n} \varepsilon(w) K_{\nu\lambda_w} \Big) D_\nu$$

$$= \sum_{w \in S_n} \varepsilon(w) \Big(\sum_{\nu \vdash n} K_{\nu\lambda_w} D_\nu \Big)$$

$$(\ast\ast\ast) \quad = \sum_{w \in S_n} \varepsilon(w) \frac{[n]_q!}{[\lambda_1 - 1 + w(1)]_q! \cdots [\lambda_n - n + w(n)]_q!}$$

$$= [n]_q! \det(m_{ij}) \quad \text{where} \quad m_{ij} = \frac{1}{[\lambda_i - i + j]_q!},$$

where we agree $\dfrac{1}{[k]_q!} := 0$ when $k < 0$.

This gives an explicit formula for the generic degrees D_λ in type A. The formula makes it clear that these are polynomials in q, as it is just an alternating sum of q-multibinomial coefficients.