1. Motivation: Hecke Algebras in Nature

Let \( G \) be a finite group with the properties:
1) \( 3 \) subgroups \( B, N, G \) s.t. \( B N N = N \)
2) \( W = N/BN \) is generated by a set of involutions \( S \subseteq W \) s.t. \( (W,S) \) is a Coxeter system
3) Have double coset decomposition
   \[ G = \bigsqcup_{w \in W} BwB \]
4) If \( \ell \) is the length function for \( (W,S) \), have:
   \[ C(s)C(w) \in \begin{cases} C(sw) & \text{if } \ell(sw) > \ell(w) \\ C(sw) \cup C(w) & \text{if } \ell(sw) < \ell(w) \end{cases} \]
   where \( C(w) = BwB \).

Example: Let \( G \) be a connected, reductive linear algebraic group over an algebraically closed field \( k \) of char \( p > 0 \).
A Frobenius map on \( G \) is an endomorphism \( F: G \to G \) s.t. \( \exists n > 1 \) and an embedding \( \iota: G \to GL_n(k) \) s.t. the diagram
\[
\begin{array}{ccc}
G & \xrightarrow{\iota} & GL_n(k) \\
F & \downarrow & \\
G & \xrightarrow{\iota} & GL_n(k)
\end{array}
\]
\( F^n \) is standard Frobenius
\( \iota(G) \) commutes, where standard Frobenius is a map \( (a_{ij}) \mapsto (a_{ij}^p) \) where \( g = p \), see \( e > 0 \).
If \( B \) is an \( F \)-stable Borel subgroup, \( T \subset B \)
is an \( F \)-stable maximal torus, and \( N = N_G(T) \) is the normalizer of \( T \), then \( (G^F, B^F, N^F) \)
satisfy 1). \( G^F \) is finite and called a finite group of Lie type.
Sub-example. Let $G = GL_n(F_p)$, $F(a_{ij}) = (a_{ij}^q)$ with $q = p^k$, some $k \geq 1$. Then $G^F = GL_n(F_{q})$ can take $B = \text{Unip}$, $T = \text{Diag}$, so $B^F$ has same description, and $N^F = \text{monomial matrices}/F_{q}$, $W = S_n$, in fact $N^F = T^F \times S_n$.

Let $(G, B, N)$ satisfy $\Phi$ w/ Cox system $(W, S)$. We associate the Hecke algebra

$$H = \text{End}_G(1^G_B).$$

We have $1^G_B = \{ f : G \rightarrow C : f(bg) = f(g) \}$ is the set of all left-$B$-invariant $C$-valued functions on $G$.

The representation is by $(gf)(x) = f(xg^{-1})$. This is the permutation representation of $G$ on $B \backslash G$.

We recall:

Mackey's Theorem. Let $H_1, H_2 \subset G$ be finite groups, $V_1, V_2$ reps of $H_1, H_2$. For $\Delta : G \rightarrow \text{Hom}(V_1, V_2)$ s.t. $\Delta(h_2gh_1) = h_2 \Delta(g) h_1$, we get a map $\ 
\text{V}_1 \rightarrow \text{V}_2$

$f \mapsto \Delta \ast f$, where

$$\Delta \ast f(g) = \frac{1}{\# H_1} \sum_{x \in H_1} \Delta(x)f(x^{-1}g).$$

Note $\Delta \ast f : G \rightarrow V_2$ and $(\Delta \ast f)(gg') = g' \circ (\Delta \ast f)(g),$ clearly $f \mapsto \Delta \ast f$ comes w/ $G$-action on right, so $\Delta \ast \in \text{Hom}_G(\text{V}_1, \text{V}_2).$

Mackey's theorem:

$$\Delta : G \rightarrow \text{Hom}(V_1, V_2), \Delta(h_2gh_1) = h_2 \Delta(g) h_1 \Rightarrow \Delta \ast \in \text{Hom}_G(\text{V}_1, \text{V}_2).$$

$\Delta \mapsto \Delta \ast$.
Thus \( H = \text{End}_G(1_G) \cong \mathbb{B}\text{-bi-invariant } f: G \to \mathbb{C} \)
as vector spaces. But have convolution
\[
(\Delta_1 * \Delta_2)(g) = \frac{1}{|\mathbb{B}|} \sum_{x \in G} \Delta_1(x) \Delta_2(x^{-1}g),
\]
and \( \mathbb{B} \) this product the iso is as algebras.

Thus \( H \) has a natural basis given by
indicitator functions of the double cosets \( C(w) = \mathbb{B}w\mathbb{B} \).
Let \( T_w^3 \) be these indicator function, so
\( |T_w| = \text{dim } \mathbb{C} H \).

We would like to understand how to multiply the \( T_w \).

For \( s \in S \), let \( g_s = [\mathbb{B}s\mathbb{B}] / |\mathbb{B}| \).

Proposition \( \mathbb{B}w\mathbb{B} / |\mathbb{B}| = g_s_1 \cdots g_s_k \) when \( w = s_1 \cdots s_k \) reduced.
Proof By induction, need to show
\[ s w > w \Rightarrow [\mathbb{B}s\mathbb{B} / |\mathbb{B}|] [\mathbb{B}w\mathbb{B} / |\mathbb{B}|] = [\mathbb{B}sw\mathbb{B} / |\mathbb{B}|] \]
We have

\( \text{surjective mult. map } C(s) \times C(w) \to C(sw) \).
We need to check all fibers have size \( |\mathbb{B}| \).
The size of the fiber containing \( (x, y) \) is \# \{ \( g \in G \) : \( xg, g'y \) \in C(s) \times C(w) \}.
By Bruhat decomp, given \( g \in C(s) \cap C(w') \) with \( g \in C(w) \). Then
\[ xg \in C(s) \cap C(w) \cap C(sw) \cap C(w) \Rightarrow \Rightarrow \text{se } \{ s w, w' \} \]
\[ w \leftrightarrow s \in S \). But can't have \( w' = s \) since then \( g \in C(w) \cap C(sw) \cap C(w) \)
\[ = C(w) \cap C(sw) = \emptyset \], so get \( w' = 1 \), so \( g \in \mathbb{B} \), which works. 

Thus \( g_s_1 \cdots g_s_k \) only depends on \( s_1 \cdots s_k \) when reduced,
so can define \( g_w \).

Note \( s \mapsto g_s \) is thus a class function on \( S \).
Proposition: The G-linear map \( H \to C, \, Tw \mapsto g_w \) is a map of algebras.

Proof: Consider \( \varepsilon : H \to C, \, \varepsilon(f) = \frac{1}{|B|} \sum_{g \in G} f(g) \).

Then \( \varepsilon(f \circ f') = \frac{1}{|B|} \sum_{x \in G} (f \circ f')(x) \).

\[
= \frac{1}{|B|} \sum_{x \in G} \frac{1}{|B|} \sum_{y \in G} f(y) f'(y^{-1} x)
\]

\[
= \left( \frac{1}{|B|} \sum_{x \in G} f(x) \right) \left( \frac{1}{|B|} \sum_{y \in G} f'(y) \right) = \varepsilon(f) \varepsilon(y).
\]

But \( \varepsilon(T_w) = \frac{1}{|B|} |BwB| = g_w \).

Proof: If \( s \in C \) and \( Tw = T_s w \),

Post: Note \( (T_s T_w)(g) = \frac{1}{|B|} \sum_{x \in G} T_s(x) T_w(x^{-1} g) \).

But \( T_s(x) T_w(x^{-1} g) \neq 0 \)

\[
\Rightarrow x \in C(s), \, x^{-1} g \in C(w)
\]

\[
\Rightarrow g \in C(s) \cap C(w) = C(sw),
\]

so \( T_s T_w \) is supported on \( C(sw) \).

\[
\Rightarrow T_s T_w = c(sw) T_s.
\]

for some \( c(sw) \in C \). Applying \( \varepsilon \), get

\[
\varepsilon(T_s T_w) = c(sw) \Rightarrow c(sw) = 1.
\]

Proof: \( T_s^2 = g_s T_s + (g_s - 1) T_s \).

Post: Since \( C(s) C(s) \subset C(s) \cup C(s) \) as above, get

\[
\exists x, y \in C \text{ such that } T_s^2 = x T_s + \mu T_s.
\]

Evaluating at 1, get

\[
\mu = T_s^2(1) = \frac{1}{|B|} \sum_{x \in G} T_s(x) T_s(x^{-1}) = \frac{1}{|B|} |C(s)| = g_s,
\]

so

\[
T_s^2 = g_s T_s + \mu T_s.
\]

Applying \( \varepsilon \), get

\[
g_s^2 = g_s + g_s \mu \Rightarrow \mu = g_s - 1.
\]
2. Formal Parameters + Freeness

Note that for the algebras $H$ we just constructed, $H$ depended only on the Coxeter system $(W,S)$, and we have, by the relation we found, a surjection of algebras

$$H' = C < Tw: T_s T_w = \begin{cases} T_w & \text{if } sw > w \\ a_s T_w + b_s T_w & \text{if } sw < w \end{cases} \rightarrow H$$

We see $H$ is spanned by the $T_w$, so $\dim H' \leq \dim H$ and hence the above is an iso, and $H'$ is free on the $T_w$.

We can replicate this result abstractly for any Coxeter system $(W,S)$. Let $A$ be a commutative ring, and let $a, b : S \rightarrow A$ be class functions, and write $a_s = a(s), b_s = b(s)$. We then define the generic algebra $H(W,S,a,b)$ to be the $A$-algebra generated by $\{ T_w : w \in W \}$ with relations:

1) $T_s^2 = a_s 1 + b_s T_s$
2) $T_s T_w = T_w$ when $sw \geq w$.

Note then if $sw < w$, we have

$$T_s T_w = T_s T_s T_w = T_s T_s T_w = (a_s + b_s T_s) T_w = a_s T_w + b_s T_w,$$

so we could have as well said

$$T_s T_w = \begin{cases} T_w & \text{if } sw > w \\ a_s T_w + b_s T_w & \text{if } sw < w \end{cases}.$$

Note if we take $W$ as before, $a_s = b_s = 0$, $b_s = 0$, $A = C$, we get the algebras we already saw.
Theorem. \( H(W,S,a,b) \) is free on \( \{ Tw \} \) over \( A \).

Proof. Consider the free \( A \)-module \( E := \oplus_{w \in W} ATw \).

Define for \( s \in S \) the operators \( \lambda_s, \rho_s \in \text{End}_A(E) \) by:

\[ \lambda_s(Tw) = \begin{cases} 
Tsw & \text{if } sw < w \\
ast Tsw + bs Tw & \text{if } sw = w \\
ast Tsw & \text{if } sw > w
\end{cases} \]

\[ \rho_s(Tw) = \begin{cases} 
Tsw & \text{if } sw > w \\
d Tsw + bs Tw & \text{if } sw = w \\
ast Tsw & \text{if } sw < w.
\end{cases} \]

These are optimistic left-and right-multiplication operators.

Let \( L \subset \text{End}_A(E) \) be the \( A \)-algebra generated by the \( \lambda_s \). Suppose we knew \([\lambda_s, \rho_t] = 0 \) \( \forall s,t \).

We have the evaluation-at-\( T \) map

\[ \text{ev}: L \to E. \]

It is clearly surjective, since if \( w = s, \cdots, s_k \) reduced then \( \text{ev}(\lambda_s, \cdots, \lambda_{s_k}) = Tw \) by induction on \( \ell(w) \).

I claim it is also injective. Let \( f \in \ker(\text{ev}) \). Then \( f(Tw) = 0 \). But if \( f(Tw) = 0 \) and \( ws > w \), we have \( f(Tws) = f(\rho_s Tw) = \rho_s f(Tw) = 0 \) since \([L, \rho_s] = 0 \).

So by induction on \( \ell(w) \), \( f(Tw) = 0 \) \( \forall w \Rightarrow f = 0 \Rightarrow \text{ev} \) is an isomorphism.

I claim \( \lambda_s^2 = \ast Tsw + bs \lambda_s \). We show this by evaluating on the \( Tw \). Suppose first \( sw > w \). Then

\[ \lambda_s^2 Tw = \lambda_s Tsw = \ast Tsw + bs Tsw = (\ast + bs \lambda_s) Tw. \]

If \( sw < w \),

\[ \lambda_s^2 Tsw = \lambda_s (\ast Tsw + bs Tw) = \ast Tsw + bs Tw = (\ast + bs \lambda_s) Tw, \]

as needed.

Since \( \text{ev} \) is an isomorphism, \( L \) has a basis given by the \( \lambda_s \). \( \ast \lambda_s \) where \( w = s, \cdots, s_k \) is any reduced expression. But then if \( sw > w \) obviously \( \lambda_s \lambda_s = \lambda_s Tsw \).

Since \( \lambda_s \lambda_s = \lambda_s Tsw \), it follows that \( \lambda_s \lambda_s = \lambda_s Tsw \).
Thus, we have a map $H(W,S,a,b) \to L$ of $A$-algebras which is surjective, $Tw \to Ax$. But the $Tw$ span $H(W,S,a,b)$ and the $Ax$ are $A$-linearly independent in $L$, so this is injective too. Thus $H(W,S,a,b)$ is free on the $Tw$.

It remains to check that $[Ax, g+t]=0 \forall s,t$. This is by comparing the action of $Axg+t$ and $g+Ax$ on the $Tw$. We need a lemma:

**Lemma.** If $\ell(sw+t)=\ell(w)$, $\ell(sw)=\ell(w+t)$, we have $sw+t=w$, $sw=w+t$.

**Proof.** The numbers $\ell(sw+t)=\ell(w)$, $\ell(sw)=\ell(w+t)$ differ by 1. We can assume $\ell(w)$ is the small one, since we have symmetry by setting $w'=sw$, so $w+t=sw'$, $sw=w'+t$, $w=sw'$, and $sw'=w+t \iff w=sw'$.

Thus $\ell(sw') \leq \ell(w)$, so can make $sw=ss; \ldots; sx$ reduced. But $sw < sw$, so by the exchange lemma $sw+t$ has a reduced expression by deleting a term from $ss; \ldots; sx$. But it can't be an $s$ since then we'd have $\ell(w+t)=\ell(w)$, so it's $s$. But then $w=sw+t$.

Back to the proof now.
We do 6 cases:

1) \( l(w) < l(sw) = l(sw+) \leq l(sw+w) \).
   Then \( \lambda s s + Tw = Tsw+ = s + \lambda s Tw \).

2) \( l(sw+) < l(sw) = l(sw+) \leq l(w) \).
   Then \( \lambda s s + Tw = \lambda s (a + Tw + b + Tw) = a + (as Tsw+ + bs Tw) \)
   \( = a + (a + Tsw + b + Tsw) + bs (a + Tw + b + Tw) \)
   \( = a + Tsw + bs (Tw) \)
   \( = s + (as Tsw + bs Tw) = s + \lambda s Tw \).

3) \( l(sw) < l(w) = l(sw+) < l(sw) \).
   \( \lambda s s + Tw = \lambda s Tsw+ = as Tsw+ + bs Tw \)
   \( s + \lambda s Tw = s + (as Tsw+ + bs Tw) = as Tsw+ + bs Tw+ \).

4) \( l(sw) = l(sw+) < l(sw) = l(w) \).
   \( \lambda s s + (Tw) = \lambda s (a + Tw + b + Tw) = a + Tsw + b + Tsw \)
   \( s + \lambda s Tw = s + Tsw = a + Tsw + b + Tsw \).

5) \( l(sw) = l(sw+) < l(sw+) = l(w) \).
   Then \( s + \lambda s Tw = s + (as Tsw + bs Tw) = as Tsw+ + bs (a + Tw + b + Tw) \)
   while \( \lambda s s + Tw = \lambda s (a + Tw + b + Tw) = a + Tsw+ + b + (as Tsw + bs Tw) \).
   But also \( sw = wT, sw+ = w, s = wT + w' \), so \( a = aT, b = b + bT \), and w have equality.

6) \( l(w) = l(sw) = l(sw+) = l(w+) \).
   \( s + \lambda s Tw = s + Tsw = a + Tsw + b + Tsw \) while
   \( \lambda s s + Tw = \lambda s Tw = as Tsw + bs Tw \).
   But \( sw = wT, as = aT, bT = bT \).
3. Specializations and Semisimplicity

Let \((W,S)\) be a finite Coxeter system, and let \(H = H(W,S,a,b)\) be the generic Hecke algebra over \(A := \mathbb{C}[s,bs^{-1}]\) so we have one formal variable for each conjugacy class of \(S\). Give a \(C\)-algebra homomorphism \(\sigma : A \rightarrow C\), which amounts to choosing \(a,s,bs^{-1} \in \mathbb{C}\), we have the specialization by \(\sigma\) defined by

\[ H_{\sigma} := H \otimes_{A} C \]

This is a \(C\)-algebra, and from our freeness result we see it is a \(\mathbb{C}[W]\)-algebra with generators and relations as in section (2), i.e.,

\[ H_{\sigma} \cong H(W,S, a,s,bs^{-1}) \]

**Basic Question** What does \(H_{\sigma}\) look like?

When/how often is it semisimple?

We've seen something already. When \(W\) is a Weyl group and we set \(a = p_s, b = p_s^{-1}\), we get an endomorphism algebra of a representation of a finite group, hence is semisimple.

For general finite \(W\), setting \(a = 1, b = 0\) we get \(CW\), which is semisimple.

**Proposition** \(H_{\sigma}\) is semisimple for generic \(a,s,bs^{-1} \in \mathbb{C}\).

**Proof** Given a finite dimensional \(C\)-algebra and a basis \(\{b_i\}_i\), one can consider the discriminant

\[ \det(\text{Tr}(b_i b_j)) \]

This is well-defined up to nonzero multiple. I claim semisimple \(\iff\) disc \(\neq 0\). If \(x\) is in the
radical, then \( xy \) is in the radical of \( H_0 \), so mult by \( xy \) is a nilpotent operator, so \( Tr(xy) = 0 \) for \( H_0 \).

Thus \( x \) is in the kernel of the trace form. But \( \text{disc} \neq 0 \Rightarrow \) Trace form nondegen \( \Rightarrow x = 0 \Rightarrow \) semisimplicity.

Conversely, if an algebra is semisimple it is a product of complete matrix algebras over \( C \), and you can check the discriminant is nonzero.

The discriminant of \( H \) is a poly in the \( a_s, b_s \), and the disc of \( H_{a_s} \) is the specialization of this poly at \( a_s, b_s \). Thus \( H_{a_s} \) semisimple \( \iff a_s, b_s \) outside of a Zariski closed set. But \( H_{a_s} \) where \( \sigma_1(a_s) = 1, \sigma_1(b_s) = 0 \) is \( \cong CW \) so is semisimple, so \( H_{a_s} \) semisimple at a nonempty Zariski open set.

So what do these semisimple specializations look like? Let's check out type A.

Set \( W = S_n, g = P, G = GL_n(F_q) \), \( B = \Delta_{\alpha} \), and \( H = \text{End}_G(F^G) \). For \( x \in n \), let \( P_x \) be the corresponding standard parabolic. Let \( R = \text{G} \text{r} \text{o} \text{t} \text{h} \text{ } \text{g} \text{r} \text{ou} \text{p} \text{ of cat of } \leq \text{dim} \text{ } C \text{-reps of } S_n \) \( R(g) = \text{G} \text{r} \text{o} \text{t} \text{h} \text{ } \text{g} \text{r} \text{ou} \text{p of cat of } \leq \text{dim} \text{ } C \text{-reps of } S_n \).

For \( x = (\lambda_1 \geq \lambda_2 \geq \ldots) \in n \), let \( S_\lambda = s_{\lambda_1} \times s_{\lambda_2} \times \ldots \subset S_n \).

Then from symmetric function theory we know the \( h_\lambda : = \text{Ind}_{s_\lambda}^{S_n} 1 \) form a \( Z \)-basis for \( R \). So define the linear operator \( R \rightarrow R(g) \) by \( h_\lambda \rightarrow \text{Ind}_{s_\lambda}^G 1 \).

By Mackey and Brauer decomposition, we have

\[
\langle \text{Ind}_{s_\lambda}^G 1, \text{Ind}_{s_\mu}^G 1 \rangle = \# P_{\lambda} / \text{G} / P_{\mu} = \# S_n / S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_k} = \langle h_\lambda, h_\mu \rangle.
\]
Thus \( R \rightarrow R(g) \) is an isometry. It follows it sends irreps to irreps, and since every irrep occurs in some \( h_\alpha \) w/ positive multiplicity, we see it is \( \text{irrep} \rightarrow \text{irrep} \). But we then see that if \( \text{Ind}_{G}^{\text{1G}} = \bigoplus n_i V_i(g) \) is the decomp into irreps, then \( \text{Ind}_{S_n}^{\text{1S_n}} = \bigoplus n_i V_i (g) \) (with \( V_i \rightarrow V_i (g) \) irreps). Thus \( H = \text{End}(G) \cong \text{End}(1_{S_n}) = \text{End}_{S_n}(C_{S_n}) \cong C_{S_n} \). So all these \( H \) for different \( g = p^k \) are not only all semisimple, but also isomorphic!

This is a general thing:

**Theorem (Tits')** If \( \sigma, \sigma' : A \rightarrow C \) are so that \( H_\sigma, H_{\sigma'} \) are semisimple, then \( H_\sigma \cong H_{\sigma'} \).

So all semisimple specializations of \( H \) are isomorphic.

**Proof** Let \( F = \text{Frac}(A) \), and \( \overline{F} \) be an algebraic closure. Let \( H_{\overline{F}} = H \otimes F \). We've seen \( \text{disc}(H_{\overline{F}}) \) is a nonzero poly in the \( \alpha_i \)'s, so \( H_{\overline{F}} \) is semisimple. So it is a product of complete matrix algebras \( / F \) of size \( n_1 \cdots n_k \). Call these the numerical invariants of \( H_{\overline{F}} \). It suffices to show if \( \sigma : A \rightarrow C \) is such that \( H_\sigma \) is semisimple, then \( H_\sigma \) has numerical invariants \( n_1 \cdots n_k \) also.

Now we adjoin more formal variables \( x_w \) for \( w \in W \), and consider \( H_{\overline{F}} \otimes F(x_w: w \in W) \) and consider the "generic element" \( \alpha = \sum w x_w a_w \). If \( P(t) \)
is the characteristic poly for left mult by \(a\), we can factor
\[
P(t) = \prod P_i(t)^{e_i}
\]
in \(F(A)[t]\) where the \(P_i(t)\) are distinct irreps and the \(e_i \geq 1\).

But \(H = \bigoplus F(x_i)\) also has a direct sum decomp as \(\bigoplus M_{n_i}(F(x_i))\), so has a basis \(\{E_{ij} : 1 \leq i \leq k, 1 \leq j \leq n_i\}\).

So we can write
\[
a = \sum_{i,j} y_{ij} E_{ij}.
\]

The change of basis matrix between the \(T\) and \(E_{ij}\) has coeffs in \(F\) (since we can adjoin all this over \(F\)), so we conclude \(F(A) = F(y_{ij})\)

so the \(y_{ij}\) are alg ind by transc degree reasons. But working in the \(E_{ij}\) basis we see
\[
P(t) = \prod \det(tI - y_{ij} E_{ij}).
\]

Any poly \(g(t)\) of degree \(n_e\) is the determinant of some matrix

Since \(E\) irreps of all degrees and the \(y_{ij}\) are alg ind, we can specialize \(y_{ij} \neq t\). \(\det(tI - y_{ij} E_{ij})\) is irreducible, thus \(\det(tI - y_{ij} E_{ij})\) is irreducible. Clearly they are distinct for distinct \(e\), so we conclude
\[
P_e(t) = \det(tI - y_{ij} E_{ij})\] and \(e_e = y_e = \deg P_e(t)\).
Consider the coefficients of $P_e(t)$. They are polynomials in the roots of $P(t)$, so polynomials in the roots of $P(t)$, which are hence integral over the coefficients of $P(t)$, which lie in $\mathbb{A}[x_w]$. 

So, the coefficients of $P_e(t)$ lie in the integral closure of $\mathbb{A}[x_w]$ in $\mathbb{F}(x_w)$. If $\mathcal{I}$ is the integral closure of $\mathbb{A}$ in $\mathbb{F}$, from commutative algebra we have $\mathbb{A}[x_w] = \mathcal{I}[x_w]$, so coefficients of $P_e(t)$ lie in $\mathcal{I}[x_w]$.

I claim $\sigma : A \to C$ can be extended to an algebra homomorphism $\sigma : \mathcal{I} \to C$. By Zorn, it suffices to add one element at a time. But since $C$ is algebraically closed, we just send this elt to a root of its minimal polynomial.

Now consider the specialized algebra $H_0$. Consider the generic element $\alpha = \sum w^i t^w \in H_0 \otimes \mathbb{C}(x_w)$. Let $P(t)$ be its characteristic poly. Clearly this lies in the specialization of $P(t)$ by $\sigma$. Thus we have, using the extension $\sigma : \mathcal{I} \to C$

$$P_{\sigma}(t) = \prod_{c} P_{E_{\sigma}(c)}^{m_c}$$

where $P_{E_{\sigma}}$ is the specialization of $P_c$ by $\sigma$.

But since $H_0$ is semisimple, we know each irreducible factor of $P_{E_{\sigma}}(t)$ occurs with multiplicity = degree, by 1st argument. Since $m_c = \deg P_{E_{\sigma}}$, the $P_{E_{\sigma}}(t)$ must therefore be irreducible and distinct. Thus the $\mu_c$ are the numerical invariants for $H_0$ and we win.
4. Hecke Algebras as Symmetric Algebras

Let \((W,S)\) be a finite Coxeter system and let \(A\) be a commutative ring, and let \(a,b: S \to A\) be class functions so that \(a, b \in A^S\). We will see now that \(H = H(W,S,a,b)\) admits a non-degenerate symmetric bilinear form \((\cdot,\cdot): H \otimes A H \to A\) such that

\[(xy,z) = (x,yz).\]

This gives \(H\) the structure of a symmetric algebra.

Let \(\tau: H \to A\) be the A-linear functional defined by \(\tau(T_1) = 1, \tau(T_w) = 0\) for \(w \neq 1\). So \(\tau\) gives the coefficient of \(T_1\) in an expression in the \(T_w\) basis.

Define \((\cdot,\cdot): H \otimes A H \to A\) by

\[(x,y) = \tau(xy).\]

Theorem \((\cdot,\cdot)\) is symmetric non-degenerate bilinear form with \((xy,z) = (x,yz)\). We have explicitly

\[\tau(T_w T_{w'}) = \begin{cases} aw & \text{if } w' = w^{-1} \\ 0 & \text{otherwise} \end{cases}\]

where \(aw = a_{w_1} \cdots a_{s_1} \cdots a_{s_n}\) whenever \(s_1, \ldots, s_n\) is a reduced for \(w\). This is well-defined since \(a\) is a class function. The dual basis is \(T_w^\vee = a_{w_1} \cdots a_{s_1} \cdots a_{s_n} T_{w^{-1}}\).

In particular \(\tau(xy) = \tau(yx)\), so \(\tau\) is a trace function.
Proof. Clearly $(xy, z) = (x, yz)$. The rest follow immediately from the explicit formula. We prove this by induction on $\ell(w)$.

If $\ell(w) = 0$, $w = 1$ so nothing to prove.

Let $\ell(w) > 0$. Then $\exists s \leq t$, $ws < w$. We then have for $w' < w$,

$$\tau(TwTw') = \tau(TwTsTw').$$

Case 1. $sw' > w'$. Then $\tau(TwTw') = \tau(TwTsTw')$.

Now $sw' = (sw')s' \Rightarrow w' = s'. But then $ws < w'$

$$\Rightarrow sw' = sw' < w' = s'. \text{ contradicting, so } w' = w \text{ and }$$

$$\tau(TwTw') = \tau(TwTsTw') = 0 \text{ by induction, so the formula holds.}$$

Case 2. $sw' < w'$. Then

$$\tau(TwTw') = \tau(TwTsTw') = \tau(Tw(aTsTw' + bsTw'))$$

$$= a\tau(TwTsTw') + bs\tau(TwTw').$$

Note $w' = (sw')s' \Rightarrow w' = sw' < w' = sw'$. contradicting so 2nd term = 0 by induction.

As before $ws = (sw')s' \Rightarrow w' = w$, so if $w' = w'$

get 0, and otherwise get $a\tau(aWs = aw$ since $ws < w$.}