Category of Sheaves on $X_{\acute{e}t}$

Let X be a scheme/variety. Let Et/X denote the category of étale morphisms $\varphi: U \to X$, where morphisms in this category are the X-morphisms. Recall that there is an associated site, denoted $X_{\acute{e}t}$ and called the "small étale site of X," whose underlying category is Et/X. The coverings of an étale morphism $U \to X$ in this site are the families $\{\varphi_i : U_i \to U\}$ of étale X-morphisms which are surjective as a family (images cover U). That $X_{\acute{e}t}$ is a site reflects the facts that the class of étale morphisms is closed under base change and composition, and contains identity morphisms.

Recall that a presheaf on $X_{\acute{e}t}$ is a contravariant functor $P: Et/X \to \mathcal{C}$, where $\mathcal{C}$ is some category. Today we will only consider presheaves of abelian groups, where we take $\mathcal{C} = Ab$.

A morphism of presheaves is a morphism of functors. Unfortunately, as Et/X is not a small category, as we've defined things hom spaces of presheaves may fail to form sets. So, for technical reasons we replace Et/X by the smaller category of étale morphisms $\varphi: U \to X$ where U is obtained by patching varieties/schemes of the form $\operatorname{Spec} A[T_1, T_2, \ldots]$ where $A = \Gamma(\mathcal{O}_X, V)$ for some open affine $V \subset X$, and $T_1, T_2, \ldots$ are some fixed symbols. This is not a serious constraint, and we'll ignore it. But in this setting the category of presheaves of abelian groups on $X_{\acute{e}t}$ becomes an additive category. Denote it by $PSh(X_{\acute{e}t})$.

Prop $PSh(X_{\acute{e}t})$ is abelian. Kernels, cokernels, products, direct sums, inverse limits, direct limits exist and are formed by doing so on each étale $U \to X$.

1 side 2

Def The category of sheaves on $X_{ét}$, denoted $Sh(X_{ét})$, is the full subcategory of $PSh(X_{ét})$ consisting of objects F such that for every cover $\{U_i \to U\}$ in $X_{ét}$ the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact.

Clearly $Sh(X_{ét})$ is an additive category, so we have a notion of exactness. We'll see today that it is abelian and understand exactness more explicitly.

Def A morphism $\alpha: F \to F'$ of sheaves or presheaves on $X_{ét}$ is called locally surjective if for any section $s \in F'(U)$, there exists a cover $\{U_i \to U\}$ such that each $s|_{U_i}$ is in the image of $\alpha: F(U_i) \to F'(U_i)$.

Lemma Let $\alpha: F \to F'$ be a morphism of sheaves. TFAE:

a) The sequence $F \xrightarrow{\alpha} F' \to 0$ is exact.
b) α is locally surjective
c) For each geometric point $\bar{x}$, $\alpha_{\bar{x}}: F_{\bar{x}} \to F'_{\bar{x}}$ is surjective.

Comment/Explanation: This is entirely analogous to the Zariski situation, but now geometric points play the role of points. The reason geometric points play the same role for sheaves on $X_{ét}$ as points do for sheaves on X_{Zar} is that sheaves on a point in the Zariski sense are equivalent to Ab, but this is false for étale sheaves unless $k(x)$ is separably closed.
When $k = k^{sep}$, we get

$$Sh(\mathrm{Spec}(k)_{ét}) \xrightarrow{\sim} Ab$$
$$F \longmapsto F(\mathrm{id})$$

Like for usual points, we can form the stalk of a presheaf/sheaf at a geometric point, as the direct limit $\varinjlim_{(U,\bar{u})\to(X,\bar{x})} \mathcal{F}(U)$ over "étale neighborhoods of $\bar{x}$", which are the commutative diagrams (for $k = k^{sep}$)

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\bar{u}"] \arrow[dr, "\bar{x}"'] & U \arrow[d, "\text{étale}"] \\ & X \end{tikzcd}$$

This defines the stalk functor $\mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$ associated to $\bar{x}$. Clearly it is exact for presheaves.

Similarly, we have the functor

$$\begin{aligned} Ab &\longrightarrow Sh(X_{et}) \\ M &\longmapsto M^{\bar{x}}, \end{aligned}$$

where $M^{\bar{x}}$ is the sheaf defined by

$$M^{\bar{x}}(U \xrightarrow{\varphi} X) = \bigoplus_{\mathrm{Hom}_X(\bar{x}, U)} M$$

with obvious restriction maps. This is the étale analogue of skyscraper sheaves. As for usual sheaves, we have an adjunction

$$\mathrm{Hom}(\mathcal{F}, M^{\bar{x}}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}_{\bar{x}}, M).$$

<u>Proof of Lemma</u>

(b) $\Rightarrow$ (a). Suppose $\beta : \mathcal{F}' \longrightarrow \mathcal{T}$ is some morphism of sheaves such that $\beta \circ \alpha = 0$. We need $\beta = 0$. So let $s \in \mathcal{F}'(U)$ be some section. As α is locally surjective, $\exists$ a cover $\{U_i \to U\}$ such that the $s|_{U_i}$ are in the image of α. But then $\beta(s|_{U_i}) = 0$, and since $\beta(s)|_{U_i} = \beta(s|_{U_i})$ and $\mathcal{T}$ is a sheaf, we get $\beta(s) = 0$. So $\beta = 0$ as needed.

(a) $\Rightarrow$ (c). Well we just saw that the stalk functor is a left adjoint, so is right exact. But more explicitly, suppose some $\alpha_{\bar{x}} : \mathcal{F}_{\bar{x}} \longrightarrow \mathcal{F}'_{\bar{x}}$ is not surjective. Let $M = \operatorname{coker}(\alpha_{\bar{x}})$. Then $\mathcal{F}'_{\bar{x}} \longrightarrow M$ is nonzero, so gives a nonzero morphism $\mathcal{F}' \to M^{\bar{x}}$. But $\mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}} \to M$ is zero, so $\mathcal{F} \to \mathcal{F}' \to M^{\bar{x}}$ is also, which contradicts that $\mathcal{F} \to \mathcal{F}' \to 0$ is exact.

(c) $\Rightarrow$ (b). First, notice that if $\mathcal{U} \to X$ is étale and $\bar{u}$ is a geometric point of $\mathcal{U}$, then by composition we get a geometric point of X, say $\bar{x}$, and a natural map $\mathcal{F}_{\bar{u}} \to \mathcal{F}_{\bar{x}}$. It is easy to see that this map is an isomorphism, as the family of étale neighborhoods of $\bar{x}$ in X coming from étale neighborhoods of $\bar{u}$ in $\mathcal{U}$ via φ are clearly cofinal in the direct limit defining $\mathcal{F}_{\bar{x}}$ (just pull back).

So, let $s \in \mathcal{F}(\mathcal{U})$ be a section. For each $u \in \mathcal{U}$, choose a geometric point $\bar{u}$ mapping to u. Then we get $\alpha_{\bar{u}}: \mathcal{F}_{\bar{u}} \to \mathcal{F}'_{\bar{u}}$ is surjective. This means we can find an étale $V_u \to \mathcal{U}$ such that s is in the image of $\mathcal{F}(V_u) \to \mathcal{F}'(V_u)$. Taking sufficiently many V_u to cover $\mathcal{U}$, we get our claim. ∎

<u>Prop</u> Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be a sequence of sheaves on X_{et}. TFAE:

a) The sequence is exact.

b) The sequence $0 \to \mathcal{F}'(\mathcal{U}) \to \mathcal{F}(\mathcal{U}) \to \mathcal{F}''(\mathcal{U})$ is exact $\forall \mathcal{U}$.

c) The sequence $0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}}$ is exact for all geometric points $\bar{x}$.

<u>Proof</u> We'll soon see the forgetful functor $Sh(X_{et}) \to PSh(X_{et})$ has a left adjoint (sheafification), which gives the equivalence (a) $\Leftrightarrow$ (b).

Direct limits preserve exactness, so (b) $\Rightarrow$ (c). That (c) $\Rightarrow$ (b) follows from an argument like the one in the previous proposition. ∎

Sheafification

For a presheaf $\mathcal{F}$, the sheafification of $\mathcal{F}$ is a sheaf $a\mathcal{F}$ with a morphism $\mathcal{F} \to a\mathcal{F}$ such that any morphism $\mathcal{F} \to \mathcal{F}'$ with $\mathcal{F}'$ a sheaf factors uniquely as

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & a\mathcal{F} \arrow[d, dashed, "\exists!"] \\ & \mathcal{F}' \end{tikzcd}$$

So, the morphism $\mathcal{F} \to a\mathcal{F}$ is unique up to unique isomorphism, when it exists. In other words, a is the left adjoint to forget: $Sh(X_{et}) \to PSh(X_{et})$.

We have the following characterization of $\mathcal{F} \to a\mathcal{F}$:

Proposition Let $\mathcal{P}$ be a presheaf, $\mathcal{F}$ a sheaf, and $\alpha : \mathcal{P} \to \mathcal{F}$ a morphism so that

a) The only sections of $\mathcal{P}$ which have the same image under α are those that are locally equal, i.e. have equal restrictions on some refinement

b) α is locally surjective.

Then α is the sheafification of $\mathcal{P}$.

Proof Let $\beta : \mathcal{P} \to \mathcal{G}$ be another morphism with $\mathcal{G}$ a sheaf. Let $s \in \mathcal{F}(U)$ be a section. By (b), $\exists$ refinement $\{U_i \to U\}$ and $s_i \in \mathcal{P}(U_i)$ s.t. $\alpha(s_i) = s|_{U_i}$ $\forall i$. By (a) and the fact that $\mathcal{G}$ is a sheaf, their images $\beta(s_i)$ are independent of choice and agree on overlaps, so glue to some section of $\mathcal{G}(U)$. This clearly defines the needed morphism $\mathcal{F} \to \mathcal{G}$, and it is clearly unique by (b).

3
Side 2

Subsheaf Generated By A Subpresheaf of a Sheaf

Prop Let $\mathcal{P} \subset \mathcal{F}$ be a subpresheaf of the sheaf $\mathcal{F}$. For each U, let $\mathcal{P}'(U)$ be the set of $s \in \mathcal{F}(U)$ which are "locally in $\mathcal{P}$," i.e., $\exists$ cover $\{U_i \to U\}$ s.t. $s|_{U_i} \in \mathcal{P}(U_i)$ $\forall i$. Then $\mathcal{P}'$ is a subsheaf of $\mathcal{F}$, and the minimal one containing $\mathcal{P}$.

Also, $\mathcal{P} \to \mathcal{P}'$ is locally surjective.

Proof Obvious.

(✲) So, to construct the sheafification of a presheaf $\mathcal{P}$, it suffices to construct a morphism $\mathcal{P} \to \mathcal{F}$ with $\mathcal{F}$ a sheaf satisfying only (a) of the previous prop.

Theorem Sheafification $\mathcal{P} \to a\mathcal{P}$ exists. The induced map on stalks at geometric points $\bar{x}$

$$\mathcal{P}_{\bar{x}} \longrightarrow (a\mathcal{P})_{\bar{x}}$$

is an isomorphism $\forall \bar{x}$. $\mathcal{P} \longmapsto a\mathcal{P}$ is exact.

Proof Let $\mathcal{P}^* = \prod (\mathcal{P}_{\bar{x}})^{\bar{x}}$. The natural map $\mathcal{P} \to \mathcal{P}^*$ satisfies (a), so letting $a\mathcal{P}$ be the sheaf generated by the image, we get by (✲) that $\mathcal{P} \to a\mathcal{P}$ is the sheafification of $\mathcal{P}$.

The statement about stalks follows from (a), (b) easily, and the statement $\mathcal{P} \mapsto a\mathcal{P}$ is exact follows from the statement about stalks. ∎

Corollary $Sh(X_{et})$ is abelian.

Proof Construct cokernels etc in $PSh(X_{et})$ and sheafify. That

$$\text{coimage} \xrightarrow{\sim} \text{image}$$

can be checked on stalks. ∎

Example If M is an abelian group, $U \longmapsto M$ defines the constant presheaf, which sheafifies to $U \longmapsto M^{\pi_0(U)}$, the constant sheaf associated to M.

Direct Images

Recall that for a morphism $f: Y \to X$ of topological spaces, we can associate the direct image functor

$$f_*: PSh(Y) \longrightarrow PSh(X)$$
$$\mathcal{F} \longmapsto (U \longmapsto \mathcal{F}(f^{-1}(U))),$$

with the obvious restriction maps.

We want to do the same on the étale sites, when f is a morphism of varieties/schemes.

To see how to generalize, note in the usual case we have

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d, phantom, "\subseteq"] \arrow[dr, phantom, "\square"] & U \arrow[d, phantom, "\subseteq"] \\ Y \arrow[r, "f"] & X \end{tikzcd}.$$

So in the étale setting, we should do exactly the same thing, using diagrams

$$\begin{tikzcd} U \times_X Y \arrow[r] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "\text{étale}"] \\ Y \arrow[r] & X \end{tikzcd}$$

This suggests:

Definition If $f: Y \to X$ is a morphism,
$$f_*: PSh(Y_{\acute{e}t}) \longrightarrow PSh(X_{\acute{e}t})$$
is defined by $(f_*\mathcal{F})(U) = \mathcal{F}(U_Y)$, where by U_Y I mean the associated étale morphism $U \times_X Y \to Y$.

Lemma f_* preserves the sheaf property.

Proof This is because base changing étale covers gives étale covers: for $\{U_i \to U\}$ a covering in $X_{\acute{e}t}$, the sequence
$$f_*\mathcal{F}(U) \longrightarrow \prod_i f_*\mathcal{F}(U_i) \rightrightarrows \prod_{i,j} f_*\mathcal{F}(U_i \times_U U_j)$$
is equal to the sequence
$$\mathcal{F}(U_Y) \longrightarrow \prod_i \mathcal{F}(U_{i,Y}) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{i,Y} \times_{U_Y} U_{j,Y})$$
which is exact because $\mathcal{F}$ is a sheaf.

4
side 2

Lemma Clearly $f_*: PSh(Y_{\acute{e}t}) \longrightarrow PSh(X_{\acute{e}t})$ is exact, so $f_*: Sh(Y_{\acute{e}t}) \longrightarrow Sh(X_{\acute{e}t})$ is left exact. In general it is NOT right exact (consider X = alg var / $k = \bar{k}$, and $X \longrightarrow \operatorname{Spec} k$).

Example If $i: \bar{x} \longrightarrow X$ is a geometric point and M is an abelian group, then the $M^{\bar{x}}$ defined earlier is just $i_*(M)$, where we identify

$$\begin{aligned} Sh(\bar{x}_{\acute{e}t}) &\xrightarrow{\sim} Ab \\ \mathcal{F} &\longmapsto \mathcal{F}(id). \end{aligned}$$

Behavior on Stalks

$\pi: Y \rightarrow X$

Suppose $\bar{y}$ is a geometric point of Y mapping to a geometric point $\bar{x}$ of X. Then we have an induced map $(\pi_* \mathcal{F})_{\bar{x}} \longrightarrow \mathcal{F}_{\bar{y}}$.

In general, this will be neither injective nor surjective. But sometimes we can say something:

Prop (a) Let $\pi: V \hookrightarrow X$ be an open immersion. Then

$$(\pi_* \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } x \in V \\ ?? & \end{cases}$$

(b) Let $\pi: Z \hookrightarrow X$ be a closed immersion. Then

$$(\pi_* \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } x \in Z \\ 0 & \end{cases}$$

(c) Let $\pi: Y \longrightarrow X$ be finite map. Then

$$(\pi_* \mathcal{F})_{\bar{x}} = \bigoplus_{y \mapsto x} \mathcal{F}_{\bar{y}}^{d},$$

where d is the separable degree of $k(y)$ over $k(x)$.

Proof (a) If $x \in V$, the by base changing to V it is clear that the étale neighborhoods of $\bar{x}$ in X coming from étale neighborhoods of $\bar{x}$ in V are cofinal for the direct limit defining $\mathcal{F}_{\bar{x}}$, and (a) follows.

b) A similar argument shows $(\pi_* F)_{\bar{x}} = 0$ for $x \notin Z$.

For $x \in Z$, we need to show that every étale neighborhood $\bar{\varphi}: \bar{U} \longrightarrow Z$ extends to an étale morphism $U \longrightarrow X$. In terms of rings, this means for an étale morphism $\bar{\varphi}: \bar{A} \longrightarrow \bar{B}$ with $\bar{A} = A/\mathfrak{a}$, there needs to be an étale lift $\varphi: A \longrightarrow B$. Because all étale morphisms are locally standard etale morphisms, we can assume

$$\bar{B} = (\bar{A}[T]/\bar{f}(T))_{\bar{b}}$$

where $\bar{f}(T)$ is a monic polynomial such that $\bar{f}'(T)$ is invertible in $\bar{B}$ (where $\bar{b}$ is some localizing element). But then just choose $f(T) \in A[T]$ lifting $\bar{f}(T)$ and set $B = (A[T]/f(T))_b$ for an appropriate b.

Leave as exercise: is Cor 3.5 in Milne's book

c) Let $\bar{x}$ be a geometric point of X with image x. Then let $\tilde{X} = \operatorname{Spec} \mathcal{O}_{X,\bar{x}}$. Recall $\mathcal{O}_{X,\bar{x}}$ is a (strict) Henselian local ring. Let $\tilde{Y} = Y \times_X \operatorname{Spec} \mathcal{O}_{X,\bar{x}}$. Then $\tilde{Y}$ is finite over $\tilde{X}$, and as $\mathcal{O}_{X,\bar{x}}$ is Henselian this implies that the connected components of $\tilde{Y}$ are indexed by $\pi^{-1}(x)$ and

$$\tilde{Y} = \coprod_{y \in \pi^{-1}(x)} \operatorname{Spec}(\mathcal{O}_{X,\bar{x}}^{d(y)}).$$

The claim follows.

Cor If π is closed or finite, π_* is exact.

Pf Use previous prop + check exactness on geometric stalks.

Prop For any morphisms $Z \xrightarrow{\pi} Y \xrightarrow{\pi'} X$,

$$(\pi' \circ \pi)_* = \pi'_* \circ \pi_*.$$

Pf Obvious.

5
Side 2

Inverse Images

Recall in the setting of presheaves on topological spaces, for $\pi : Y \to X$ we define

$$(\pi^*_{p.s.}\mathcal{F})(U) = \varinjlim_{V \supset \pi(U)} \mathcal{F}(V).$$

This amounts to limits over diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, hook] \\ Y \arrow[r] & X \end{tikzcd}$$

with $U \to V$ any morphism,

so do exactly the same thing for the étale setting, replacing $\hookrightarrow$ by $\to$ étale.

Def Let $\pi : Y \to X$ be a morphism. Define the functor $\pi^*_{p.s.} : PSh(Y_{et}) \to PSh(X_{et})$ by

$$(\pi^*_{p.s.}\mathcal{F})(U) = \varinjlim_{\begin{tikzcd}[sep=small] U \arrow[r] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\text{étale}"] \\ Y \arrow[r] & X \end{tikzcd}} \mathcal{F}(U).$$

with obvious restriction maps.

It is easy to see that if P is a presheaf on X_{et}, Q a presheaf on Y_{et}, then there are natural identifications of:

1) Morphisms $\pi^*_{p.s.}P \to Q$

2) Families of maps $P(U) \to Q(V)$ indexed by $\begin{tikzcd}[sep=small] V \arrow[r] \arrow[d, "et"] & U \arrow[d, "et"] \\ Y \arrow[r] & X \end{tikzcd}$ compatible with restrictions

3) Morphisms $P \to \pi_* Q$.

Thus:

Prop $\pi^*_{p.s.}$ is left adjoint to π_*.

Cor $(\pi' \circ \pi)^*_{p.s.} = \pi^*_{p.s.} \circ \pi'^*_{p.s.}$

Pf Both are left adjoint to $(\pi' \circ \pi)_* = \pi'_* \circ \pi_*$.

Def: $\pi^*: Sh(X_{et}) \longrightarrow Sh(Y_{et})$ is $a \circ \pi^*_{p.s.}$

Thus π^* is left adjoint to π_*.

Let $\pi: Y \longrightarrow X$ be a morphism, and $i: \bar{y} \longrightarrow Y$ be a geometric point, and let $\bar{x}$ be the geometric point of X defined by $\pi \circ i$. Then note

$$(\pi^* \mathcal{F})_{\bar{y}} = (i^* \pi^* \mathcal{F})(\bar{y}) = \mathcal{F}_{\bar{x}}.$$

Thus:

Prop π^* is exact. π_* sends injectives $\longrightarrow$ injectives.

Cor Exactness is checked on stalks, using the previous comment. Preservation of injectives follows from being right adjoint to an exact functor. ∎

Cor $Sh(X_{et})$ has enough injectives.

Pf For each $x \in X$, choose a geometric point $\bar{x}$ mapping to x. Let $\mathcal{F}$ be a sheaf on X_{et}. Embed $\mathcal{F}_{\bar{x}} \hookrightarrow I(x)$, $I(x)$ some injective abelian group. Identifying $Sh(\bar{x}_{et}) \cong Ab$, this is an injective sheaf on $\bar{x}_{et}$, so $\pi_* I(x)$ is injective sheaf on X_{et}, so the product $\prod \pi_* I(x)$ is also. Then the embedding $\mathcal{F} \hookrightarrow \prod \pi_* I(x)$ does the trick. ∎

Extension by Zero

Let $j: U \hookrightarrow X$ be an open immersion. Define the functor

$$j_{!,p.s.}: PSh(U_{et}) \longrightarrow PSh(X_{et}) \text{ by}$$

$$(j_{!,p.s.} \mathcal{F})(\varphi: V \xrightarrow{\text{etale}} X) = \begin{cases} \mathcal{F}(V) & \text{if } \varphi(V) \subset U \\ 0 & \end{cases}.$$

Clearly $j_{!,p.s.}$ is left adjoint to $j^*_{p.s.}$, the restriction.

For sheaves, define $j_! = a \circ j_{!,p.s.}$, where a is sheafification. Then $j_!$ is left adjoint to j^*.

<u>Prop</u> We have $(j_! \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } x \in U \\ 0 & \underline{\qquad\qquad} \end{cases}$.

Thus:

<u>Prop</u> $j_!$ is exact, so its right adjoint j^* preserves injectives.

Let Z be the complement of U in X, and denote the associated closed immersion by $i : Z \hookrightarrow X$.

<u>Prop</u> For any sheaf $\mathcal{F}$, the natural sequence
$$0 \longrightarrow j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0$$
defined by the adjunctions is exact.

<u>Pf</u> Check exactness on stalks. 2 cases:
For $x \in U$, sequence reads
$$0 \longrightarrow \mathcal{F}_{\bar{x}} \xrightarrow{\text{id}} \mathcal{F}_{\bar{x}} \longrightarrow 0 \longrightarrow 0,$$
and for $x \notin U$, sequence reads
$$0 \longrightarrow 0 \longrightarrow \mathcal{F}_{\bar{x}} \xrightarrow{\text{id}} \mathcal{F}_{\bar{x}} \longrightarrow 0.$$
$\blacksquare$

<u>Remark</u> Can do a similar construction for $j : U \to X$ <u>any</u> étale map, not nec an open immersion. We define for $\varphi : V \to X$ étale,
$$(j_{!,p.s.} \mathcal{F})(V) = \bigoplus_{\alpha} \mathcal{F}(V)$$
where the direct sum is over morphisms $\alpha : V \to U$ s.t.
$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & U \arrow[d, "j"] \\ & X \end{tikzcd}$$
commutes.

We define $j_! \mathcal{F}$ to be the sheafification. Again, $j_!$ is left adjoint to j^*, and is exact, so j^* always preserves injectives.

Sheaves on $X = U \cup Z$

We conclude with a description of sheaves on $X = U \cup Z$ in terms of sheaves on U, Z and some compatibility. See Theorem 3.10 in Milne's book for full details.

Let $i : Z \hookrightarrow X \hookleftarrow U : j$ such that $X = i(Z) \amalg j(U)$, and let $\mathcal{F}$ be a sheaf on X_{et}. Then we get sheaves $i^*\mathcal{F}, j^*\mathcal{F}$ on Z_{et}, U_{et}.

By adjunction, we get a morphism
$$\mathcal{F} \longrightarrow j_*j^*\mathcal{F}.$$
Applying i^*, we get a morphism
$$\Phi_{\mathcal{F}} : i^*\mathcal{F} \longrightarrow i^*j_*j^*\mathcal{F}.$$

Now, let $Tr(X, U, Z)$ be the category whose objects are triples $(\mathcal{F}_1, \mathcal{F}_2, \phi)$ where $\mathcal{F}_1, \mathcal{F}_2$ are sheaves on Z_{et}, U_{et} respectively, and $\phi : \mathcal{F}_1 \longrightarrow i^*j_*\mathcal{F}_2$ is a morphism.

Morphisms in $Tr(X, U, Z)$ are defined in the obvious way, i.e. pairs (ψ_1, ψ_2) of morphisms of the sheaves on Z_{et}, U_{et} compatible with the compatibility morphism.

<u>Thm</u> The functor
$$Sh(X_{et}) \longrightarrow Tr(X, U, Z)$$
$$\mathcal{F} \longmapsto (i^*\mathcal{F}, j^*\mathcal{F}, \Phi_{\mathcal{F}})$$
is an equivalence. An essential inverse is given by
$$(\mathcal{F}_1, \mathcal{F}_2, \Phi) \longmapsto i_*\mathcal{F}_1 \times_{i_*i^*j_*\mathcal{F}_2} j_*\mathcal{F}_2.$$

<u>Corollary</u> For any closed immersion $i : Z \hookrightarrow X$, i_* defines an equivalence between $Sh(Z_{et})$ and sheaves on X_{et} <u>supported</u> on Z.

(means $j^*\mathcal{F} = 0$)