Summary of D-Modules

\[ D = \text{sheaf of differential operators on a variety } X, \]
\[ D\text{-Module} = \text{sheaf of modules for } D. \]

Assume all varieties are smooth over \( \mathbb{C} \).

Grothendieck's Definition of Differential Operators

\[ \text{Def Let } D_0(M,N) = \mathcal{O}_X, \text{ and for } n \geq 0: \]
\[ D_n(M,N) = \{ L \in \text{Hom}_X(M,N) : [L,f] \in D_{n-1}(M,N) \text{ \forall } f \in \mathcal{O}_X \}. \]
\[ \text{Set } D(M,N) = \bigcup_{n \geq 0} D_n(M,N). \]

Note \( D_i(M,N) \subseteq D_j(M,N) \) for \( i \leq j \), and \( D_*(M,N) \) is naturally an \( \mathcal{O}_X \)-bimodule.

When \( M = N \), it is easy to see that \( D(M) = D(M,M) \) has the structure of an algebra, and in fact
\[ D_n(M) \cdot D_m(M) \subseteq D_{n+m}(M), \]
so \( D(M) \) is a filtered algebra, and we can consider the associated graded algebra
\[ \text{gr } D(M) = \bigoplus_{n \geq 0} D_n(M)/D_{n+1}(M). \]
For \( M = \mathcal{O}_X \)

Note \( [D_n(R), D_m(R)] \subseteq D_{n+m-1}(R) \),
so \( \text{gr } D(R) \) is a graded commutative algebra.

Important special case: \( M = N = \mathcal{O}_X \). \( D(R) \) is called the algebra of differential operators on \( R \).

Example \( D(\mathbb{C}[[x_1, \ldots, x_n]]) = \text{usual alg of poly diff ops } \mathbb{C}[[x_1, \ldots, x_n]] \).

Example \( D^*(\mathbb{R}) = \mathbb{R} \). For \( L \in D^*_1(\mathbb{R}) \), easy to see
\[ L = \text{ad } L : \mathbb{R} \rightarrow \mathbb{R} \text{ is a derivation.} \]
and since \( Lf = Lf_1 = \left[ L, f_1 \right] + fL(1) = \left( \text{ad}L + L(1) \right)f \), we see \( D_1(R) = R \oplus \text{Der}(R) \).

Note \( D(M, N) \) is a filtered right \( D(M) \)-module and a filtered left \( D(N) \)-module.

Sheaves of Differential Operators:

Let \( X \) be smooth, affine \( G \)-variety.

We want to attach a quasi-coherent sheaf of associative algebras \( D_X \) to \( X \) such that for open affine \( U \subseteq X \), we have \( D_X(U) \equiv D(U) \).

2 Ways to Think of This:

1. Ore conditions. Note for \( f \in \mathcal{O}(X) \), \( f \) is locally ad-nilpotent in \( D(X) \). This means the quasi-coherent sheaf \( D_X \) on \( X \) associated to the left \( \mathcal{O}(X) \)-module \( D(X) \) naturally has the structure of a sheaf of associative algebras. This is a bit unsatisfying, not clear what \( D_X(U) \) is.

2. Define \( D_{\mathcal{O}_U} x_n^i = D_n(\mathcal{O}(X), \mathcal{O}(U)) \) for open affine \( U \subseteq X \).

Lemma: For \( L = D_{\mathcal{O}_U} x_n^i \) and \( f \in M_{n+1}^I \), we have \( Lf = 0 \), and conversely.

So, we get a natural map
\[
D_{\mathcal{O}_U} x_n^i \rightarrow (m_I^I/m_{n+1}^I)^* = (S^n T^*_U)^* = S^n T^*_U.
\]
This is clearly regular in \( \mathbb{Z} \), so get a map
\[
o : D_{\mathcal{O}_U} x_n^i \rightarrow \Gamma(U, S^n T^*_U)
\]
By Lemma again, this is \( 0 \) on \( D_{\mathcal{O}_U} x_{n-1}^i \), so get
\[
o : \text{gr}_n D_{\mathcal{O}_U} x_n^i \rightarrow \Gamma(U, S^n T^*_U).
\]
Easy to see \( \sigma : \text{gr } D_{x,u} \rightarrow \Gamma(U, STU) = \sigma(T^*U) \).

This is the symbol map.

Also easy to see for \( \mathcal{L} \in \text{gr } D_{x,u}, \mathcal{L}' \in \text{gr } D_{x,u} \), we have \( \sigma(\mathcal{L} \mathcal{L}') = \sigma(\mathcal{L}) \sigma(\mathcal{L}') \).

The Lemma implies \( \sigma \) is injective, so only need surjective. For this, it suffices to treat case \( X = \mathcal{U}, \mathcal{L} \in \text{STU } \) is quasi-coherent, \( X \) affine, and \( \sigma \) map of \( \mathcal{O}(U) \)-modules. For this, note \( \Gamma(I(U, STU)) \) is generated in degree 1/degree 0, and \( \sigma \) is iso in degrees \( \leq 1 \), so \( \sigma \) iso.

Cor: \( D(X) \) is generated by \( \mathcal{O}(X) \) and \( \text{Vect}(X) \).

Cor: Restriction \( D_{x,u} \rightarrow D_{x,u} \) is iso for \( \mathcal{U} \subset \mathcal{V} \subset \mathcal{U} \) \((\mathcal{U}, \mathcal{V} \text{ open affine})\). So \( D_{x,u} \equiv D(U) \).

Cor: The natural map \( \mathcal{O}(\mathcal{U}) \otimes D(X) \rightarrow D_{x,u} \equiv D(U) \) is an iso.

Cor: If \( x_1, \ldots, x_n, \xi_1, \ldots, \xi_n \) is an etale coordinate system on \( X \), the natural map

\[
\mathcal{O} \rightarrow \mathcal{O} \otimes \mathbb{R}^n_{x} \\
\mathbb{R} = (x_1, \ldots, x_n)
\]

is an isomorphism.

Any smooth \( X \) has a cover by affine open with etale coordinate systems, so this gives a pretty explicit description of \( D_x \).
The symbol map \( \sigma: \text{gr} D^x \rightarrow \mathcal{O}(T^*X) \)
is an iso of graded Poisson algebras.

\textbf{Pf} Check on stalk coordinate neighborhoods.

\underline{Car} \( D^x \) is generated by \( \partial x \) and \( \mathcal{D}v: v \in \text{Vect}(X) \)
subject to the relations:

1. \( [Dv, f] = v(f) \) for \( v \in \text{Vect}(X) \), \( f \in \mathcal{O}(X) \)
2. \( [Dv, Dw] = [Dv, w] \cdot v, w \in \text{Vect}(X) \)
3. \( Dv = fDv \).

For \textit{General Non-affine Smooth} \( \mathcal{X} \):

For affine smooth \( X \), the sheaf \( D^x \) is intrinsic, so
one can cover general \( X \) by affines and glue to get
affine sheaf of associative algebras \( D^x \). The
filtrations and symbol maps are compatible too,
so obtain is \( \text{gr} D^x \approx \mathcal{O}(T^*X) \)
as sheaves of graded Poisson algebras.

So now to smooth \( X/\mathbb{C} \) have a canonical sheaf of
affine algebras with a natural filtration. We will
study its modules.

\textbf{Assumption: All \( D^x \)-Modules are \( \mathcal{O} \)-quasi-coherent.}

\( D^x \)-\text{Modules} = \text{Sheaves of} \( \partial x \)-\text{Modules \& Flat Connection}

Given \( M \) a \( D^x \)-module, given a section \( m \in M \) and
a vector field \( v \in \text{Vect}(X) \), get another section \( v \cdot m \in M \).
Clearly have \( (f \cdot v) \cdot m = (f \cdot v) \cdot m \), so this defines a map \( \nabla : M \rightarrow M \otimes \mathbb{C}^2 \).

The relations:
\( [v, f] = f(v) \iff \nabla \text{ satisfies Liebniz, so is a connection} \)
\[ [D_v,D_w] = D_{[v,w]} \iff [\nabla_v,\nabla_w] = 0 \iff \nabla \text{ flat.} \]

**Special Case:** \( \mathcal{O} \)-coherent \( D \)-modules.

Thus a \( \mathcal{O} \)-coherent \( D \)-module is a vector bundle (finite rank) with a flat connection.

**PF:** Need to explain why a vector bundle; use \( D_x \)-action to reduce relations among generators \( \implies \) contradiction.

In vector bundles w/ flat connection have monodromy/parallel transport indep of path up to homotopy. So get \( \{ \text{vector bundles w/ flat connection} \} \)

\[ \implies \{ \text{local systems} = \text{loc constant sheaves in analytic top.} \} \]

\[ \implies \text{reps of } \pi_1. \]

Under reasonable assumptions ("regular singularities"), step 2 is an equivalence too.

**Example:** let \( X = \mathbb{C}^x \), \( M = D_x \langle z^{-1/2} \rangle = \bigoplus_{n \in \mathbb{Z}} z^{-n - 1/2} \), the \( D_x \)-mod generated by a branch of \( z^{-1/2} \).

As \( \mathcal{O} \)-module, \( M \cong \mathcal{O}, \quad f z^{-1/2} \iff f \). Since

\[ z (f z^{-1/2}) = f' z^{1/2} - \frac{f z^{1/2}}{2z} = (f' - \frac{f}{2}) z^{-1/2} \]

we see \( M \) corresponds to trivial vector bundle w/ fiber \( \mathbb{C} \) w/ nontrivial connection \( \nabla = d - \frac{dz}{2z} \).

Differential equation for flat sections is

\[ \frac{df}{dz} = \frac{f}{z} \iff f = cz^{1/2} \]

Get \( \pi_1 \)-rep given by \( n \mapsto (-1)^n \).
So $O$-coherent $D$-mods $\iff$ local systems.

Question: So $\iff$ constructible sheaves?

Answer: Nothing, but can make equivalence at the derived level using holonomic $D$-modules. Need notion of singular support first.

**Good Filtrations on $Dx$-Modules**

- **Def** A filtration of a $Dx$-module $M$ is good if $\text{gr} \ M$ is finitely generated/graded.

  Fact $M$ fin gen $\iff$ exists good filtration for $M$.

- **Def** Two filtrations are equivalent if each embeds in a shift of the other.

  Fact Any two good filtrations on $M$ are equivalent.

- **Car** For $F^*M$ a good filtration of $M$, the homogeneous ideal $\sqrt{\text{Ann} \ gr_{F^*}M} \subset gr_{D} = D(\mathcal{T}^* X)$

  is independent of the filtration $F$.

  Thus, we can define singular support:

  $\text{ss}(M) = V(\sqrt{\text{Ann} \ gr_{F^*}M}) \subset T^*X$, a (set theoretic) closed, $C^*$-stable subvariety of $T^*X$.

  **Example** $O$-coherent $\iff$ $D$-fin gen, $+ \text{ sing supp is } \subset O$-section of $T^*X$. 
Def. Let $A$ be a Poisson algebra. Then an ideal $IA$ of $A$ is called coisotropic if $I, I^2 \subset IA/I$. Geometrically, if $A = \mathcal{O}(X)$, $X$ a Poisson affine alg var, a closed subvariety $Z \subset X$ is called coisotropic if $I(Z)$ is coisotropic.

**Lemma.** $Z$ coisotropic $\iff \forall z \in Z$ smooth point, and for $\eta \in \Lambda^2 T_x X$ the Poisson bi-vector, $\eta(T_{z,Z}^{-1}) \subset T_{z,Z}$. 

Cor. If $X$ symplectic, $Z \subset X$ coisotropic, then every irreducible component of $Z$ has dimension $\geq \frac{1}{2} \dim X$. 

Thm. (Gabber) Let $A$ be a filtered associative algebra such that $grA$ is finitely generated commutative. Let $M$ be a finitely generated $A$-module. Then we have the ideal $J := \sqrt{Ann grM} \subset grA$, indep of good filtration $F$. $J$ is coisotropic.

Applying Gabber, prev corollary to finitely generated $D_x$-modules, using the fact $T^* X$ is symplectic, we see

Thm. For $M$ a f.g. $D_x$-mod, every irreducible component of $ss(M)$ has \( \dim \geq \dim X = \frac{1}{2} \dim T^* X \).

So, now we define holonomic $D_x$-modules as those with the smallest possible support.
Def. A $D_X$-module $M$ is called holonomic if it is finitely generated and either:
- $M = 0$
- $\dim ss(M) = n' = \dim X$.

Easy. Holonomic $D_X$-modules form a Serre subcategory (i.e., closed under sums, quotients, extensions) of full sets of $D_X$-mols.

Let's play with filtrations.

Example. $\Theta$-coherent $\Rightarrow$ holonomic.
If $\Theta$-coherent, $D$-flat gen + $ss = 0$ section of $T^*X$.

Actually, also have

Prop. Holonomic $\Rightarrow$ generically $\Theta$-coherent
If $H_j$ must have
- $\dim x \in X \colon \dim ss(M)_x \geq i_j \leq n - i_j$
or else $\dim ss(M) > n$. So $ss(M)_x = \emptyset$ for $x \in X$ outside a closed lower-dimensional subvariety, and $M$ $\Theta$-coherent on this open set.

Can squeeze even more info out of singular support:

Characteristic Cycles:

Corollary of the Jantzen Filtration: Let $A$ be a filtered algebra, and let $F, F'$ be two equivalent filtrations on an $A$-module $M$. Then the modules $gr^FM$ and $gr^F'M$ admit finite filtrations with subquotients occurring in the opposite order.
A Piece of Commutative Algebra (Hartshorne, I Prop 7.4):

Let $S$ be a Noetherian graded commutative ring, $M$ a finitely generated graded $S$-module. Then $M$ admits a finite filtration with subquotients of the form $(S/p)^I I^L$ with $I^L = R$ and $p \in S$ a homogeneous prime ideal. Furthermore:

1. The minimal $p$ appearing in these subquotients are the minimal primes containing $\text{Ann} M$.
2. For each such minimal prime, the # times $S/p$ appears $= \text{length}_{S/p} M_p$. So # times $S/p$ appears is indep of filtration.

So, in this setting have a well-defined "multiplicity" of a minimal prime, i.e., irreducible component of $\text{Supp} M$.

These all over short exact sequences, so together with the Car of saturated we have a well-defined characteristic cycle:

$$\chi(M) = \sum_{z \in \text{SS}(M)} \text{mult}(z) z$$

Car Holonomic $D$-modules have finite length.

Fact If $X$ is irreducible (= connected, b/c smooth), $D_X$ is simple, of infinite length.

Fact If $A$ a simple algebra of $\infty$ length, any $A$-mod of finite length is cyclic.

Car Holonomic $D$-modules are cyclic.
Left vs Right $D^x$-Modules

In general, $D^x$ and $D_x^{op}$ are not isomorphic, but nonetheless they are Morita equivalent, i.e.
we have a canonical equivalence

$$M^L(D_x) \cong M^R(D_x).$$

This is via the side-changing sheaf, the canonical sheaf $K(x) = \Sigma^{top}(X)$.

Prop $\Sigma^{top}(X)$ is a right $D^x$-module via the action:

$$v \cdot w = -L_v w \quad v \in \Sigma^{top}(X), \quad w \in \text{Vect}(X).$$

Constructions Can do the following things to left/right $D^x$-mods:

left $\otimes$ left = left

right $\otimes$ left = right

$\text{Hom}_{D^x}$(left, left) = left

$\text{Hom}_{D^x}$(right, right) = left

$\text{Hom}_{D^x}$(left, right) = right

in natural ways.

$$\begin{align*}
\text{Car} & : M^L(D_x) \to M^R(D_x), \quad M \mapsto M \otimes_{D^x} K(x) \\
\text{Car} & : M^R(D_x) \to M^L(D_x), \quad M \mapsto \text{Hom}_{D^x}(K(X), M)
\end{align*}$$

define quasi-inverse equivalences of cats of left and right $D^x$-modules.
Functors:
We bothered to talk about right vs left because some of the important functors for D-modules are defined more naturally for right (or left) D-modules.

Let $\pi : X \to Y$ be a morphism of affine smooth C-varieties.

Pullback. Want a functor $\pi^{\ast} : \mathcal{M}^e(D_Y) \to \mathcal{M}^e(D_X)$.
This is $\pi^{\ast} \mathcal{M} := \mathcal{O}_X \otimes_{\pi^\ast \mathcal{O}_Y} \pi^{-1} \mathcal{M}$.

Clearly it is $\mathcal{O}_X$-mod, as usual. But how a $\mathcal{O}_Y$-mod?
Need to define action of $\text{Vect}(Y)$.
Problem: Can't push forward vector fields!
Solution: In AG, can push forward vector fields on $X$ to vector fields on $Y$ with coefficients in $\mathcal{O}_X$.

- Have natural map $\text{Vect}(X) \to \mathcal{O}_X \otimes_{\pi^\ast \mathcal{O}_Y} \text{Vect}(Y)$, to define, for example, choose $\mathcal{O}_X$-basis $v_1, \ldots, v_n$ on $Y$, say $y_1, \ldots, y_n$, $\partial_1, \ldots, \partial_n$, and send $\text{Vect}(X) \ni v \to \sum_{i=1}^n v(\pi x_i, y_i) \partial_i$.

Easy to check is independent of choice of vectors (and comes from dualizing $\pi^{\ast} \Omega_Y^1 \to \Omega_X^1$).

Fact: This map defines $\mathcal{O}_Y$-mod structure on $\pi^{\ast} \mathcal{M}$.

Clearly $\pi^{\ast} \mathcal{M}$ is local on source + target, so can remove the assumption that $X, Y$ affine. Clearly right exact.

Notation $D_X \to Y := \pi^{\ast} \mathcal{O}_Y = \mathcal{O}_X \otimes_{\pi^\ast \mathcal{O}_Y} \pi^{-1} \mathcal{O}_Y$

This is a $D_X - \pi^{-1} \mathcal{O}_Y$ bimodule.

Fact: $\pi^{\ast} \mathcal{M} = D_X \to Y \otimes_{\pi^{-1} \mathcal{O}_Y} \pi^{-1} \mathcal{M}$. 
Pushforward: Assume $\pi: X \to Y$ is an affine map.

Bimodules $\to$ Functors, so $D_x \to y$

defines a functor $\pi_{\to}: M^\pi(D_x) \to M^\pi(D_y)$ by

$$M \mapsto M \otimes_{D_x} D_y \to \pi_\# \text{sheaf } (M \otimes_{D_x} D_y \to y).$$

(Latter is naturally a $\text{sheaf } \pi^{-1}(D_y)$-mod (right), but have adjunction map $D_y \to \pi^\# \text{sheaf } \pi^{-1}(D_y)$.

Note $\pi_{\to}$ is local on target but not on source.

Example Thinking of $K(X) = \Sigma^\text{top}_x$ as a right $D_x$-module, we see pushforward of $D_x$-modules is a generalization of "integration over fibers."

Using side-changing, can view as a functor

$$\pi_{\to}: M^\pi(D_x) \to M^\pi(D_y).$$

Note Homologically speaking, $\pi_{\to}$ is an ugly functor, it is the composition of a right exact functor (tensor $w/ D_x \to y$) and a left exact functor ($\pi_\# \text{sheaf}$). But recall from AG that for affine maps $\pi$, $\pi_{\#} \text{sheaf}$ is exact on $D_x$-quasicoherent modules, so actually $\pi_{\to}$ is right exact.

Thus For $\pi: X \to Y$ an affine map of smooth $C$-varieties, the functors $\pi_{\#0}$ and $\pi_{\to0}$ and their higher derived functors $L_i \pi_{\#0}, L_i \pi_{\to0}$ preserve the categories of holonomic $D$-modules.

Also, $(\pi_{\#0})_{\#0} = \pi_{\#0} \circ \pi_{\#0}$, $(\pi_{\to0})_{\to0} = \pi_{\to0} \circ \pi_{\to0}$. 
Let \( \pi : X \hookrightarrow Y \) be a closed embedding. Then we can define a functor
\[
\pi^!_0 \colon \mathcal{M}^r(\mathcal{O}_Y) \to \mathcal{M}^r(\mathcal{O}_X)
\]
\[
\pi^!_0(M) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, M) = \mathfrak{S}_{m \mathcal{O}_X} = \mathcal{O}_X
\]
where \( \mathfrak{S} \) is the defining ideal of \( X \) (so \( \pi^!_0 \)
takes these sections scheme-theoretically supported on \( X \)).

\( \pi^!_0 \) is a \( \mathcal{O}_X \)-module by defining the action of
vector fields on \( X \) by
1. lifting \( \text{vf}(\mathcal{O}_X) \) to \( \mathcal{O}_Y \text{vf}(\mathcal{O}_Y) \) preserving \( \mathfrak{S} \)
2. using action of \( \text{vf}(\mathcal{O}_Y) \) on \( M \).

Thm (Kashiwara)

1. \( \pi^!_0 \) is right adjoint to \( \pi^\circledast_0 \)
2. If we restrict \( \pi^!_0 \) to \( \mathcal{M}^r(\mathcal{O}_Y) \), i.e., the full
   subcat of \( \mathcal{M}^r(\mathcal{O}_Y) \) of modules set-theoretically supported
   on \( X \), then \( \pi^!_0, \pi^\circledast_0 \) are mutually quasi-inverse equivalences.

Det/Ch can define the category of \( D \)-modules on a
singular space by embedding in a smooth space
and considering cat of \( D \)-mods supported on the
image of the embedding. Can check this cat is
indep of choice of embedding.

Derived Cat of \( D \)-Modules, Formalism of 6 Functors:
The theory discussed so far becomes richer at the
level of the derived category.

Let \( X \) be a smooth \( \mathbb{C} \)-variety, \( \mathcal{D}_X \) the sheaf of
differential operators on \( X \).
We now collect some technical facts/definitions needed for the derived machinery to work.

**Def.** A $D_x$-module $M$ is coherent if:
1. $M$ is locally finitely generated over $D_x$.
2. For any open $U \subset X$, any locally finitely generated submodule of $M|_U$ is locally finitely presented.

**Prop.** Let $M \in \text{Mod}(D_x)$ be a coherent $D_x$-module. Then $M$ has a resolution

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

by locally free $D_x$-modules of finite rank.

Furthermore, there exist a finite resolution

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

by locally projective $D_x$-modules of finite rank.

**Prop.** If $R$ is a sheaf of rings on a topological space $X$, then $\text{Mod}(R)$ has enough injectives and flashts.

Let $D^b(D_x)$ denote the bounded derived category of all $D_x$-modules, let $D^{\geq 0}(D_x)$ and $D^{\leq 0}(D_x)$ denote the full subcategories of those objects with $D_x$-quasi-coherent $D_x$-coherent (respectively) cohomology. Let $\text{Mod}^q(D_x)$ and $\text{Mod}(D_x)$ denote the cats of $D_x$-quasi-coherent $D_x$-coherent (resp.) $D_x$-modules.
Thus (1) Any object in \( D^b(D_X) \) is represented by a bounded complex of flat \( D_X \)-modules. Any object of \( D^b_{\text{coh}}(D_X) \) is represented by a bounded complex of locally projective \( D_X \)-modules belonging to \( M^\text{coh}(D_X) \).

(2) The natural functors
\[
D^b(M_{\text{coh}}(D_X)) \to D^b(D_X)
\]
\[
D^b(M_{\text{coh}}(D_X)) \to D^b_{\text{coh}}(D_X)
\]
are equivalences.

(3) The natural functors
\[
D^b(D_X) \to D^b_{\text{hol}}(D_X)
\]
is an equivalence of categories. In particular,
\[
\text{For } M, N \in D_{\text{hol}}(D_X),
\]
\[
\text{Ext}^n_{D_{\text{hol}}(D_X)}(M, N) \cong \text{Ext}^n_M(D_X)(M, N).
\]

**Duality Functor**

Define \( D : D^b(D_X) \to D^b(D_X) \),
\[
D(M) := R \text{Hom}^n_{D_X}(M, D_X) \otimes K^{(-n)}_{D_X} \quad (n = \dim X)
\]
where \( \text{Hom} \) is sheaf hom/\( D_X \), \([n]\) is the shift functor, and \( K^{(-n)}_{D_X} \) is the shift-changing sheaf.

**Theorem**

(1) \( D^2 \cong \text{id} \) on \( D^b(M_{\text{coh}}(D_X)) \)

(2) For \( M, N \in D^b(M_{\text{hol}}(D_X)) \), we have
\[
\text{Hom}(M, DN) \cong \text{Hom}(N, DM).
\]

**Inverse Image Functor**

Let \( \pi : X \to Y \) be a morphism of smooth alg \( C \)-vars. Then we define the inverse image functor
\[
\pi^! : D^b(D_Y) \to D^b(D_X)
\]
by
\[
\pi^!(M) = D_X \to Y \otimes_{D_Y} \pi^* M [\dim X - \dim Y]
\]
\( \pi^! \) sends \( D\text{-gsh} \to \Omega\text{-gsh} \), and holonomic \( \to \) holonomic. It due not in general preserve \( D\text{-coherence} \).

**Direct Image** Have functor \( \pi_* : D^b(D_x) \to D^b(D_y) \),

\[
\pi_* M = R^\text{Tor}\text{sheaf} (M \otimes_{D_x} D_x \to y)
\]

(twist by side changing sheaf to get story for top D-mods).

**Properties**
- Both \( \pi^! \), \( \pi_* \) preserve \( D\text{-gsh} \text{coherence} \)
- If \( \pi \) smooth, \( \pi^! \) preserves \( D\text{-coherence} \)
- If \( \pi \) projective, \( \pi_* \) preserves \( D\text{-coherence} \)
- If \( \pi \) projective, \( f_* \) is left adjoint to \( f^! \) on \( D\text{-coherent complexes, and } f^* \) commutes with \( D\text{-} \)

We want to define \( \pi^{\star} := D \pi^! \text{D} \) and
\( \pi^{\star} := D \pi_* \text{D} \), but the problem is \( \pi^! (\text{coh}) \) may not be coh, same for \( \pi_* \).

For the holonomic case, these problems go away:

Then The functors \( \pi^\star, \pi^!, \text{D}, \text{ and } \boxtimes \) preserve (cohomological) holonomicity, \( \text{D} \) preserves actual holonomicity.

**Cor** Can define \( \pi^{\star} := D \pi^\star \text{D} \), \( \pi^{\star} := D \pi_* \text{D} \) between bounded derived cats of holonomic \( D\text{-} \\text{modules} \).

**Fact** This satisfies the "formalism of 6 functors"

**Prop**
1. \( \pi^\star \) is left adjoint to \( \pi_* \)
2. \( \pi^! \) is right adjoint to \( \pi^! \)
Classification of Irreducible Holonomic D-Modules: Minimal Extension:

Thin (Existence of Minimal/Intermediate Gevrey-MacPherson-Deligne Extension)

Let $X$ be irreducible, $U \subset X$ open subset. For every holonomic $D_X$-module $N$, there exists a unique holonomic $D_X$-module, denoted $j!_{\ast}N$ (if $j:U \to X$) such that
1) $j!_{\ast}N \supseteq N$ (so $j!_{\ast}N$ is an extension of $N$)
2) $j!_{\ast}N$ has no subs or quotients supported on $X \setminus U$.

Prop $N$ irred $\Rightarrow$ $j!_{\ast}N$ irred.

PF All but possibly one composition factor of $N$ must be supported on $X \setminus U$ (to see this, take $j!_{\ast}$ which is just restriction and use (1)).

So if $\geq 2$ comp factors, either one at top or one at bottom is supported on $X \setminus U$, contradiction.

So any irred holonomic on U has a unique holonomic irred extension to $X$.

Idea of Construction

3 morphism $j!_{\ast}N \to j\ast N$.

- $j!_{\ast}N \in D_{\text{hol}}^0(X)$, but $j\ast N \in D_{\text{hol}}^\infty(X)$, so morphism $j!_{\ast}N \to j\ast N$ factorize as

$j!_{\ast}N \to H^0(j!_{\ast}N) \to H^0(j\ast N) \to j\ast N$.

$j!_{\ast}N$ is the image here.

FACT $j\ast$ commutes with $D$. 
So now we can see how to classify/describe all irreducible holonomic $D$-modules.

Let $M$ be an irreducible holonomic $D_X$-module. Let $Y \subseteq X$ be its support, which is an irreducible closed subvariety of $X$. By Kashiwara's Theorem, $M$ is obtained as the pushforward of a holonomic $D_Y$-module, $M_Y$. $M_Y$ is a local system on some open subset $U \subseteq Y$.

So $M_Y$ must be the minimal extension of this local system.

So, we see irreducible holonomic $D_X$-modules are given by starting with a local system $N$ on a locally closed subspace $Z \subset X$, taking the minimal extension of $N$ to $\overline{Z}$, and then pushing forward to $X$.

The equiv relation on such pairs $(Z, N)$ generated by $Z', Z' \cap Z = N \implies (Z, N) - (Z', N)$ clearly coincides with the equiv relation given by isomorphism of corresponding holonomic on $X$, so holonomic irreducible $D$-mods are parameterized by sets of pairs equivalence classes of such pairs.