Category of Sheaves on $X\text{ét}$

Let $X$ be a scheme/variety. Let $\mathcal{E}_X$ denote the category of étale morphisms $\mathcal{U} \to X$, where morphisms in this category are the $X$-morphisms. Recall that there is an associated site, denoted $X\text{ét}$ and called the "small étale site of $X"$, whose underlying category is $\mathcal{E}_X$. The coverings of an étale morphism $\mathcal{U} \to X$ in this site are the families $\{\mathcal{U}_i \to \mathcal{U}\}$ of étale $X$-morphisms which are surjective as a family (images cover $U$). That $X\text{ét}$ is a site reflects the facts that the class of étale morphisms is closed under base change and composition, and contains identity morphisms.

Recall that a presheaf on $X\text{ét}$ is a covariant functor $\mathcal{P} : \mathcal{E}_X \to \mathcal{C}$, where $\mathcal{C}$ is some category. Today we will only consider presheaves of abelian groups, where we take $\mathcal{C} = \text{Ab}$.

A morphism of presheaves is a morphism of functors. Unfortunately, as $\mathcal{E}_X$ is not a small category, as we've defined things hom spaces of presheaves may fail to form sets. So, for technical reasons we replace $\mathcal{E}_X$ by the smaller category of étale morphisms $\mathcal{U} \to X$ where $\mathcal{U}$ is obtained by patching varieties/schemes of the form $\text{Spec } A[T_1, \ldots, T_n]_Y$ where $A = \Gamma(\mathcal{O}_X, V)$ for some open affine $V \subseteq X$, and $T_1T_n$ are some fixed symbols. This is not a serious constraint, and we'll ignore it. But in this setting the category of presheaves of abelian groups on $X\text{ét}$ becomes an additive category. Denote it by $\text{PSh}(X\text{ét})$.

Prop $\text{PSh}(X\text{ét})$ is abelian. Kernels, cokernels, products, direct sums, inverse limits, direct limits exist and are formed by doing so on each étale $\mathcal{U} \to X$. 
Def. The category of sheaves on $\mathbb{X}(\text{et})$, denoted $\mathbf{Sh}(\mathbb{X}(\text{et}))$, is the full subcategory of $\mathbf{PSh}(\mathbb{X}(\text{et}))$ consisting of objects $\mathcal{F}$ such that for every cover $\mathcal{U} : U \rightarrow U$, in $\mathbb{X}(\text{et})$, the sequence
\[ \mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightarrow \prod_{ij} \mathcal{F}(U_i \times U_j) \]

is exact.

Clearly $\mathbf{Sh}(\mathbb{X}(\text{et}))$ is an additive category, so we have a notion of exactness. We'll see today that it is abelian and understand exactness more explicitly.

Def. A morphism $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ of sheaves or presheaves on $\mathbb{X}(\text{et})$ is called locally surjective if for any section $s \in \mathcal{F}(U)$, there exist a cover $\mathcal{U} : U \rightarrow U$, such that each $s|U_i$ is in the image of $\alpha : \mathcal{F}(U) \rightarrow \mathcal{F}'(U_i)$.

Lemma. Let $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ be a morphism of sheaves.

TFAE:
\begin{enumerate}
  \item The sequence $\mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0$ is exact.
  \item $\alpha$ is locally surjective.
  \item For each geometric point $\xi$, $\alpha_{\xi} : \mathcal{F}_{\xi} \rightarrow \mathcal{F}'_{\xi}$ is surjective.
\end{enumerate}

Comment/Explanation: This is entirely analogous to the Zariski situation, but now geometric points play the role of points. The reason geometric points play the same role for sheaves on $\mathbb{X}(\text{et})$ as points do for sheaves on $\mathbb{X}(\text{zar})$ is that sheaves on a point in the Zariski sense are equivalent to $\text{Ab}$, but this is false for étale sheaves unless $\mathbb{X}(\mathbb{R})$ is separably closed.

When $k = k^{\text{sep}}$, we get
\[
\text{Sh}(\text{Spec}(k)_{\text{et}}) \longrightarrow \text{Ab}
\]
\[
\mathcal{F} \longrightarrow \mathcal{F}(\text{id})
\]
Like for usual points, we can form the stalk of a presheaf/sheaf at a geometric point, as the direct limit $\varprojlim F(U)$ over "etale neighborhoods of $k$", which are the commutative diagrams (for $k = k^\text{sep}$)

$$\begin{array}{ccc}
\text{Spec } k & \to & U \\
\downarrow & & \downarrow \text{etale} \\
\times & & X
\end{array}$$

This defines the stalk functor $F \mapsto F_x$ associated to $x$. Clearly it is exact for presheaves.

Similarly we have the functor

$$\begin{array}{ccc}
\text{Ab} & \to & \text{Sh}(X_{\text{et}}) \\
M & \mapsto & \underline{M}^X
\end{array}$$

where $\underline{M}^X$ is the sheaf defined by

$$\underline{M}^X(U \to X) = \bigoplus_{\text{Hom}_X(F, U)} M$$

with obvious restriction maps. This is the etale analogue of skyscraper sheaves. As for usual sheaves, we have an adjunction

$$\text{Hom}(F, \underline{M}^X) \cong \text{Hom}(F_x, M).$$

**Proof of Lemma**

$(b) \implies (a)$. Suppose $\beta : F' \to T$ is some morphism of sheaves such that $\beta_0 = 0$. We need $\beta = 0$. So let $s \in F'(U)$ be some section. As $x$ is locally surjective, $\exists$ a cover $\mathbf{U} : \mathbf{U}^\beta \to U_\beta$ such that the $s|_{U_i}$ are in the image of $x$. But then $\beta(s|_{U_i}) = 0$, and since $\beta(s)|_{U_i} = \beta(s|_{U_i})$ and $T$ is a sheaf, we get $\beta(s) = 0$. So $\beta = 0$ as needed.

$(a) \implies (c)$. Well we just saw that the stalk functor $i^!$ is a left adjoint, so is right exact. But more explicitly, suppose some $\alpha_x : F_x \to F'_x$ is not surjective. Let $M = \text{coker}(\alpha_x)$. Then $F'_x \to M$ is nonzero, so gives
a nonzero morphism $F' \to M$. But $F'' \to F'_x \to M$ is zero, so $F \to F' \to M$ is also, which contradicts that $F \to F' \to 0$ is exact.

(c) $\Rightarrow$ (b). First, notice that if $U \to X$ is étale and $\overline{u}$ is a geometric point of $U$, then by composition we get a geometric point of $X$, say $x$, and a natural map $F_u \to F_x$. It is easy to see that this map is an isomorphism, as the family of étale neighborhoods of $\overline{x}$ in $X$ coming from étale neighborhoods of $\overline{u}$ in $U$ via $\eta$ are clearly cofinal in the direct limit defining $F_x$ (just pull back).

So, let $s \in F(U)$ be a section. For each $x \in U$, choose a geometric point $\overline{u}$ mapping to $x$. Then we get $s_{\overline{u}} : F_\overline{u} \to F_x$ is surjective. This means we can find an étale $V_u \to U$ such that $s$ is in the image of $F(V_u) \to F(U)$.

Taking sufficiently many $V_u$ to cover $U$, we get our claim.

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Let $O \to F' \to F \to F''$ be a sequence of sheaves on $X_{et}$. TFAE:

a) The sequence is exact.

b) The sequence $O \to F'(U) \to F(U) \to F''(U)$ is exact for all $U$.

c) The sequence $O \to F'_x \to F_x \to F''_x$ is exact for all geometric points $x$.

Proof. We'll soon see the forgetful functor $\text{Sh}(X_{et}) \to \text{PSh}(X_{et})$ has a left adjoint (sheafification), which gives the equivalence (a) $\iff$ (b).

Direct limits preserve exactness, so (b) $\Rightarrow$ (c).

That (c) $\Rightarrow$ (b) follows from an argument like the one in the previous proposition.
Sheafification

For a presheaf \( F \), the sheafification of \( F \) is a sheaf \( aF \) with a morphism \( F \rightarrow aF \) such that any morphism \( F \rightarrow F' \) with \( F' \) a sheaf factors uniquely as \( F \rightarrow aF \rightarrow F' \).

So, the morphism \( F \rightarrow aF \) is unique up to unique isomorphism, when it exists. In other words, \( a \) is the left adjoint to the forget: \( \text{Sh}(X_{et}) \rightarrow \text{PSh}(X_{et}) \).

We have the following characterization of \( F \rightarrow aF \):

**Proposition** Let \( P \) be a presheaf, \( F \) a sheaf, and \( \alpha : P \rightarrow F \) a morphism so that

a) The only sections of \( P \) which have the same image under \( \alpha \) are those that are locally equal, i.e. have equal restrictions on some refinement.

b) \( \alpha \) is locally surjective.

Then \( \alpha \) is the sheafification of \( P \).

**Proof** Let \( B : P \rightarrow G \) be another morphism with \( G \) a sheaf. Let \( s \in F(U) \) be a section. By (b), \( B \) refines \( \psi : U_1 \rightarrow U_2 \) and \( s \in P(U_1) \) s.t. \( \alpha(s) = \beta(U_1, U_2) \). By (a) and the fact that \( G \) is a sheaf, their images \( B(s) \) are independent of choice and agree on overlaps, so glue to some section of \( G(U_1) \). This clearly defines the needed morphism \( F \rightarrow G \), and it is clearly unique by (b).
Subsheaf Generated By A Subpresheaf of a Sheaf

Prop. Let $P \subseteq F$ be a subsheaf of the sheaf $F$. For each $U$, let $P'(U)$ be the set of $s \in F(U)$ which are "locally in $P"$, i.e., $\exists$ cover $\mathcal{U} \rightarrow U$ s.t. $s|_{U_i} \in P(U_i)$ $\forall i$. Then $P'$ is a subsheaf of $F$, and the minimal one containing $P$.

Also, $P \twoheadrightarrow P'$ is locally surjective.

Proof. Obvious.

[\text{(b)}]

So, to construct the sheafification of a presheaf $P$ it suffices to construct a morphism $P \twoheadrightarrow aP$ with $aP$ a sheaf satisfying only (a) of the previous prop.

Theorem. Sheafification $P \twoheadrightarrow aP$ exists. The induced map on stalks at geometric point $\overline{x}$

$$P_{\overline{x}} \twoheadrightarrow (aP)_{\overline{x}}$$

is an isomorphism $\forall \overline{x}$. $P \twoheadrightarrow aP$ is exact.

Proof. Let $P^* = \Pi(P_{\overline{x}})^{\overline{x}}$. The natural map $P \twoheadrightarrow P^*$ satisfies (a), so letting $aP$ be the sheaf generated by the image, we get by (b) that $P \twoheadrightarrow aP$ is the sheafification of $P$.

The statement about stalks follows from (a), (b) easily, and the statement $P \twoheadrightarrow aP$ is exact follows from the statement about stalks.

Corollary. $\text{Sh}(X_{et})$ is abelian.

Proof. Construct cokernels etc in $P\text{Sh}(X_{et})$ and sheafify. That coimage $\sim$ image can be checked on stalks.

Example. If $M$ is an abelian group, $U \hookrightarrow M$ defines the constant presheaf which sheafifies to $U \rightarrow M^{U_{\text{et}}(U)}$, the constant sheaf associated to $M$.
Direct Images

Recall that for a morphism \( f : \mathcal{Y} \to \mathcal{X} \) of topological spaces, we can associate the direct image functor
\[
\tilde{f}^* : \text{PSh}(\mathcal{Y}) \to \text{PSh}(\mathcal{X})
\]
\[
F \mapsto (U \mapsto F(f^{-1}(U)));
\]
with the obvious restriction maps.

We want to do the same on the étale site, when \( f \) is a morphism of varieties/schemes.

To see how to generalize, note in the usual case we have
\[
\begin{align*}
\mathcal{Y} & \xrightarrow{f} \mathcal{X} \\
\mathcal{U} & \xrightarrow{\tilde{f}} \mathcal{U} \\
\end{align*}
\]
So in the étale setting, we should do exactly the same thing, using diagrams
\[
\begin{align*}
\mathcal{U} \times \mathcal{Y} & \xrightarrow{f} \mathcal{U} \\
\mathcal{U} & \xrightarrow{\tilde{f}} \mathcal{U} \\
\end{align*}
\]
This suggests:

**Definition** If \( f : \mathcal{Y} \to \mathcal{X} \) is a morphism,
\[
\tilde{f}^* : \text{PSh}(\mathcal{Y}_{\text{ét}}) \to \text{PSh}(\mathcal{X}_{\text{ét}})
\]
is defined by \((\tilde{f}^*F)(U) = F(\mathcal{U}_Y))\), where by \( \mathcal{U}_Y \) I mean the associated étale morphism \( \mathcal{U} \times \mathcal{Y} \to \mathcal{Y} \).

**Lemma** \( \tilde{f}^* \) preserves the sheaf property.

**Proof** This is because base changing étale covers gives étale covers: for \( \mathcal{U}_1 \to \mathcal{U} \) a covering in \( \mathcal{X}_{\text{ét}} \),

the sequence \( \tilde{f}^*F(U) \to \tilde{f}^*F(U_1) \to \tilde{f}^*F(U_1 \times_U U_2) \)
is equal to the sequence
\[
F(\mathcal{U}_Y) \to \tilde{f}^*F(U_1 \times_U \mathcal{Y}) \to \tilde{f}^*F(\mathcal{U}_1 \times_U \mathcal{Y} \times_U \mathcal{Y}),
\]
which is exact because \( F \) is a sheaf.
Lemma Clearly \( f^* : \text{PSh}(Y_{et}) \to \text{PSh}(X_{et}) \) is exact, so \( f^* : \text{Sh}(Y_{et}) \to \text{Sh}(X_{et}) \) is left exact. In general it is NOT right exact (consider \( X = \text{alg } \mathbb{F}/k = \mathbb{F}, \text{and } X \to \text{Spec } k \)).

Example If \( i : \tilde{x} \to X \) is a geometric point and \( M \) is an abelian group, then the \( M^\tilde{x} \) defined earlier is just \( i^*(M) \), where we identify \( \text{Sh}(X_{et}) \to \text{Ab} \)
\[ F \to F(i^*). \]

**Behavior on Stalks**

\[ \pi : Y \to X \]

Suppose \( \tilde{y} \) is a geometric point of \( Y \) mapping to a geometric point \( \tilde{x} \) of \( X \). Then we have an induced map \( (\pi^* F)_x \to F_{\tilde{y}} \).

In general, this will be neither injective nor surjective. But sometimes we can say something.

**Prop (a)** Let \( \pi : V \to X \) be an open immersion. Then
\[ (\pi^* F)_x = \begin{cases} F_{\tilde{X}} & \text{if } x \in V \\ 0 & \text{otherwise} \end{cases} \]

(b) Let \( \pi : Z \to X \) be a closed immersion. Then
\[ (\pi^* F)_x = \begin{cases} F_{\tilde{X}} & \text{if } x \in Z \\ 0 & \text{otherwise} \end{cases} \]

(c) Let \( \pi : Y \to X \) be finite map. Then
\[ (\pi^* F)_x = \bigoplus_{y \to x} F_{\tilde{Y}} \]

where \( d \) is the separable degree of \( k(y) \) over \( k(x) \).

**Proof** (a) If \( x \in V \), then by base changing to \( V \) it is clear that the étale neighborhoods of \( \tilde{x} \) in \( X \) coming from étale neighborhoods of \( \tilde{x} \) in \( V \) are cofinal for the direct limit defining \( F_{\tilde{x}} \), and (a) follows.
b) A similar argument shows \((\pi \circ F)'_x = 0\) for \(x \notin \mathcal{Z}\).

For \(x \in \mathcal{Z}\) we need to show that every étale neighborhood \(\overline{\varphi} : \overline{U} \to \mathcal{Z}\) extends to an étale morphism \(U \to X\). In terms of rings, this means for an étale morphism \(\overline{\varphi} : \overline{A} \to \overline{B}\) with \(\overline{A} = A_{\mathfrak{m}_x}\), there needs to be an étale lift \(\tilde{\varphi} : \tilde{A} \to \tilde{B}\). Because all étale morphisms are locally standard étale morphisms, we can assume

\[
\overline{B} = \left( \frac{\overline{A[T]}}{f(T)} \right)_B
\]

where \(f(T)\) is a monic polynomial such that \(f'(T)\) is invertible in \(B\) (where \(B\) is some localizing element). But then just choose \(f(T) \in A[T]\) lifting \(f(T)\) and set \(B = \left( \frac{A[T]}{f(T)} \right)_B\) for an appropriate \(B\).

c) Let \(\tilde{x}\) be a geometric point of \(X\) with image \(x\). Then let \(\tilde{X} = \text{Spec} \overline{O_{x,\tilde{x}}}\). Recall \(\overline{O_{x,\tilde{x}}}\) is a (strict) Henselian local ring. Let \(\tilde{Y} = \tilde{X} \times \tilde{X} \text{Spec} \overline{O_{x,\tilde{x}}}\). Then \(\tilde{Y}\) is finite over \(\tilde{X}\), and as \(\overline{O_{x,\tilde{x}}}\) is Henselian this implies that the connected components of \(\tilde{Y}\) are indexed by \(\pi^{-1}(x)\), and

\[
\tilde{Y} = \bigsqcup_{x' \in \pi^{-1}(x)} \text{Spec} \left( \frac{O_{x,x'}^h}{\mathfrak{m}_{x'}^n} \right)
\]

The claim follows.

Car If \(\pi\) is closed or finite, \(\pi^{\circ}\) is exact.

**Pf** Use previous prop + check exactness on geometric stalks.

Prop For any morphisms \(Z \xrightarrow{f} Y \xrightarrow{\pi'} X\),

\[
(\pi' \circ \pi)^* = \pi'^* \circ \pi^*
\]

**Pf** Obvious.
Inverse Images

Recall in the setting of presheaves on topological spaces, for \( \pi : Y \to X \) we define
\[
(\pi^*_p \cdot F)(U) = \lim_{U \to \pi(U)} F(V).
\]
This amounts to limits over diagrams
\[
\begin{array}{ccc}
U & \longrightarrow & V \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\]
and \( G \to \phi \) with \( U \to V \) any morphism.

So do exactly the same thing for the étale setting, replacing \( \to \) by \( \to \) étale.

**Def.** Let \( \pi : Y \to X \) be a morphism. Define the functor \( \pi^*_p \cdot \cdot : \text{PSh}(Y) \to \text{PSh}(X) \) by
\[
(\pi^*_p \cdot F)(U) = \lim_{U \to V} F(U)
\]

with obvious restriction maps.

It is easy to see that if \( P \) is a presheaf on \( Y \), \( Q \) a presheaf on \( X \), then there are natural identifications of:

1) Morphisms \( \pi^*_p \cdot P \to Q \)

2) Families of maps \( P(U) \to Q(V) \) indexed by \( \text{Set} \)

3) Morphisms \( P \to \pi^* \cdot Q \).

Thus:

\[\text{Prop.} \quad \pi^*_p \cdot \text{ is left adjoint to } \pi^* \cdot \]

\[\text{Cor.} \quad (\pi^* \circ \pi^*_p)^* = \pi^*_p \circ \pi^* \cdot \]

\[\text{Pf.} \quad \text{Both are left adjoint to } (\pi^* \circ \pi^*_p)^* \text{ and } \pi^*_p \circ \pi^* .\]
**Def:** $\pi^*: \text{Sh}(X^\text{et}) \rightarrow \text{Sh}(Y^\text{et})$ is $\alpha \circ \pi^*$.  

Thus $\pi^*$ is left adjoint to $\pi^*$.  

Let $\pi: X \rightarrow Y$ be a morphism, and $i: \tilde{Y} \rightarrow Y$ be a geometric point, and let $\tilde{x}$ be the geometric point of $X$ defined by $\pi \circ i$. Then note $(\pi^*F)\tilde{y} = (i^*\pi^*F)(\tilde{y}) = F_{\tilde{x}}$.

Thus:

**Prop:** $\pi^*$ is exact. $\pi^*$ sends injectives $\rightarrow$ injectives.  

Car Exactness is checked on stalks, using the previous comment. Preservation of injectives follows from being right adjoint to an exact functor.

**Car:** $\text{Sh}(X^\text{et})$ has enough injectives.  

For each $x \in X$, choose a geometric point $\tilde{x}$ mapping to $x$. Let $F$ be a sheaf on $X^\text{et}$.  

Embed $F_{\tilde{x}} \hookrightarrow \pi(\tilde{x})$, $\pi(\tilde{x})$ some injective abelian group. Identifying $\text{Sh}(X^\text{et}) \cong \text{Ab}$, this is an injective sheaf on $X^\text{et}$, so $\pi^*(\pi(\tilde{x}))$ is injective sheaf on $X^\text{et}$ so the product $\pi^* \pi^*(\pi(\tilde{x}))$ is also.  

Then the embedding $F \hookrightarrow \pi^* \pi^*(\pi(\tilde{x}))$ does the trick.

**Extension by Zero**  

Let $j: U \hookrightarrow X$ be an open immersion.  

**Define the functor**  

$j_!\text{ps.}: P\text{Sh}(U^\text{et}) \rightarrow P\text{Sh}(X^\text{et})$ by  

$(j_!\text{ps.}, F)(\varphi: V^\text{et} \rightarrow X) = \begin{cases} 0 & \text{if } \varphi(U) \subset U \\ F(V) & \text{otherwise} \end{cases}$

Clearly $j_!\text{ps.}$ is left adjoint to $j^!\text{ps.}$, the restriction.

For sheaves, define $j_! = \alpha \circ j_!\text{ps.}$ where $\alpha$ is sheafification. Then $j_!$ is left adjoint to $j^*$.
We have \((\widetilde{j_! F})_x \cong \sum F_x\) if \(x \in U\).

Thus:

Prop \(j_!\) is exact, so its right adjoint \(j^*\) preserves injectives.

Let \(Z\) be the complement of \(U\) in \(X\), and denote the associated closed immersion by \(i: Z \hookrightarrow X\).

Prop For any sheaf \(F\), the natural sequence

\[ 0 \rightarrow j_! j^* F \rightarrow F \rightarrow i_* i^* F \rightarrow 0 \]

defined by the adjunctions is exact.

Check exactness on stalks. 2 cases:

For \(x \in U\), the sequence reads

\[ 0 \rightarrow \mathcal{F}_x \rightarrow \mathcal{F}_x \rightarrow 0 \rightarrow 0 \]

and for \(x \not\in U\), the sequence reads

\[ 0 \rightarrow 0 \rightarrow \mathcal{F}_x \rightarrow \mathcal{F}_x \rightarrow 0. \]

Remark Can do a similar construction for \(j^* U \rightarrow X\) any étale map, not nec an open immersion. We define for \(\psi: V \rightarrow X\) étale,

\[ (j^!, \psi^* F)(V) = \bigoplus \mathcal{F}(\psi(V)) \]

where the direct sum is over morphisms \(\alpha: V \rightarrow U\) s.t.

\[ V \xrightarrow{\alpha} U \xrightarrow{\psi} X \]

We define \(j^! F\) to be the sheafification. Again, \(j_!\) is left adjoint to \(j^*\) and \(i^*\) exact, so \(j^*\) always preserves injectives.
Sheaves on $X = U \cup \bar{Z}$

We conclude with a description of sheaves on $X = U \cup \bar{Z}$ in terms of sheaves on $U, \bar{Z}$ and some compatibility. See Theorem 3.10 in Milne's book for full details.

Let $i : \bar{Z} \to X \leftarrow U, j$ such that $X = i(\bar{Z}) \cup j(U)$, and let $\mathcal{F}$ be a sheaf on $X_{et}$. Then we get sheaves $i^* \mathcal{F}, j^* \mathcal{F}$ on $\bar{Z}_{et}, U_{et}$.

By adjunction, we get a morphism $\mathcal{F} \to j_! j^* \mathcal{F}$.

Applying $i^*$, we get a morphism $\mathcal{F} \to i^! i_! i^* \mathcal{F}$.

Now, let $\text{Tr}(X, U, \bar{Z})$ be the category whose objects are triples $(\mathcal{F}_1, \mathcal{F}_2, \phi)$ where $\mathcal{F}_1, \mathcal{F}_2$ are sheaves on $\bar{Z}_{et}, U_{et}$ respectively, and $\phi : \mathcal{F}_1 \to i^! j_! j^* \mathcal{F}_2$ is a morphism.

Morphisms in $\text{Tr}(X, U, \bar{Z})$ are defined in the obvious way, i.e., pairs $(\phi_1, \phi_2)$ of morphisms of the sheaves on $\bar{Z}_{et}, U_{et}$ compatible with the compatibility morphism.

Thus the functor $\text{Sh}(X_{et}) \to \text{Tr}(X, U, \bar{Z})$

$\mathcal{F} \mapsto (i^! \mathcal{F}, j_! j^* \mathcal{F}, \phi)$

is an equivalence. An essential inverse is given by

$(\mathcal{F}_1, \mathcal{F}_2, \phi) \mapsto i^! \mathcal{F}_1 \xrightarrow{\phi} i^! \mathcal{F}_2$.

Corollary: For any closed immersion $i : Z \to X$,

$i^*$ defines an equivalence between $\text{Sh}(Z_{et})$ and sheaves on $X_{et}$ supported on $Z$.

(m.e.s. $j^* \mathcal{F} = 0$)