DERIVED CATEGORIES, DERIVED FUNCTORS, AND D-MODULE INVERSE AND DIRECT IMAGE

SETH SHELLEY-ABRAHAMSON

Abstract. These are notes for a graduate seminar on D-Modules. Our main reference is *Methods of Homological Algebra* by Gelfand and Manin, and Pavel Etingof's notes from a Fall 2013 MIT course on D-modules.

Contents

1. Derived Categories

1.1. **Basic Motivation.** One starting point for introducing derived categories is the observation that many naturally occurring functors between abelian categories, for example taking global sections of a sheaf, tensor and hom constructions, invariants, etc., are only left or right exact. As exact functors have significantly better properties and are easier to work with, it is desirable to produce from an abelian category $\mathcal{A}$ another category $D(\mathcal{A})$, its *derived category*, where all these functors become exact, in a sense. Motivating the definition that we will discuss tonight, consider the case $\mathcal{A} = A - mod$ is the category of modules over some ring A. Then objects $M \in \mathcal{A}$ admit projective resolutions $P^\bullet \to M \to 0$. Furthermore, we have the following important fact:

Date: April 12, 2015.

Theorem 1. *Let $\mathcal{A}$ be an abelian category, and let $X, Y \in \mathrm{Ob}(\mathcal{A})$. Let $P^\bullet \to X$ and $Q^\bullet \to Y$ by projective resolutions and let $f : X \to Y$ be a morphism. Then there exists a morphism of resolutions $R(f) : P^\bullet \to Q^\bullet$ extending f in the sense that the diagram*

$$\begin{array}{ccc} P^0 & \longrightarrow & X \\ {\scriptstyle R(f)^0}\downarrow & & \downarrow{\scriptstyle f} \\ Q^0 & \longrightarrow & Y \end{array}$$

commutes. Furthermore, any two such $R(f)$ are homotopic maps of complexes.

So, we see that when projective resolutions (or similarly injective resolutions) exist, they are unique up to canonical isomorphism in the homotopy category of chain complexes over $\mathcal{A}$, and that maps between objects are naturally the same as homotopy classes of maps between their projective resolutions. Thus, the basic idea is to work with complexes rather than honest objects, and to replace objects by their projective/injective resolutions. In order to make this work, we will want to identify complexes that are *quasi-isomorphic*, i.e. whose cohomology can be identified through some map of complexes. Now let's turn to details.

1.2. **Derived Categories: Definition via Universal Property.** Let $\mathcal{A}$ be an abelian category. Let $Kom(\mathcal{A})$ be the category of complexes over $\mathcal{A}$. It is again an abelian category, with images, kernels, quotients, etc. all realized in the obvious ways. Let $Kom^+(\mathcal{A})$ denote the full subcategory of complexes $K^\bullet$ such that $K^i = 0$ for $i << 0$, and let $Kom^-(\mathcal{A})$ be the full subcategory of complexes $K^\bullet$ such that $K^i = 0$ for $i >> 0$, and let $Kom^b(\mathcal{A})$ be the full subcategory of complexes $K^\bullet$ such that $K^i = 0$ for both $i >> 0$ and $i << 0$. Let $Kom_0(\mathcal{A}) = \prod_{i \in \mathbb{Z}} \mathcal{A}$ be the full subcategory of complexes with zero differential. Then taking cohomology induces a functor

$$H : Kom(\mathcal{A}) \to Kom_0(\mathcal{A}).$$

We say that a map of complexes $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if $H(f)$ is an isomorphism.

Theorem 2. *Let $\mathcal{A}$ be an abelian category. Then there exists a pair $(D(\mathcal{A}), Q)$ with $D(\mathcal{A})$ a category and $Q : Kom(\mathcal{A}) \to D(\mathcal{A})$ a functor such that*

(1) if $f : K^\bullet \to L^\bullet$ is a quasi-isomorphism then $Q(f)$ is an isomorphism

(2) if $(\tilde{D}(\mathcal{A}), \tilde{Q})$ is any other such pair, then there exists a unique functor $G : D(\mathcal{A}) \to \tilde{D}(\mathcal{A})$ such that the diagram

$$\begin{array}{ccc} Kom(\mathcal{A}) & \xrightarrow{Q} & D(\mathcal{A}) \\ & {\scriptstyle \tilde{Q}}\searrow & \downarrow{\scriptstyle G} \\ & & \tilde{D}(\mathcal{A}) \end{array}$$

commutes.

$D(\mathcal{A})$ is called the *derived category* of $\mathcal{A}$, and the functor Q should be viewed as a sort of localization. We can give a nearly identical construction of categories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ with functors from $Kom^+(\mathcal{A})$, $Kom^-(\mathcal{A})$, and $Kom^b(\mathcal{A})$ universal for inverting quasi-isomorphisms in those categories.

Before proceeding to properties of $D(\mathcal{A})$, it is worth giving a brief, although perhaps not overly enlightening, proof of the above theorem. In fact, we can prove equally easily the following generalization:

Theorem 3. *Let $\mathcal{A}$ be an abelian category, and let S be any class of morphisms in $\mathcal{A}$. Then there exists a universal functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ transforming elements of S into isomorphisms, in the sense of the previous theorem.*

Proof. Define $\mathrm{Ob}(\mathcal{A}[S^{-1}]) = \mathrm{Ob}(\mathcal{A})$ and Q to be the identity on objects. To define the morphisms, first introduce a variable x_s for every morphism $s : X \to Y$ in S. We then construct an oriented graph Γ in which the vertices are the objects of $\mathcal{A}$ and edges are either morphisms $f : X \to Y$, which is oriented from X to Y, or a variable x_s, which is oriented oppositely from s. Then we define a morphism in $\mathcal{A}[S^{-1}]$ from X to Y to be an equivalence class of finite paths in this graph, starting at X and ending at Y, under the equivalence relation generated by replacing a concatenation of genuine morphisms with their composition and identifying sx_s and $x_s s$ with the identity morphism on the appropriate object. Composition of morphisms is given by concatenation of representing paths. It is easy to see, modulo some set theoretic considerations, e.g. that $\mathcal{A}$ is equivalent to a small category, that this gives a category, and that there is a natural morphism $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ sending elements of S to isomorphisms and which is clearly universal for this property. $\square$

1.3. **Shift Functors.** Let $i \in \mathbb{Z}$. Then we have an exact functor $\bullet[i] : Kom(\mathcal{A}) \to Kom(\mathcal{A})$ defined by $K^\bullet[i]^j = K^{i+j}$ and $d_{K[i]} = (-1)^i d_K$. This is called the *shift by i* functor. Clearly $\bullet[i] \circ \bullet[j] = \bullet[i+j]$ and $\bullet[0] = id_{Kom(\mathcal{A})}$, so these all give autoequivalences of $Kom(\mathcal{A})$. Furthermore they clearly restrict to autoequivalences of the subcategories Kom^+, Kom^-, Kom^b. It is clear that shifts commute with cohomology and send quasi-isomoprhisms to quasi-isomorphisms. Thus they lift uniquely to autoequivalences of the derived category $D(\mathcal{A})$, and similarly for its bounded variants.

1.4. **Cone and Cylinder.** In this section we introduce two fundamental constructions related to morphisms of complexes and prove some of their basic properties. The main purpose of these constructions is to allow us to define a notion of *distinguished triangle*, which will replace the role of short exact sequences in derived categories, as in general derived categories are not abelian. They also provide convenient tools for various proofs in homological algebra.

Let $\mathcal{A}$ be an abelian category and let $f : K^\bullet \to L^\bullet$ be a morphism in $Kom(\mathcal{A})$. Then we can form the complex $C(f)$, called the *cone of f*, by

$$C(f) = K[1]^\bullet \oplus L^\bullet$$

(at the level of objects) with differential given by

$$d(k^{i+1}, l^i) = (-d_K(k^{i+1}), f(k^{i+1}) + d_L(l^i)).$$

This is conveniently expressed by the matrix

$$d_{C(f)} = \begin{pmatrix} d_{K[1]} & 0 \\ f[1] & d_L \end{pmatrix}.$$

That $d_{C(f)}^2 = 0$ is the statement that f is a morphism of complexes.

Let us take a moment to see where the name comes from and a connection with algebraic topology. Let $f : X \to Y$ be a simplicial map of simplicial complexes. Then we can form the associated *mapping cone* MC_f. This is another simplicial complex. Its i-simplices are in bijection with the union of the i-simplices of Y and the $(i-1)$-simplices of X, except we have one extra 0-simplex. This complex is built from the complex Y by attaching the cone on each $(i-1)$-simplex of X, which is then an i-simplex, by identifying the cone point with the extra 0-simplex and the opposite face with the image in Y under f of the original $(i-1)$-simplex of X. Then one sees immediately that if we take the cone associated to the map of simplicial chain complexes

$$f : C^\bullet_\Delta(X) \to C^\bullet_\Delta(Y)$$

(put homological degrees negative, unlike usually in topology, so that things are consistent with previous definitions) we obtain the *reduced* simplicial chain complex for MC_f. So the complex $C(f)$ computes the reduced homology of the associated mapping cone, hence the name of the construction.

Exercise 4. *Let $f : K^\bullet \to L^\bullet$ be an inclusion of a subcomplex. Then $C(f)$ is quasi-isomorphic to $L^\bullet/K^\bullet$.*

Proof. This basically follows immediately from the definitions. Note that we have a natural map $C(f) \to L^\bullet$ induced by projection $K[1]^\bullet \oplus L^\bullet \to L^\bullet/K^\bullet$. For $(k^{i+1}, l^i) \in C(f)^i$ we see $d(k^{i+1}, l^i) = (-dk^{i+1}, k^{i+1} + dl^i)$. This is 0 if and only if $-k^{i+1} = dl^i$ and $dk^{i+1} = 0$. The latter condition follows from the first, and so we see projection to $L^\bullet$ is injective on cycles and hence on boundaries. We obtain identifications

$$Z^i(C(f)) = \{l^i \in L^i : dl^i \in K^{i+1}\}$$
$$B^i(C(f)) = B^i(L^\bullet) + K^i.$$

At the same time, we have

$$Z^i(L^\bullet/K^\bullet) = \{l^i + K^i \in L^i/K^i : dl^i \in K^{i+1}\}$$
$$B^i(L^\bullet/K^\bullet) = \{dl^{i-1} + K^i : l^{i-1} \in L^{i-1}\}.$$

The statement then follows immediately. □

In this case of the inclusion of a subcomplex, we also have the associated short exact sequence of chain complexes

$$0 \to K^\bullet \to L^\bullet \to L^\bullet/K^\bullet \to 0$$

and hence the associated long exact sequence in cohomology

$$\cdots \to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(L^\bullet/K^\bullet) \to H^{i+1}(K^\bullet) \cdots$$

where the connecting homomorphism $H^i(L^\bullet/K^\bullet) \to H^{i+1}(K^\bullet)$ is given by lifting a cycle to $L^\bullet$ and applying the differential (see proof of previous exercise to see why this works). Via the previous exercise, we have an identification $H^i(C(f)) \cong H^i(L^\bullet/K^\bullet)$ so we also have a long exact sequence in cohomology

$$\cdots \to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(C(f)) \to H^{i+1}(K^\bullet) \cdots$$

Considering the connecting homomorphism of the original long exact sequence and the explicit identification of cohomologies, we see that in this second long exact sequence this connecting homomorphism is induced by the natural projection

$$C(f) \to K[1].$$

While $f : K^\bullet \to L^\bullet$ will not in general be an inclusion of complexes, we nonetheless still have a long exact sequence in cohomology of the type we just derived:

Theorem 5. *Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. Then we have the associated long exact sequence in cohomology*

$$\cdots \to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(C(f)) \to H^{i+1}(K^\bullet) \cdots$$

where the connecting homomorphism is again induced by the natural map $C(f) \to K[1]$.

Before proving this theorem, it is useful to introduce another complex depending on f, the *cylinder of f*, denoted $Cyl(f)$. We have, at the level of objects,

$$Cyl(f) = K^\bullet \oplus K[1]^\bullet \oplus L^\bullet$$

with differential given by

$$d_{Cyl(f)}(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L l^i).$$

Again, this is conveniently expressed via the matrix

$$d_{Cyl(f)} = \begin{pmatrix} d_K & -id_K[1] & 0 \\ 0 & d_{K[1]} & 0 \\ 0 & f[1] & d_L \end{pmatrix}.$$

That this is a complex again follows from the fact that f is a morphism of complexes. Here is the key interaction between f and its cone and cylinder:

Theorem 6. *Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. Then we have the following commutative diagram in $Kom(\mathcal{A})$ with exact rows*

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & L^\bullet & \xrightarrow{\overline{\pi}} & C(f) & \xrightarrow{\delta} & K[1]^\bullet \longrightarrow 0 \\
 & & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle=} & & \\
0 & \longrightarrow & K^\bullet & \xrightarrow{\overline{f}} & Cyl(f) & \xrightarrow{\pi} & C(f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle\beta} & & & & \\
 & & K^\bullet & \xrightarrow{f} & L^\bullet & & & &
\end{array}$$

This diagram is functorial in f. α and β are quasi-isomorphisms; more specifically, $\beta\alpha = id_L$ and $\alpha\beta$ is homotopic to $id_{Cyl(f)}$. In particular, we see that $Cyl(f)$ is canonically isomorphic to $L^\bullet$ in the derived category.

Proof. The proof is by easy diagram chasing, and can be found in *Methods of Homological Algebra*, III.3 Lemma 3. However, we do need to say what the maps are in the diagram above. They are (note we can use elements to describe them, thanks to the Freyd-Mitchell theorem):

$$\overline{\pi}(l^i) = (0, l^i)$$
$$\delta(k^{i+1}, l^i) = k^{i+1}$$
$$\alpha(l^i) = (0, 0, l^i)$$
$$\overline{f}(k^i) = (k^i, 0, 0)$$
$$\pi(k^i, k^{i+1}, l^i) = (k^{i+1}, l^i)$$
$$\beta(k^i, k^{i+1}, l^i) = f(k^i) + l^i.$$

The only map that is not the obvious map is the map β. □

Proof. (Of Theorem 5) We have the long exact sequence

$$0 \to K^\bullet \to Cyl(f) \to C(f) \to 0$$

which gives rise to a corresponding long exact sequence in cohomology. We have the canonical quasi-isomorphism between $Cyl(f)$ and $L^\bullet$, so we get a long exact sequence with the cohomology objects we want, and the only remaining question is the connecting homomorphism. The statement about that again follows by diagram chasing. □

1.5. **Distinguished Triangles and Exact Functors.** In the last section we associated to any morphism of complexes $f : K^\bullet \to L^\bullet$ complexes $C(f)$ and $Cyl(f)$. Recall $Cyl(f)$ was canonically isomorphic to $L^\bullet$ in $D(\mathcal{A})$ and that these objects fit into a natural short exact sequence

$$0 \to K^\bullet \to Cyl(f) \to C(f) \to 0$$

which gave rise to a long exact sequence in cohomology. The maps on cohomology in this sequence can be seen as arising from the *triangle*

$$K^\bullet \to Cyl(f) \to C(f) \to K[1]$$

where the maps are as in Theorem 5 in the preceding section (all maps are the obvious maps). Out of deference to the importance of this sequence, we make the following definition:

Definition 7. *A* triangle *in the derived category $D(\mathcal{A})$ is a diagram of the form*

$$K^\bullet \to L^\bullet \to M^\bullet \to K[1]^\bullet.$$

A morphism of triangles is the obvious sort of commutative diagram. A triangle is said to be a distinguished triangle *if it is isomorphic to one of the form*

$$K^\bullet \to Cyl(f) \to C(f) \to K[1].$$

Distinguished triangles play the role of short exact sequences in derived categories. For example, as we have seen, to every distinguished triangle there is a long exact sequence in cohomology induced by the morphisms in the triangle. As some basic examples, we see that for any morphism of complexes $f : K^\bullet \to L^\bullet$ the triangle

$$K^\bullet \to L^\bullet \to C(f) \to K[1]^\bullet$$

is distinguished - this follows from the canonical isomorphism between $Cyl(f)$ and $L^\bullet$ in the derived category. As another example, this canonical isomorphism, in combination with Exercise 4, shows that for any *inclusion* of complexes $K^\bullet \to L^\bullet$ the triangle

$$K^\bullet \to L^\bullet \to L^\bullet/K^\bullet \to K[1]^\bullet$$

is distinguished. So we see in particular that every short exact sequence of complexes gives rise to a distinguished triangle - in this sense distinguished triangles are generalizations of short exact sequences of complexes.

The need for introducing the notion of distinguished triangle arises in part from the problem that the cone of a morphism $f : X \to Y$ of objects in the derived category is not canonically defined.

Definition 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A functor $F : D(\mathcal{A}) \to D(\mathcal{B})$ is called* exact *if it commutes with the shift functors and maps distinguished triangles to distinguished triangles.*

1.6. **The Homotopy Category and Ore Conditions.** We showed at the very beginning that the derived category of an abelian category exists using an unsatisfying argument about general localization of categories. This argument gives essentially no information about what morphisms in $D(\mathcal{A})$ look like, and in fact we can't even see that $D(\mathcal{A})$ is an additive category, which turns out to be true. We now give an alternative construction of the derived category which is a bit more involved and much more involved to prove, but which makes morphisms much easier to understand and makes additivity clear.

Definition 9. *Let $\mathcal{A}$ be an abelian category. The* homotopy category *of $\mathcal{A}$ is the category $K(\mathcal{A})$ with objects that are complexes of objects in $\mathcal{A}$ and morphisms that are homotopy classes of morphisms of complexes.*

This definition makes sense because if f g then fh gh and hf hg for any morphism of complexes h when these expressions make sense. Also, cohomology is still defined at the level of the homotopy category, and so we still have a notion of quasi-isomorphism.

Lemma 10. *The localization functor $Q : Kom(\mathcal{A}) \to D(\mathcal{A})$ factors through the homotopy category $K(\mathcal{A})$, i.e. $f\ g \implies Q(f) = Q(g)$.*

Proof. Let $f, g : K^\bullet \to L^\bullet$ be homotopic maps of complexes. Then we can write $f = g + dh + hd$. Define a morphism

$$c(h) : C(f) \to C(g)$$

by

$$c(h)(k^{i+1}, l^i) = (k^{i+1}, l^i + h(k^{i+1})).$$

A simple calculation shows that $c(h)$ is a morphism of complexes. Similarly, we define a morphism of complexes

$$cyl(h) : Cyl(f) \to Cyl(g)$$

by the formula

$$cyl(h)(k^i, k^{i+1}, l^i) = (k^i, k^{i+1}, l^i + h(k^{i+1})).$$

$c(h)$ provides a morphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^\bullet & \longrightarrow & C(f) & \longrightarrow & K[1]^\bullet & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle c(h)}\downarrow & & {\scriptstyle =}\downarrow & & \\ 0 & \longrightarrow & L^\bullet & \longrightarrow & C(g) & \longrightarrow & K[1]^\bullet & \longrightarrow & 0 \end{array}$$

By the 5-lemma, this implies $c(h)$ is a quasi-isomorphism.

Similarly, we have the morphism of short exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^\bullet & \longrightarrow & Cyl(f) & \longrightarrow & C(f) & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle cyl(h)}\downarrow & & {\scriptstyle c(h)}\downarrow & & \\ 0 & \longrightarrow & K^\bullet & \longrightarrow & Cyl(g) & \longrightarrow & C(f) & \longrightarrow & 0 \end{array}$$

and by the 5-lemma again we conclude $cyl(h)$ is a quasi-isomorphism.

Next, we construct the NONCOMMUTATIVE diagram

$$\begin{array}{ccc} & \overset{f}{\nearrow} & L^\bullet \\ & & \downarrow{\scriptstyle \alpha_f} \\ K^\bullet & \xrightarrow{\overline{f}} & Cyl(f) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle cyl(h)} \\ K^\bullet & \xrightarrow{\overline{g}} & Cyl(g) \\ & \underset{g}{\searrow} & \downarrow{\scriptstyle \beta_g} \\ & & L^\bullet \end{array}$$

All maps are as in the earlier subsection on the cone and cylinder. The middle square and bottom triangle DO COMMUTE, but the top triangle in general DOES NOT COMMUTE IN $Kom(\mathcal{A})$. However, we do have

$$f = \beta_f \circ \overline{f}$$

in $Kom(\mathcal{A})$. But we've seen before that in the derived category $Q(\beta_f)$ and $Q(\alpha_f)$ are mutually inverse, from which we get

$$Q(\overline{f}) = Q(\alpha) \circ Q(f)$$

in the derived category. SO IN THE DERIVED CATEGORY THE ABOVE DIAGRAM IS COMMUTATIVE. By direct computation, we have

$$\beta_g \circ cyl(f) \circ \alpha_f = id_L.$$

It follows then that

$$Q(f) = Q(g)$$

as needed. □

From this we get the following the key result:

Theorem 11. *The localization of $K(\mathcal{A})$ by the class of quasi-isomorphisms is canonically equivalent to the derived category of $\mathcal{A}$. The same holds for the various bounded versions of these categories.*

Proof. Let $\overline{S}$ denote the class of quasi-isomorphisms in $K(\mathcal{A})$. Then by the universal property of localization of categories we have the following commutative diagram

$$\begin{array}{ccc} Kom(\mathcal{A}) & \xrightarrow{Q} & D(\mathcal{A}) \\ \downarrow & \nearrow & \exists!\downarrow \uparrow\exists! \\ K(\mathcal{A}) & \xrightarrow{Q} & K(\mathcal{A})[\overline{S}^{-1}] \end{array}$$

where the diagonal arrow is the factorization of the previous lemma. It follows that the composition of the vertical dotted arrows in either order is the identity functor, so that $D(\mathcal{A})$ and $K(\mathcal{A})[\overline{S}^{-1}]$ are canonically equivalent, as needed. □

The motivation to consider the derived category as this localization of the homotopy category is that this latter localization is much better behaved. This is reminiscent of a situation in noncommutative algebra - if we have a ring R and a multiplicative subset $S \subset R$, one can always form the noncommutative localization $R[S^{-1}]$, defined by the obvious universal property and constructed, for example, by adjoining new variables to invert elements of S and adding appropriate relations, like in our first proof of the existence of $D(\mathcal{A})$. However, when S satisfies the *Ore condition*:

(*Ore*) For any $a \in R$ and $s \in S$, there exist $b, c \in R$ and $t, u \in S$ such that $as = tb$ and $sa = cu$.

This condition allows us to just invert elements of S on one side, so that the localization looks like it does in the noncommutative case. Note in particular that if $f \in R$ is *locally ad-nilpotent* then the set $\{f^n\}_{n\geq 0}$ satisfies the Ore condition. We can consider the noncommutative localization $R[f^{-1}]$, OR we can view R as a $\mathbb{Z}[x]$-module where x acts by left (or right) multiplication by f, and can consider the localization $R[f^{-1}]$ in the usual sense of commutative algebra. The Ore condition guarantees that these localizations are the same. This, in a sense, is what allows us to sheafify the algebra of differential operators on a smooth variety, as by Grothendieck's definition functions are locally nilpotent in the algebra of differential operators.

The same game works for categories. We need some condition to replace the Ore conditions. This is the notion of a *localizing* class of morphisms:

Definition 12. *Let $\mathcal{C}$ be a category, and let S be a class of morphisms in $\mathcal{C}$. We say S is* localizing *if:*

(1) S contains all identity morphisms and all compositions of morphisms in S when defined.

(2) For any $f \in Mor(\mathcal{C}), s \in S$ such that the following diagrams make sense, there exist $g \in Mor(\mathcal{C}), t \in S$ such that the following commute

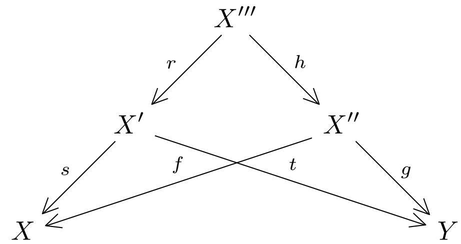
$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array} \qquad\qquad \begin{array}{ccc} W & \xleftarrow{g} & Z \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle s} \\ X & \xleftarrow{f} & Y \end{array}$$

(3) For morphisms $f, g : X \to Y$, the existence of $s \in S$ with $sf = sg$ is equivalent to the existence of $t \in S$ with $ft = gt$.

In this setting, the localization $\mathcal{C}[S^{-1}]$ becomes much better, in the sense that we only need "right fractions." In other words, we have the following construction. Again, let the objects be just the objects of $\mathcal{C}$. Now define morphisms $X \to Y$ to be equivalence classes of diagrams (s, f) of the following form (called "roofs"):

$$\begin{array}{ccccc} & & X' & & \\ & {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} & \\ X & & & & Y \end{array}$$

(for $s \in S$ and f and morphism) under the equivalence relation $(s, f)\ (t, g)$ if and only if there exists a third roof forming the following commutative diagram:



The composition of equivalence classes of roofs is defined using concatenation and the first square in (2) in the definition of a localizing class. This is a category, call it $\widetilde{\mathcal{C}[S^{-1}]}$, and we have a natural functor

$$Q : \mathcal{C} \to \widetilde{\mathcal{C}[S^{-1}]}$$

inverting all morphisms in S. Moreover, we have

Theorem 13. *The unique induced functor*

$$\mathcal{C}[S^{-1}] \to \widetilde{\mathcal{C}[S^{-1}]}$$

is an equivalence of categories.

This is relevant for us because:

Theorem 14. *Let $\mathcal{A}$ be an abelian category. The class S of quasi-isomorphisms in the homotopy category $K(\mathcal{A})$ is localizing.*

This gives us a more manageable description of morphisms in $D(\mathcal{A})$ as roofs of homotopy classes of maps of complexes. Note that we may as well have taken "left roofs" rather than "right roofs" as in the above construction.

In particular, since now all localizing can be done on one side and since we can find "common denominators." More concretely, suppose (s, f) and (s', f') are roofs from X to Y in the homotopy category $K(\mathcal{A})$, say with "chimneys" Z and Z'. Then we have two quasi-isomorphisms $s : Z \to X$ and $s' : Z' \to X$. Using the fact that the class of quasi-isomorphisms in the homotopy category is localizing, we can find morphisms $r : U \to Z$ (with $r \in S$ a quasi-isomorphism) and $r' : U \to Z'$ for some object U such that $sr = r's'$. It follows that r' is also a quasi-isomorphism. Then

$$(s, f)\ (sr, fr) \qquad (s', f')\ (sr, f'r').$$

Then we can define addition of morphisms by

$$(s, f) + (s', f') = (sr, fr + f'r').$$

It is not hard to check that we get:

Proposition 15. *With the above addition of morphisms $D(\mathcal{A})$ is an additive category.*

1.7. Canonical Equivalence Between $\mathcal{A}$ and H^0-complexes in $D(\mathcal{A})$.

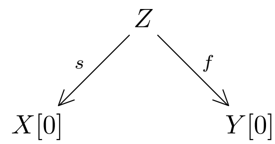
Definition 16. *Let $\mathcal{A}$ be an abelian category and $D(\mathcal{A})$ its derived category. An object $X \in D(\mathcal{A})$ is called an* H^0-complex *if $H^i(X) = 0$ for $i \neq 0$.*

We can now see formally how our original category appears inside its derived category:

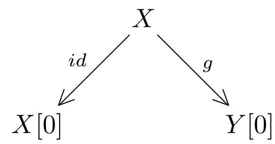
Theorem 17. *The composition $\mathcal{A} \to Kom(\mathcal{A}) \to D(\mathcal{A})$ where the first map sends objects to the associated 0-complexes in $Kom(\mathcal{A})$ induces an equivalence between $\mathcal{A}$ and the full subcategory of H^0-complexes.*

Proof. The inverse functor is given by zeroth cohomology H^0. It is clear that the composition on $\mathcal{A}$ is naturally isomorphic to the identity functor, so we must consider the reverse composition. We will use the realization of $D(\mathcal{A})$ as the localization of the homotopy category as in the previous section. We note that the functor in question lands in the full subcategory of 0-complexes. So first consider a morphism $X[0] \to Y[0]$

in $D(\mathcal{A})$ of 0-complexes. By our construction, this is represented by a roof



where s is a quasi-isomorphism. In particular, Z is an H^0-complex. Under our composition of functors this is sent to the roof



where $g = H^0(f) \circ H^0(s)^{-1}$. To show that our functor is fully faithful we need to show that the two roofs above are equivalent.

Consider the truncation V of Z defined by

$$V^i = Z^i \text{ for } i < 0,$$
$$V^0 = \ker(d_Z : Z^0 \to Z^1),$$

and

$$V^i = 0 \text{ for } i > 0.$$

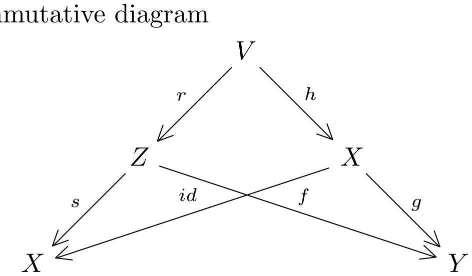
Then the natural inclusion map $r : V \to Z$ is a quasi-isomorphism (it is always an isomorphism on nonpositive cohomology, and in particular is a quasi-isomorphism because Z and V are H^0-complexes). We also have the morphism

$$h : V \to X[0]$$

which in degree 0 is given by the composition

$$V^0 \to Z^0 \to X.$$

These maps then fit into the commutative diagram

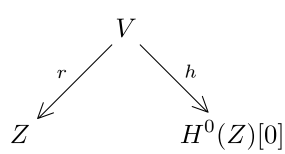


and in particular we see that the two roofs above are equivalent and the functor

$$\mathcal{A} \to D(\mathcal{A})$$

is an equivalence of $\mathcal{A}$ with its image, the full subcategory of 0-complexes in $D(\mathcal{A})$.

To finish the proof, we need to check that every H^0-complex is isomorphic to a 0-complex in $D(\mathcal{A})$. But we basically already know this, because for Z any H^0-complex and with the same notations as above, the roof



gives an isomorphism of Z with $H^0(Z)[0]$ because both r and h are quasi-isomorphisms. □

1.8. **Ext as Hom in the Derived Category.** Derived categories also give a convenient framework to define Ext objects. We have seen that the original category $\mathcal{A}$ embeds as H^0-complexes (or 0-complexes) in $D(\mathcal{A})$. By applying shift functors, this works just as well for any fixed homological degree. By embedding different objects of $\mathcal{A}$ in different in different homological degrees and studying their interaction, we discover additional structure. In particular, we have the following definition:

Definition 18. *Let $X, Y \in \mathcal{A}$. Then we define*

$$Ext^i_{\mathcal{A}}(X,Y) := Hom_{D(\mathcal{A})}(X, Y[i]).$$

When $\mathcal{A}$ has enough injectives or projectives, this definition of Ext coincides with the usual definition. This definition has the additional pleasant property of making obvious the existence of the multiplication maps

$$Ext^i_{\mathcal{A}}(X,Y) \times Ext^j_{\mathcal{A}}(Y,Z) \to Ext^{i+j}_{\mathcal{A}}(X,Z).$$

In this setting these maps are just given by composition of morphisms!

1.9. Splitting of the Exact Sequence $0 \to L^\bullet \to C(f) \to K[1]^\bullet \to 0$.

Proposition 19. *Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. Then the natural short exact sequence*

$$0 \to L^\bullet \to C(f) \to K[1]^\bullet \to 0$$

of complexes is split if and only if f is homotopic to 0. Furthermore, the naive map on objects

$$K[1]^\bullet \oplus L^\bullet \to C(f)$$

is a splitting of $C(f)$ as a complex if and only if $f = 0$.

Proof. This is an exercise in definitions. Let us consider the naive map

$$K[1]^\bullet \oplus L^\bullet \to C(f).$$

This is a map of complexes if and only if

$$(-d_K k^{i+1}, dl^i) =: d_{K[1]^\bullet \oplus L^\bullet}(k^{i+1}, l^i) = d_{C(f)}(k^{i+1}, l^i) := (d_K k^{i+1}, f(k^{i+1}) + dl^i)$$

for all $k^{i+1} \in K^i$ and $l^i \in L^i$. Clearly this is true if and only if $f = 0$.

Now let us consider when the exact sequence is split. To give a splitting is to give a map of complexes $h : K[1]^\bullet \to L^\bullet$ such that the resulting map

$$(id, h) : K[1]^\bullet \to C(f)$$

is a map of complexes. This holds exactly when

$$(-d_K k^{i+1}, h(-d_K k^{i+1})) = (id, h)(d(k^{i+1})) = d((id, h)k^{i+1}) = (-d_K k^{i+1}, f(k^{i+1}) + d_L h k^{i+1})$$

for all k^{i+1} and l^i. This just says $f = -(hd_K + d_L h)$, i.e. $-h$ gives a homotopy f 0, as needed. □

2. Derived Functors

2.1. Basic Motivation.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Then we see immediately that F induces functors

$$Kom^*(F) : Kom^*(\mathcal{A}) \to Kom^*(\mathcal{B})$$

between the associated categories of complexes and functors

$$K^*(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B})$$

between the associated homotopy categories, where * denotes any of our boundedness conditions. But in general these functors DO NOT send quasi-isomorphisms to quasi-isomorphisms, and in particular do not, at least in this naive way, give rise to corresponding functors at the level of the derived categories. An exception to this is the following:

Proposition 20. *Suppose F is exact. Then F in particular commutes with taking cohomology, and so sends quasi-isomorphisms to quasi-isomorphisms. Thus F gives rise to a functors*

$$D^*(F) : D(\mathcal{A}) \to D(\mathcal{B})$$

between the derived category with any boundedness condition. The functor $D^*(F)$ is exact.

The purpose of this section is to make some similar construction work for functors that are only left or right exact. The basic idea is that we cannot apply the functor term-wise to just any complex - we must first take a quasi-isomorphic complex whose objects are "adapted" to our functor, and then apply term-wise. We now make this precise.

2.2. **Universal Property of Derived Functors.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. The following is a definition of derived functors via a universal property:

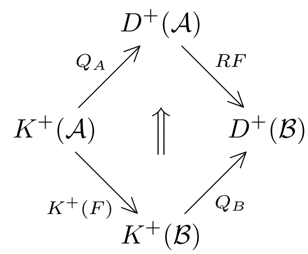
Definition 21. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. The* derived functor *of F, if it exists, is a pair (RF, ϵ_F) of an exact functor*

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

and a morphism of functors

$$\epsilon_F : Q_B \circ K^+(F) \to RF \circ Q_A$$

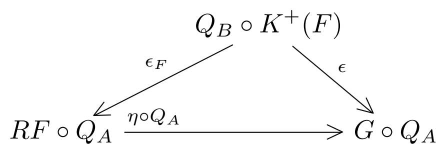
as in the following diagram



satisfying the following universality property: for any other such pair (G, ϵ), these exists a unique morphism of functors

$$\eta : RF \to G$$

such that the diagram



commutes.

It follows immediately that if RF exists it is unique up to a unique isomorphism of functors.

Similarly, we have a definition for the left derived functor of a right exact functor, where now some arrows are reversed and the derived functor is between the bounded above derived categories:

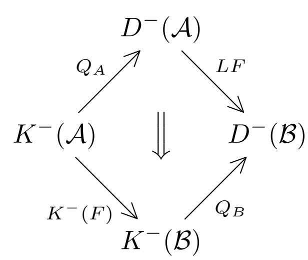
Definition 22. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. The* derived functor *of F, if it exists, is a pair (LF, ϵ_F) of an exact functor*

$$RF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$$

and a morphism of functors

$$\epsilon_F : LF \circ Q_A \to Q_B \circ K^-(F)$$

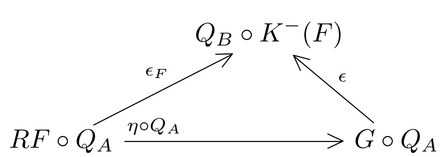
as in the following diagram



satisfying the following universality property: for any other such pair (G, ϵ), these exists a unique morphism of functors

$$\eta : G \to LF$$

such that the diagram



commutes.

Again, the functor LF is clearly unique up to a unique isomorphism of functors if it exists. It is nontrivial to show that the derived functor exists. However, we will see that under mild hypotheses it does indeed exist. For this we need the notion of an adapted class of objects.

2.3. **Adapted Classes of Objects.**

Definition 23. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. We say that a class of objects*

$$\mathcal{R} \subset Ob(\mathcal{A})$$

is adapted to F *if*

(1) $\mathcal{R}$ is closed under finite direct sums
(2) F maps any acyclic complex in $Kom^+(\mathcal{R})$ to an acyclic complex
(3) Any object in $\mathcal{A}$ is a subobject of an object in $\mathcal{R}$.

Similarly, for a right exact functor we have:

Definition 24. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. We say that a class of objects*

$$\mathcal{R} \subset Ob(\mathcal{A})$$

is adapted to F *if*

(1) $\mathcal{R}$ is closed under finite direct sums
(2) F maps any acyclic complex in $Kom^-(\mathcal{R})$ to an acyclic complex
(3) Any object in $\mathcal{A}$ is a quotient of an object in $\mathcal{R}$.

Here is an important and commonly applicable example:

Example 25.

Suppose $\mathcal{A}$ has enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Then the class of injective objects in $\mathcal{A}$ is adapted to F. The only thing to check is that left exact functors send exact complexes $0 \to I^\bullet$ to exact complexes, and this follows by an easy induction using the fact that a complement of an injective object in an injective object is an injective object.

Similarly, for right exact functors one may always take the class of projective objects, when there are enough projectives.

The following fact about adapted-to-F classes will be crucial to the construction of derived functors:

Proposition 26. *Let $\mathcal{R}$ be a class of objects adapted to a left exact functor $F : \mathcal{A} \to \mathcal{B}$, and let $S_\mathcal{R}$ be the class of quasi-isomorphisms in the homotopy category $K^+(\mathcal{R})$. Then $S_\mathcal{R}$ is a localizing class of morphisms in $K^+(\mathcal{R})$ and the canonical functor*

$$K^+(\mathcal{R})[S_\mathcal{R}^{-1}] \to D^+(\mathcal{A})$$

is an equivalence of categories.

A similar statement holds for right exact functors F, with the bounded below categories replaced by the bounded above categories.

Proof. Proof can be found in [GM] III.6 Proposition 4. □

2.4. **Construction of the Derived Functor.**

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, and let $\mathcal{R}$ be a class of objects adapted to F. Then we saw that the class $S_\mathcal{R}$ of quasi-isomorphisms in $K^+(\mathcal{R})$ is localizing and that the canonical functor

$$K^+(\mathcal{R})[S_\mathcal{R}^{-1}] \to D^+(\mathcal{A})$$

is an equivalence of categories. Our strategy for defining the derived functor RF of F will be to define RF on $K^+(\mathcal{R})[S_\mathcal{R}^{-1}]$ and then to transport the definition to $D^+(\mathcal{A})$ via an inverse equivalence.

Consider the functor

$$RF : K^+(\mathcal{R}) \to K^+(\mathcal{B})$$

be defined by term by term application of F - this is the restriction of the functor $K^+(F)$ to the full subcategory of complexes of objects in our adapted class R. Here is the significance:

Proposition 27. *RF sends quasi-isomorphisms to quasi-isomorphisms, so we have a unique induced functor*

$$RF : K^+(\mathcal{R})[S_\mathcal{R}^{-1}] \to D^+(\mathcal{B}).$$

Proof. Let
$$f : K^\bullet \to L^\bullet$$
be any morphism of complexes. We then have the associated distinguished triangle
$$K^\bullet \to L^\bullet \to C(f) \to K[1]^\bullet.$$
It follows from the corresponding long exact sequence in cohomology that f is a quasi-isomorphism if and only if $C(f)$ is acyclic. But then if f is a quasi-isomorphism and $K^\bullet, L^\bullet$ are complexes of objects from $\mathcal{R}$, then $C(f)$ is an acyclic complex with objects from $\mathcal{R}$, and hence as $\mathcal{R}$ is adapted to F we have that $F(C(f))$ is acyclic as well. But there is a canonical isomorphism
$$F(C(f)) \cong C(F(f))$$
of complexes, so $C(F(f))$ is acyclic too, and hence $F(f)$ is a quasi-isomorphism, as needed. □

So let
$$\Phi : D^+(\mathcal{A}) \to K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$$
be a quasi-inverse to the canonical equivalence
$$K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^+(\mathcal{A}).$$
We can now define RF at the level of the bounded below derived categories
$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$
by the composition
$$D^+(\mathcal{A}) \xrightarrow{\Phi} K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \xrightarrow{RF} D^+(\mathcal{B})$$
where the second map RF is (by abuse of notation) the functor constructed in the previous proposition.

We note that there are some non-canonical choices in this definition of RF, namely we needed to choose an adapted class $\mathcal{R}$ and the quasi-inverse Φ. However, these choices turn out to be unimportant, as we see in the following theorem:

Theorem 28. *The functor*
$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$
satisfies the universal property of the right derived functor for R (so in particular there exists such a functor morphism ϵ). A similar construction works for right exact functors. If F is exact, then both RF and LF are obtained by term by term application of F to complexes.

Proof. Proof can be found in sections III6.8-III6.11 of [GM]. □

2.5. **First Properties and Examples.** Under certain circumstances, the derived functor of the composition of left (or right) exact functors is simply described:

Proposition 29. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Suppose there exist classes of objects $\mathcal{R}_{\mathcal{A}}$ in $\mathcal{A}$ and $\mathcal{R}_{\mathcal{B}}$ in $\mathcal{B}$ which are adapted to F and G, respectively. Then $\mathcal{R}_{\mathcal{A}}$ is adapted to $G \circ F$, all three right derived functors RF, RG, and $R(G \circ F)$ exist, and the natural morphism of functors*
$$R(G \circ F) \to RG \circ RF$$
given by the universality property of $R(G \circ F)$ is an isomorphism.

A similar statement holds for right exact functors.

The following proposition relates the cohomology of the derived functor of a complex with the cohomology of the derived functors of the cohomology of that complex:

Proposition 30. *Let F be a left exact functor between abelian categories. Then*
$$H^n(RF(K^\bullet))$$
is a subquotient of
$$\bigoplus_{p+q=n} R^p F(H^q(K^\bullet)).$$

Example 31.

Let A be a commutative ring, M a fixed A-module, and consider the right exact functor

$$F : A - \text{mod} \to A - \text{mod} \qquad F(N) = M \otimes_A N.$$

Then we can take either the class of projective objects or the class of flat objects as an adapted class for F, so the left derived functor

$$LF = \otimes_A^L N : D^-(A - \text{mod}) \to D^-(A - \text{mod})$$

exists, and we write

$$LF(M) = M \otimes_A^L N.$$

It can be shown that this functor does not depend on which argument we derive. Its cohomology gives the familiar Tor functors.

Example 32.

For any abelian category $\mathcal{A}$ we have the Hom functor

$$\text{Hom} : \mathcal{A} \times \mathcal{A}^{op} \to \text{Ab}$$

to the abelian category of abelian groups. Fixing either slot gives a left exact functor, one defined on $\mathcal{A}$ (fixing the first slot) and the other defined on $\mathcal{A}^{op}$ (fixing the second slot). When fixing the first slot can take as an adapted class the injective objects of $\mathcal{A}$ when $\mathcal{A}$ has enough injectives, and when fixing the second slot we can take as an adapted class the injective objects of $\mathcal{A}^{op}$, i.e. the *projective* objects of $\mathcal{A}$, when $\mathcal{A}$ has enough projectives. When $\mathcal{A}$ has both enough injectives and enough projectives, it can be shown, like for $\otimes_A^L$, that it doesn't matter which argument we fix and which we derive.

3. D-Module Inverse and Direct Image

3.1. **Reminder About Categories of D-Modules.** We now turn to D-modules. Let X be a smooth variety, and let $\mathcal{D}_X$ denote the sheaf of differential operators on X. Following Hotta, we define a $\mathcal{D}_X$-module M to be a sheaf for the quasicoherent sheaf of algebras $\mathcal{D}_X$, and we say such a module is *coherent* if:

(1) M is locally finitely generated

(2) For any open subset U, and locally finitely generated submodule of $M|_U$ is locally finitely presented.

This is recast as in our class via the following proposition, also found in Hotta's book (Proposition 1.4.9):

Proposition 33. *(1) $\mathcal{D}_X$ is coherent over itself.*

(2) A $\mathcal{D}_X$-module M is coherent if and only if it is quasicoherent over $\mathcal{O}$ and locally finitely generated over $\mathcal{D}_X$.

We denote the category of such left coherent $\mathcal{D}_X$-modules by $\mathcal{M}^l(\mathcal{D}_X)$, and similarly for the category of coherent right $\mathcal{D}_X$-modules.

We will also want to know Hotta's Corollary 1.4.20:

Proposition 34. *Let $M \in \mathcal{M}^l(\mathcal{D}_X)$ be a coherent $\mathcal{D}_X$-module. Then M has a resolution*

$$P^\bullet \to M \to 0$$

by locally free $\mathcal{D}_X$-modules of finite rank. Furthermore, there exists a finite resolution

$$0 \to P_m \to \cdots \to P_0 \to M \to 0$$

by locally projective $\mathcal{D}_X$-modules of finite rank.

If we take a $\mathcal{D}_X$-module M which is quasicoherent over $\mathcal{O}$ but perhaps not coherent over $\mathcal{D}_X$, we still have resolutions as above, where we drop the condition that the P_i be of finite rank.

We also have the following general fact about sheaves of rings on topological spaces (Hotta's Lemma 1.5.2):

Proposition 35. *Let R be a sheaf of rings on a topological space X, and let $Mod(R)$ denote the category of sheaves of R-modules. Then $Mod(R)$ has enough injective objects and enough flat objects.*

In particular, one can always talk about $\text{RHom}(M, \cdot)$ and $\otimes_R^L$. Note that $\text{Mod}(R)$ may not have enough projectives - indeed this happens already for $\text{Mod}(\mathcal{O}_X)$ for non-affine varieties X.

We also have (Hotta's Proposition 1.5.6 and Theorem 1.5.7):

Proposition 36. *Let X be a smooth variety. Let $D^b(\mathcal{D}_X)$ denote the bounded derived category of all $\mathcal{D}_X$-modules, let $D^b_{qcoh}(\mathcal{D}_X)$ and $D^b_{coh}(\mathcal{D}_X)$ denote the full subcategories of those objects with $\mathcal{O}$-quasicoherent and $\mathcal{D}$-coherent cohomology, respectively. Let also $\mathcal{M}_{qcoh}(\mathcal{D}_X)$ and $\mathcal{M}_{coh}(\mathcal{D}_X)$ denote the categories of $\mathcal{O}$-quasicoherent and $\mathcal{D}$-coherent $\mathcal{D}_X$-modules, respectively.*

(1) Any object of $D^b(\mathcal{D}_X)$ is represented by a bounded complex of flat $\mathcal{D}_X$-modules. Any object of $D^b_{qcoh}(\mathcal{D}_X)$ is represented by a bounded complex of locally projective $\mathcal{D}_X$-modules belonging to $\mathcal{M}_{qcoh}(\mathcal{D}_X)$.

(2) The natural functors

$$D^b(\mathcal{M}_{qcoh}(\mathcal{D}_X)) \to D^b_{qcoh}(\mathcal{D}_X)$$

and

$$D^b(\mathcal{M}_{coh}(\mathcal{D}_X)) \to D^b_{coh}(\mathcal{D})$$

are equivalences of categories.

3.2. **Bounded Derived Category of Holonomic D-Modules.**

Theorem 37. *The natural embedding*

$$\mathcal{M}^l_{hol}(\mathcal{D}_X) \to \mathcal{M}^l(\mathcal{D}_X)$$

of the category of holonomic left $\mathcal{D}_X$-modules into the category of all quasicoherent $\mathcal{D}_X$-modules induces an equivalence

$$D^b(\mathcal{M}_{hol}(\mathcal{D}_X)) \to D^b_{hol}(\mathcal{D}_X)$$

where the later category is the full subcategory of $D^b(\mathcal{M}(\mathcal{D}_X))$ consisting of objects with holonomic cohomology.

In particular, if M, N are holonomic $\mathcal{D}_X$-modules, we have

$$Ext^i_{\mathcal{M}(\mathcal{D}_X)}(M,N) \cong Ext^i_{\mathcal{M}_{hol}(\mathcal{D}_X)}(M,N)$$

Proof. We omit the proof of the first statement. The second statement follows from the first because we can compute Ext as a Hom between shifted objects in the derived category. □

We remark that this is a fact about D-modules, and not a general fact about Ext's. More specifically, suppose $\mathcal{A} \subset \mathcal{B}$ is a Serre subcategory. Then for $M, N \in \mathcal{A}$ one can consider both $\mathrm{Ext}^i_{\mathcal{A}}(M,N)$ and $\mathrm{Ext}^i_{\mathcal{B}}(M,N)$. In general these are NOT the same. A counterexample can be seen by taking $\mathcal{A}$ to be the category of finite dimensional sl_2-modules and $\mathcal{B}$ the category of all sl_2-modules. Then

$$\mathrm{Ext}^3_{\mathcal{A}}(\mathbb{C},\mathbb{C}) = 0$$

while

$$\mathrm{Ext}^3_{\mathcal{B}}(\mathbb{C},\mathbb{C}) = \mathbb{C}.$$

3.3. **Upgraded Duality Functor.** Again let X be a smooth variety, and $\mathcal{D}_X$ its sheaf of differential operators. Denote by $D^l(\mathcal{D}_X)$ the bounded derived category of the category of quasicoherent left $\mathcal{D}_X$-modules, and by $D^r(\mathcal{D}_X)$ the bounded derived category of the category of quasicoherent right $\mathcal{D}_X$-modules.

In this setting, we will define a (contravariant) duality functor

$$\mathbb{D} : D^l(\mathcal{D}_X) \to D^r(\mathcal{D}_X)$$

by

$$M \mapsto R\underline{\mathrm{Hom}}(M, \mathcal{D}_X)$$

for any coherent left $\mathcal{D}_X$-module. Here

$$\underline{\mathrm{Hom}}(\cdot, \mathcal{D}_X) : \mathcal{M}^l(\mathcal{D}_X) \to \mathcal{M}^r(\mathcal{D}_X)$$

is the left exact functor defined by

$$\Gamma(U, \underline{\mathrm{Hom}}(M, \mathcal{D}_X)) = \mathrm{Hom}_{\mathcal{D}_U}(\Gamma(U, M), \mathcal{D}_U)$$

on affine open sets U. The right D_X-action is given through right multiplication on $\mathcal{D}_X$. Note that this is actually just the sheaf hom, over $\mathcal{D}_X$, from M to $\mathcal{D}_X$ (affine spaces are D-affine). That $\underline{\mathrm{Hom}}$ sends coherent $\mathcal{D}_X$-modules to coherent $\mathcal{D}_X$-modules follows from the fact that $\mathcal{D}(U)$ is Noetherian for any open affine U. We have seen that quasicoherent $\mathcal{D}_X$-modules have finite resolutions by locally projective $\mathcal{D}_X$-modules and that coherent $\mathcal{D}_X$-modules have finite resolutions by locally projective $\mathcal{D}_X$-modules of finite rank, and therefore we can take these classes of adapted objects to define the right derived functor $R\underline{\mathrm{Hom}}$ on either the bounded or bounded-above derived category of quasicoherent or coherent $\mathcal{D}_X$-modules. It is my belief that the embedding of the category of coherent $\mathcal{D}_X$-modules into the category of quasicoherent $\mathcal{D}_X$-modules should induce a fully faithful embedding at the level of the derived categories, although I haven't checked this (update: I just realized this follows from the fact that both of these categories embed as full subcategories of $D^b(\mathcal{D}_X)$ with the corresponding cohomology conditions, so the embedding I just mentioned is fully faithful). At any rate, the duality functor is defined at the level of the bounded derived categories of either coherent or quasi-coherent $\mathcal{D}_X$-modules.

We would like to realize the duality functor on the category $D^l(\mathcal{D}_X)$ itself. This is just degree-bookkeeping and side-changing. Recall that we have the side-changing equivalence

$$\cdot \otimes_{\mathcal{O}_X} K_X^{-1} : \mathcal{M}^r(\mathcal{D}_X) \to \mathcal{M}^l(\mathcal{D}_X).$$

This is an exact equivalence of abelian categories (on either the categories of coherent or quasicoherent $\mathcal{D}_X$-modules), so induces equivalences at the level of the derived categories, with any boundedness assumptions. So we can twist by this equivalence to get a functor just on $D^l(\mathcal{D}_X)$. Recall also that for holonomic $\mathcal{D}_X$-modules we've already constructed a duality functor

$$\mathbb{D} : \mathcal{M}^l_{hol}(\mathcal{D}_X) \to \mathcal{M}^r_{hol}(\mathcal{D}_X)$$

for holonomic left $\mathcal{D}_X$-modules on affine X given by

$$M \mapsto \mathrm{Ext}^{\dim X}_{\mathcal{D}_X}(M, \mathcal{D}_X).$$

We would like that our upgraded duality functor "restrict" to this one we've already built, where we view honest $\mathcal{D}_X$-modules as 0-complexes in the derived category. But the above formula starts with a holonomic left $\mathcal{D}_X$-module in cohomological degree 0 and gives a holonomic right $\mathcal{D}_X$-module in cohomological degree $\dim X$. Now that we are dealing with complexes we'd like to keep careful track of these degrees. So, we define the (contravariant) duality functor

$$\mathbb{D} : D^l(\mathcal{D}_X) \to D^l(\mathcal{D}_X)$$

by the formula

$$\mathbb{D}(M) = R\underline{\mathrm{Hom}}(M, \mathcal{D}_X)[n] \otimes K_X^{-1}$$

where $n = \dim X$.

We will not expect $\mathbb{D}^2 = \mathrm{Id}$ in the non-coherent case - after all $\mathbb{D}$ is built from $\underline{\mathrm{Hom}}$ and this has no chance of being related to anything involutive for not-locally-finitely-generated modules. However, we can now see that this is a genuine duality functor for bounded derived categories of *coherent* $\mathcal{D}_X$-modules:

Theorem 38. *(1) $\mathbb{D}^2 = Id$ on $D^b(\mathcal{M}^l_{coh}(\mathcal{D}_X))$.*

(2) For $M, N \in D^b(\mathcal{M}^l_{coh}(\mathcal{D}_X))$ we have

$$Hom(M, \mathbb{D}(N)) \cong Hom(N, \mathbb{D}(M)).$$

Proof. (1)We take as our adapted-to-$\underline{\mathrm{Hom}}$ class the class $\mathcal{R}$ of locally projective $\mathcal{D}_X$-modules of locally finite rank. By the contravariance of $\underline{\mathrm{Hom}}$, we see that the shift in the definition of $\mathbb{D}$ does not enter this picture, and we need to establish an isomorphism of functors

$$\mathrm{Id} \cong \mathbb{D}^2 = R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(\bullet, \mathcal{D}_X)[n] \otimes K_X^{-1}, \mathcal{D}_X)[n] \otimes K_X^{-1}$$

Note that the equivalence $\bullet \otimes K_X^{-1}$ sends our adapted class to itself (well, strictly speaking sends locally projective $\mathcal{D}_X$-modules of locally finite rank to the same type of right modules), and similarly for $\underline{\mathrm{Hom}}$. For $R^\bullet$ a bounded complex of objects from $\mathcal{R}$, we have

$$\underline{\mathrm{Hom}}(R^\bullet, \mathcal{D}_X)[n] \cong \underline{\mathrm{Hom}}(R^\bullet[-n], \mathcal{D}_X)$$

naturally by contravariance, from which one sees easily that

$$\mathbb{D}^2 \cong R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(\bullet, \mathcal{D}_X) \otimes K_X^{-1}, \mathcal{D}_X) \otimes K_X^{-1}$$

naturally, i.e. we can ignore the shifts. But also

$$\begin{aligned} &\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(R^\bullet, \mathcal{D}_X) \otimes K_X^{-1}, \mathcal{D}_X) \otimes K_X^{-1} \\ &\cong \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(R^\bullet, \mathcal{D}_X), K_X \otimes \mathcal{D}_X) \otimes K_X^{-1} \\ &\cong \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(R^\bullet, \mathcal{D}_X), \mathcal{D}_X) \end{aligned}$$

naturally in $R^\bullet$, i.e. we can ignore the twists (recall the derived functor can be constructed by applying the functor term by term to complexes of adapted objects in the homotopy category, and then transferring to the derived category by an equivalence, so this all makes sense and is functorial), so we see actually

$$\mathbb{D}^2 \cong R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(\bullet, \mathcal{D}_X), \mathcal{D}_X).$$

This is a composition of derived functors, and they have compatible adapted classes of objects, so we obtain

$$\mathbb{D}^2 \cong R(\underline{\mathrm{Hom}}(\bullet, \mathcal{D}_X) \circ \underline{\mathrm{Hom}}(\bullet, \mathcal{D}_X)).$$

But it is clear that we have an isomorphism of functors

$$\underline{\mathrm{Hom}}(\bullet, \mathcal{D}_X) \circ \underline{\mathrm{Hom}}(\bullet, \mathcal{D}_X) \cong \mathrm{Id}$$

on objects in $\mathcal{R}$ (certainly works for projective modules of finite rank - that we can take finite rank is where coherence mattered from the start), so we get

$$\mathbb{D}^2 \cong R(\mathrm{Id}) \cong \mathrm{Id}$$

as needed.

(2) For the second statement, we note that it follows from the natural isomorphism

$$Hom_{\mathcal{D}_X}(M, Hom_{\mathcal{D}_X}(N, \mathcal{D}_X)) \cong Hom_{\mathcal{D}_X-\mathrm{bimod}}(M \boxtimes N, \mathcal{D}_X)$$

and the fact that that we can exchange the roles of M and N in the second object above. Naturality means this same statement will hold when we take M and N to be complexes of adapted objects, so is upgraded to the derived functors. □

We also have:

Theorem 39.

$$RHom(M, N) = RHom(\mathbb{D}(N), \mathbb{D}(M))$$

for any $M, N \in D^b(\mathcal{M}^l_{coh}(\mathcal{D}_X))$ (note this is derived usual Hom, not derived sheaf $\underline{Hom}$).

Proof. Actually its totally not clear to me why this statement even makes sense. Does the category of coherent $\mathcal{D}_X$-modules have enough projectives or injectives? □

3.4. **Inverse Image.** Let $\pi : X \to Y$ be a morphism of smooth irreducible varieties. Let π^{-1} denote the sheaf theoretic inverse image. Recall that we have the $\mathcal{D}_X - \pi^{-1}(\mathcal{D}_Y)$-bimodule

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\mathcal{D}_Y.$$

We define the inverse image functor

$$\pi^! : D^b(\mathcal{M}^l(\mathcal{D}_Y)) \to D^b(\mathcal{M}^l(\mathcal{D}_X))$$

(note we are NOT taking coherent/locally finitely generated D-modules here) by

$$\pi^!(M^\bullet) = \mathcal{D}_{X\to Y} \otimes^L_{\pi^{-1}\mathcal{D}_Y} \pi^{-1}M^\bullet[\dim X - \dim Y].$$

Note that at the level of $\mathcal{O}$-modules we have also

$$\begin{aligned} &D_{X\to Y} \otimes^L_{\pi^{-1}\mathcal{D}_Y} \pi^{-1}M^\bullet \\ &= (\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\mathcal{D}_Y) \otimes^L_{\pi^{-1}\mathcal{D}_Y} M^\bullet \\ &= (\mathcal{O}_X \otimes^L_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\mathcal{D}_Y) \otimes^L_{\pi^{-1}\mathcal{D}_Y} M^\bullet \\ &= \mathcal{O}_X \otimes^L_{\pi^{-1}\mathcal{O}_Y} M^\bullet \end{aligned}$$

because $\mathcal{D}_Y$ is locally free over $\mathcal{O}_Y$ and hence any module which is flat for $\pi^{-1}\mathcal{D}_Y$ is flat for $\pi^{-1}\mathcal{O}_Y$.

We remark (Proposition 1.5.9 from Hotta) that the functor

$$\mathcal{O}_X \otimes^L_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}(\bullet) : D^-(\mathcal{O}_Y) \to D^-(\mathcal{O}_X)$$

sends

$$D^-_{qcoh}(\mathcal{O}_Y) \to D^-_{qcoh}(\mathcal{O}_X).$$

It follows from the computation above that $\pi^!$ sends $\mathcal{O}$-quasicoherent D-modules to $\mathcal{O}$-quasicoherent D-modules. We remark that $\pi^!$ will NOT in general send complexes with $\mathcal{D}$-coherent cohomology to complexes with $\mathcal{D}$-coherent cohomology, as one sees easily by considering $\mathcal{D}_{X\to Y}$ for $X \to Y$ a closed embedding with $\dim X < \dim Y$.

We remark that identifying 0-complexes with their 0^{th}-cohomology, we have

$$\mathcal{D}_{X\to Y} = \pi^! \mathcal{D}_Y[\dim Y - \dim X].$$

3.5. **Direct Image.** Let $\pi : X \to Y$ again be a morphism of smooth algebraic varieties. Let $M \in D^b(\mathcal{D}_X)$. We define the D-module direct image π_* as the composition of two functors between bounded derived categories. The first is

$$\bullet \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y} : D^b(\mathcal{D}_X) \to D^b(\pi^{-1}\mathcal{D}_Y)$$

(here we are using right modules, as we'd expect). To compute this one takes a $\mathcal{D}_X$-flat resolution in the category of $\mathcal{D}_X$-modules, which exists as we mentioned earlier. The second is the functor

$$R\pi_{*sheaf} : D^b(\pi^{-1}\mathcal{D}_Y) \to D^b(\mathcal{D}_Y).$$

We note that the sheaf theoretic pushforward π_{*sheaf} is a left exact functor, and the category of sheaves of (right) $\pi^{-1}\mathcal{D}_Y$-modules has enough injectives, as we mentioned earlier, so this all makes sense. In summary, we have

$$\pi_* : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$$

defined by

$$\pi_* M^\bullet = R\pi_{*sheaf}(M \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

We note that since at one point we have to take injective resolutions of $\pi^{-1}\mathcal{D}_Y$-modules, our resolutions involved will be enormous and not quasicoherent. So it is totally unclear why π_* should sent $\mathcal{O}$-quasicoherent D-modules to $\mathcal{O}$-quasicoherent D-modules. We will see this later.

We close with an example. Let Y be a point, X a smooth variety, and consider the morphism $\pi : X \to Y$. As we've seen before, $\mathcal{D}_{X\to Y} = \mathcal{O}_X$, and so we have for a $\mathcal{D}_X$-module M its direct image

$$\pi^* M = R\Gamma(M \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)$$

is the hypercohomology of the de Rham complex of M. In particular, in the complex case, by a famous result of Grothendieck for $M = \mathcal{O}_X$ (well really taking $M = K_X$ for us since we want to be dealing with right $\mathcal{D}_X$-modules) this gives the usual cohomology of the variety X, seen as a complex manifold, living between homological degrees $-\dim X$ and $\dim X$.

4. References

[GM] Gelfand and Manin, *Methods of Homological Algebra.*