Summary of D-Modules

- D = sheaf of differential operators on a variety X,
  D-Module = sheaf of modules for D.

Assume all varieties are smooth / $\mathbb{C}$.

Grothendieck's Definition of Differential Operators

- Let R be a comm ring, M, N R-modules.

Def Let $D_{-1}(M,N) = 0$, and for $n \geq 0$:
$$D_n(M,N) = \{L \in \mathrm{Hom}_{\mathbb{C}}(M,N) : [L,f] \in D_{n-1}(M,N)\ \forall f \in R\}.$$
Set
$$D(M,N) := \bigcup_{n \geq 0} D_n(M,N).$$

Note $D_i(M,N) \subset D_j(M,N)$ for $i \leq j$, and $D_i(M,N)$ is naturally an R-bimodule.

When $M = N$, it is easy to see that $D(M) := D(M,M)$ has the structure of an algebra, and in fact
$$D_n(M) \cdot D_m(M) \subset D_{n+m}(M),$$
so $D(M)$ is a filtered algebra, and we can consider the associated graded algebra

"geometric filtration"

$$\mathrm{gr}\, D(M) := \bigoplus_{n \geq 0} \frac{D_n(M)}{D_{n-1}(M)}.$$
For $M = R$,
Note $[D_n(R), D_m(R)] \subset D_{n+m-1}(R)$,
so $\mathrm{gr}\, D(R)$ is a graded commutative algebra.

Important special case: $M = N = R$. $D(R)$ is called the algebra of differential operators on R.

Example $D(\mathbb{C}[x_1, \ldots, x_n])$ = usual alg of poly diff ops / $\mathbb{C}^n$.

Example $D_0(R) = R$. For $L \in D_1(R)$, easy to see
$$\bar{L} := \mathrm{ad}\, L : R \to R \quad \text{is a derivation,}$$

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and since $Lf = Lf1 = [L,f] + fL(1) = (\operatorname{ad} L + L(1))f$, we see $D_1(R) = R \oplus \operatorname{Der}(R)$.

Note $D(M,N)$ is a filtered right $D(M)$-module and a filtered left $D(N)$-module.

<u>Sheaves of Differential Operators:</u>

Let X be smooth, affine $\mathbb{C}$-variety.
We want to attach a quasicoherent sheaf of assoc algebras D_X to X, such that for open affine $U \subset X$, we have $D_X(U) \cong D(U)$.

<u>2 Ways to Think of This:</u>

(1) Ore conditions. Note for $f \in \mathcal{O}(X)$, f is locally ad-nilpotent in $D(X)$. This means the quasicoherent sheaf D_X on X associated to the left-$\mathcal{O}(X)$-module $D(X)$ naturally has the structure of a sheaf of associative algebras. This is a bit unsatisfying, not clear what $D_X(U)$ is.

(2) Define $D_{X,U,n} := D_n(\mathcal{O}(X), \mathcal{O}(U))$ for open affine $U \subset X$.

<u>Lemma</u> For $L \in D_{X,U,n}$, $z \in U$, and $f \in m_z^{n+1}$, we have $Lf(z) = 0$, and <u>conversely</u>.

So, we get a natural map

$$D_{X,U,n} \longrightarrow \left(m_z^n / m_z^{n+1}\right)^* = (S^n T_z^* U)^* = S^n T_z U.$$

This is clearly regular in z, so get a map

$$\sigma_n : D_{X,U,n} \longrightarrow \Gamma(U, S^n TU)$$

By Lemma again, this is 0 on $D_{X,U,n-1}$, so get

$$\sigma_n : \operatorname{gr}_n D_{X,U} \longrightarrow \Gamma(U, S^n TU).$$

Easy to see $\sigma : \mathrm{gr}\, D_{X,U} \longrightarrow \Gamma(U, STU) = \mathcal{O}(T^*U)$.
is a homomorphism of graded abelian groups.
This is the symbol map.

Also easy to see for $\overline{L} \in \mathrm{gr}\, D_U$, $\overline{L'} \in \mathrm{gr}\, D_{X,U}$, we
(*) have $\sigma(\overline{LL'}) = \sigma(\overline{L})\sigma(\overline{L'})$.
The Lemma implies σ is injective, so only need surjective. For this, it suffices to treat case $X = U$, b/c STU is quasicoherent, X affine, and σ : map of $\mathcal{O}(U)$-modules. For this, note $\Gamma(X, STX)$ is generated in degree 1 / degree 0, and σ is iso in degrees ≤ 1, so σ iso.

Cor $D(X)$ is generated by $\mathcal{O}(X)$ and $\mathrm{Vect}(X)$.

Cor Restriction $D_{V,U} \longrightarrow D_{X,U}$ is iso for $X \supset V \supset U$ (U, V open affine). So $D_{X,U} \cong D(U)$.

Cor The natural map

$$\mathcal{O}(U) \otimes_{\mathcal{O}(X)} D(X) \longrightarrow D_{X,U} \cong D(U)$$

is an iso.

Cor If $x_1, .., x_n$, $\partial_1, .., \partial_n$ is an étale coordinate system on X, the natural map

$$\bigoplus_{\alpha = (\alpha_1, .., \alpha_n)} \mathcal{O}_X \partial^\alpha \longrightarrow D_X$$

is an isomorphism.

Any smooth X has a cover by affine opens w/ étale coordinate systems, so this gives a pretty explicit description of D_X.

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Cor The symbol map $\sigma: gr\, D_X \longrightarrow \mathcal{O}(T^*X)$ is an iso of graded Poisson algebras.

Pf Check on étale coordinate neighborhoods.

Cor D_X is generated by $\mathcal{O}_X$ and $\{D_v : v \in Vect(X)\}$ subject to the relations:

(1) $[D_v, f] = v(f)$ for $v \in Vect(X)$, $f \in \mathcal{O}(X)$

(2) $[D_v, D_w] = D_{[v,w]}$ $\forall v, w \in Vect(X)$

(3) $D_{fv} = fD_v$.

For General Non-affine Smooth/$\mathbb{C}$ X:

For affine smooth X, saw sheaf D_X is intrinsic, so can cover general X by affines and glue to get ~~associative~~ sheaf of associative algebras D_X. The filtrations and symbol maps are compatible too, so obtain iso

$$gr\, D_X \cong \mathcal{O}_{T^*X}$$

as sheaves of graded Poisson algebras.

So now to smooth $X/\mathbb{C}$ have a canonical sheaf of assoc algebras with a natural filtration. We will study its modules.

Assumption: All D_X-Modules are $\mathcal{O}$-quasicoherent

D_X-Modules = Sheaves of $\mathcal{O}_X$-Modules w/ Flat Connection

Given M a D_X-module, given a section $m \in M$ and a vector field $v \in Vect(X)$, get another section $vm \in M$. Clearly have $(fv)m = f(vm)$, so this defines a map

$$\nabla : M \longrightarrow M \otimes_{\mathcal{O}} \Omega^1.$$

The relations:

$[v, f] = f(v)$ $\iff$ ∇ satisfies Leibniz, so is a connection

$[D_v, D_w] = D_{[v,w]} \iff [\nabla_v, \nabla_w] = \nabla_{[v,w]}$, i.e. ∇ flat.

<u>Special Case</u>: $\mathcal{O}$-coherent D-modules.

<u>Thm</u> A $\mathcal{O}$-coherent D-module is a vector bundle (finite rank) with a flat connection.

<u>Pf</u> Need to explain why a vector bundle. Use D_X-action to reduce relations among generators $\longrightarrow$ contradiction. $\square$

In vector bundles w/ flat connection, have monodromy/parallel transport indep of path up to homotopy. So get $\{\mathcal{O}\text{-coherent D-mods}\}$

$\cong \rightsquigarrow$ {vector bundles w/ flat connection}

take flat sections/solutions

$\rightsquigarrow$ {local systems = loc constant sheaves in analytic top}

$\cong \longrightarrow$ reps of π_1.

Under reasonable assumptions ("regular singularities"), step 2 is an equivalence too.

<u>Example</u> Let $X = \mathbb{C}^\times$, $M = D_X\langle z^{-1/2}\rangle = \bigoplus_{n\in\mathbb{Z}} z^{n-1/2}$, the D_X-mod generated by a branch of $z^{-1/2}$.

As $\mathcal{O}$-module, $M \cong \mathcal{O}$, $fz^{-1/2} \leftrightarrow f$. Since

$$\partial_z(fz^{-1/2}) = f'z^{-1/2} - \frac{fz^{-1/2}}{2z} = \left(f' - \frac{f}{2z}\right)z^{-1/2},$$

we see M corresponds to trivial vector bundle w/ fiber $\mathbb{C}$ w/ nontrivial connection $\nabla = d - \frac{dz}{2z}$.

Differential equation for flat sections is

$$\frac{\partial f}{\partial z} = \frac{f}{2z} \longrightarrow f = cz^{1/2}.$$

Get π_1-rep, given by $n \longmapsto (-1)^n$.

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So $\mathcal{O}$-coherent D-mods $\longleftrightarrow$ local systems.

Question: So ________ $\longleftrightarrow$ constructible sheaves?

Answer: Nothing, but can make equivalence at the derived level using holonomic D-modules. Need notion of singular support first.

Good Filtrations on D_X-Modules

- Def A filtration of a D_X-Module M is good if $\operatorname{gr} M$ is finitely generated / $\operatorname{gr} D$.

Fact M fin gen $\Leftrightarrow \exists$ good filtration for M.

Def Two filtrations are equivalent if each embeds in a shift of the other.

Fact Any two good filtrations on M are equivalent.

Cor For $F^\bullet M$ a good filtration of M, the homogeneous ideal $\sqrt{\operatorname{Ann} \operatorname{gr}_F M} \subset \operatorname{gr} D = \mathcal{O}(T^*X)$ is independent of the filtration F.

Thus, we can define singular support:

$ss(M) = V(\sqrt{\operatorname{Ann} \operatorname{gr}_F M}) \subset T^*X$, a (set theoretic) closed, $\mathbb{C}^*$-stable subvariety of T^*X.

Example $\mathcal{O}$-coherent $\Leftrightarrow$ D-fin gen, + sing supp is $\subset$ 0-section of T^*X.

Def Let A be a Poisson algebra. Then an ideal $I \subset A$ is called coisotropic if $\{I, I\} \subset I$.

Geometrically, if $A = \mathcal{O}(X)$, X a Poisson affine alg var, a closed subvariety $Z \subset X$ is called coisotropic if $I(Z)$ is coisotropic.

Lemma Z coisotropic $\iff \forall z \in Z$ smooth point, and for $\eta \in \Lambda^2 T_X$ the Poisson bivector,

$$\eta(T_z Z^\perp) \subset T_z Z.$$

Cor If X symplectic, $Z \subset X$ coisotropic, then every irred component of Z has dimension $\geq \frac{1}{2} \dim X$.

Thm (Gabber) Let A be a filtered associative algebra such that $\operatorname{gr} A$ is finitely generated commutative. Let M be a finitely generated A-module. Then we have the ideal $J := \sqrt{\operatorname{Ann} \operatorname{gr} M} \subset \operatorname{gr} A$, indep of good filtration F.
J is coisotropic.

Applying Gabber + prev corollary to finitely generated D_X-modules, using the fact T^*X is symplectic, we see

Thm For M a fin gen D_X-mod, every irred component of $SS(M)$ has dim $\geq \dim X = \frac{1}{2} \dim T^*X$.

So, now we define holonomic D_X-modules as those with the smallest possible support:

<u>Def</u> A D_X-module M is called <u>holonomic</u> if it is finitely generated and either:

- $M = 0$
- $\dim ss(M) = n := \dim X$.

<u>Easy</u> Holonomic D_X-modules form a Serre subcategory (i.e. closed under subs, quotients, extensions) of full cat of D_X-mods.

<u>Pf</u> Play w/ filtrations.

<u>Example</u> $\mathcal{O}$-coherent $\Rightarrow$ holonomic.

<u>Pf</u> $\mathcal{O}$-coherent = D-fin gen + $ss = 0$-section of T^*X

Actually, also have

<u>Prop</u> Holonomic $\Rightarrow$ generically $\mathcal{O}$-coherent

<u>Pf</u> $\forall i$, must have

$\dim \{x \in X : \dim ss(M)_x \geq i\} \leq n - i$,

or else $\dim ss(M) > n$. So $ss(M)_x = \{0\}$ for $x \in X$ outside a closed lower-dimensional subvariety, and M $\mathcal{O}$-coherent on this open set.

Can squeeze even more info out of singular support:

<u>Characteristic Cycles:</u>

<u>Corollary of the Jantzen Filtration</u>: Let A be a filtered algebra, and let F, F' be two <u>equivalent</u> filtrations on an A module M. Then the modules $gr^F M$ and $gr^{F'} M$ admit finite filtrations with subquotients occuring in the opposite order.

A Piece of Commutative Algebra (Hartshorne, I Prop 7.4):

Let S be a Noetherian graded commutative ring, M a finitely generated graded S-module. Then M admits a finite filtration with subquotients of the form $(S/\mathfrak{p})[\ell]$ with $\ell \in \mathbb{Z}$ and $\mathfrak{p} \subset S$ a homogeneous prime ideal. Furthermore:

(1) The minimal $\mathfrak{p}$ appearing in these subquotients are the minimal primes containing $\operatorname{Ann} M$.

(2) For each such minimal prime, the # times $S/\mathfrak{p}$ appears $=$ $\text{length}_{S_\mathfrak{p}} M_\mathfrak{p}$. So # times $S/\mathfrak{p}$ appears is indep of filtration.

So, in this setting have a well-defined "multiplicity" of a minimal prime, i.e. irred component of $\operatorname{Supp} M$. These add over short exact sequences, so together with the Cor of Jantzen we have a well defined characteristic cycle:

$$CC(M) = \sum_{\substack{Z \subset SS(M) \\ \text{irred comp}}} \text{mult}(Z)\, Z$$

Cor Holonomic D-modules have finite length.

Fact If X is irreducible ($=$ connected, b/c smooth), D_X is simple, of infinite length.

Fact If A a simple algebra of ∞ length, any A-mod of finite length is cyclic.

Cor Holonomic D-modules are cyclic.

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Left vs Right D_X-Modules

In general, D_X and D_X^{op} are not isomorphic, but nonetheless they are Morita equivalent, i.e. we have a canonical equivalence

$$\mathcal{M}^l(D_X) \cong \mathcal{M}^r(D_X).$$

This is via the side-changing sheaf, the canonical sheaf $K(X) := \Omega^{top}(X)$.

Prop $\Omega^{top}(X)$ is a right D_X-module via the action:

$$\omega \cdot v = -L_v \omega \quad \forall\, \omega \in \Omega^{top}(X),\ v \in \mathrm{Vect}(X).$$

Constructions Can do the following things to left/right D_X-mods:

$$\text{left} \otimes_{\mathcal{O}_X} \text{left} = \text{left}$$

$$\text{right} \otimes_{\mathcal{O}_X} \text{left} = \text{right}$$

$$\mathrm{Hom}_{\mathcal{O}_X}(\text{left}, \text{left}) = \text{left}$$

$$\mathrm{Hom}_{\mathcal{O}_X}(\text{right}, \text{right}) = \text{left}$$

$$\mathrm{Hom}_{\mathcal{O}_X}(\text{left}, \text{right}) = \text{right}$$

in natural ways.

Cor

$$\mathcal{M}^l(D_X) \longrightarrow \mathcal{M}^r(D_X), \quad M \longmapsto M \otimes_{\mathcal{O}_X} K(X)$$

$$\mathcal{M}^r(D_X) \longrightarrow \mathcal{M}^l(D_X), \quad M \longmapsto \mathrm{Hom}_{\mathcal{O}_X}(K(X), M)$$

define quasi-inverse equivalences of cats of left and right D_X-modules.

Functors:

We bothered to talk about right vs left because some of the important functors for D-modules are defined more naturally for right (or left) D-modules.

Let $\pi : X \longrightarrow Y$ be a morphism of affine smooth $\mathbb{C}$-vars.

Pullback Want a functor $\pi^{*\circ} : \mathcal{M}^{\ell}(D_Y) \longrightarrow \mathcal{M}^{\ell}(D_X)$.
This is $\pi^{*\circ}M := \mathcal{O}_X \underset{\pi^{-1}\mathcal{O}_Y}{\otimes} \pi^{-1}M$.

Clearly is $\mathcal{O}_X$-mod, as usual. But how a D_X-mod?
Need to define action of $\mathrm{Vect}(X)$.

Problem Can't push forward vector fields!

Solution: In AG, can push forward vector fields on X to vector fields on Y with coefficients in $\mathcal{O}_X$:

- Have natural map $\mathrm{Vect}(X) \longrightarrow \mathcal{O}_X \underset{\pi^{-1}\mathcal{O}_Y}{\otimes} \pi^{-1}\mathrm{Vect}(Y)$, to define, for example, choose étale coords on Y, say $y_1, \ldots, y_n, \partial_1, \ldots, \partial_n$, and send

$$\mathrm{Vect}(X) \ni v \longmapsto \sum_{i=1}^{n} v(\pi^{\#} y_i) \otimes \partial_i.$$

Easy to check is indep of choice of coords (and comes from dualizing $\pi^{*}\Omega^1_Y \longrightarrow \Omega^1_X$).

Fact This map defines D_X-mod structure on $\pi^{*\circ}M$.

Clearly $\pi^{*\circ}$ is local on source + target, so can remove the assumption that X, Y affine. Clearly right exact.

Notation $D_{X\to Y} := \pi^{*\circ}D_Y = \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}D_Y$

This is a D_X-$\pi^{-1}D_Y$ bimodule.

Fact $\pi^{*\circ}M = D_{X\to Y} \underset{\pi^{-1}D_Y}{\otimes} \pi^{-1}M$.

Why not $\pi_\star : \mathcal{M}^\ell(D_X) \longrightarrow \mathcal{M}^\ell(D_Y)$,
$\pi_\star M = \pi_{\star}\,\mathrm{sheaf}\,\mathrm{Hom}_{D_X}(D_{X\to Y}, M)$?

This is left exact, right adjoint to $\pi^\star$....

quasicoherence?

Pushforward: Assume $\pi : X \longrightarrow Y$ is an affine map.

Bimodules $\Rightarrow$ Functors, so $D_{X\to Y}$

defines a functor $\pi_{\star 0} : \mathcal{M}^r(D_X) \longrightarrow \mathcal{M}^r(D_Y)$ by

$$\mathcal{M}^r(D_X) \longrightarrow \text{right } \pi^{-1}(D_Y)\text{-mod} \longrightarrow \mathcal{M}^r(D_Y)$$
$$M \longmapsto M \underset{D_X}{\otimes} D_{X\to Y} \longmapsto \pi_{\star,\mathrm{sheaf}}\big(M \underset{D_X}{\otimes} D_{X\to Y}\big).$$

(latter is naturally a $\pi_{\star,\mathrm{sheaf}}\,\pi^{-1}(D_Y)$-mod (right), but have adjunction map $D_Y \longrightarrow \pi_{\star\,\mathrm{sheaf}}\,\pi^{-1}(D_Y)$.

Note $\pi_{\star 0}$ is local on target but not on source.

Example Thinking of $K(X) = \Omega_X^{top}$ as a right D_X-module, we see pushforward of D_X-modules is a generalization of "integration over fibers"

[Using side-changing, can view as a functor
$\pi_{\star 0} : \mathcal{M}^\ell(D_X) \longrightarrow \mathcal{M}^\ell(D_Y)$.]

Note Homologically speaking, $\pi_{\star 0}$ is an ugly functor, it is the composition of a right exact functor (tensor w/ $D_{X\to Y}$) and a left exact functor ($\pi_{\star\,\mathrm{sheaf}}$).
But recall from AG that for affine maps π, $\pi_{\star 0,\mathrm{sheaf}}$ is exact on $\mathcal{O}_X$-quasicoherent modules, so actually $\pi_{\star 0}$ is right exact.

Thm For $\pi : X \longrightarrow Y$ an affine map of smooth $\mathbb{C}$-varieties, the functors $\pi_{\star 0}$ and $\pi^{\star 0}$ and their higher derived functors $L^i\pi_{\star 0}$, $L^i\pi^{\star 0}$ preserve the categories of holonomic D-modules.

Also, $(\pi\circ\sigma)_{\star 0} = \pi_{\star 0}\circ\sigma_{\star 0}$, $(\pi\circ\sigma)^{\star 0} = \pi^{\star 0}\sigma^{\star 0}$.

$\underline{\pi^{!\circ}}$ Let $\pi : X \hookrightarrow Y$ be a closed embedding. Then we can define a functor

$$\pi^{!\circ} : \mathcal{M}^r(D_Y) \longrightarrow \mathcal{M}^r(D_X)$$

$$\pi^{!\circ}(M) = Hom_{\mathcal{O}_Y}(\mathcal{O}_X, M) = \{m \in M : mJ = 0\}$$

where J is the defining ideal of X (so $\pi^{!\circ}$ takes those sections scheme theoretically supported on X).

$\pi^{!\circ}M$ is a D_X-module by defining the action of vector fields on X by
(1) lifting $v \in Vect(X)$ to $\tilde{v} \in Vect(Y)$ preserving J
(2) using action of $Vect(Y)$ on M.

Thm (Kashiwara)

(1) $\pi^{!\circ}$ is right adjoint to $\pi_{*\circ}$
(2) If we restrict $\pi^{!\circ}$ to $\mathcal{M}^\ell_X(Y)$, i.e. the full subcat of $\mathcal{M}^\ell(Y)$ of modules set-theoretically supported on X, then $\pi^{!\circ}, \pi_{*\circ}$ are mutually quasi-inverse equivalences.

Def/Cor Can define the category of D-modules on a singular space by embedding in a smooth space and considering cat of D-mods supported on the image of the embedding. Can check this cat is indep of choice of embedding.

Derived Cat of D-Modules, Formalism of 6 Functors:

The theory discussed so far becomes richer at the level of the derived category.

Let X be a smooth $\mathbb{C}$-variety, D_X the sheaf of differential operators on X.

If only $\mathcal{O}$-qcoh, still get such resolutions, just need to drop finite rank conditions.

We now collect some technical facts/definitions needed for the derived machinery to work.

Def A D_X-module M (=sheaf of modules for D_X) is coherent if
(1) M is locally finitely generated / D_X
(2) For any open $U \subset X$, any locally finitely generated submodule of $M|_U$ is locally finitely presented.

Prop (1) D_X is coherent / D_X
(2) A D_X-mod M is coherent $\iff$ it is quasicoherent / $\mathcal{O}_X$ and locally finitely generated over D_X.

& The cat of such modules is denoted $\mathcal{M}^c(D_X)$.

Prop Let $M \in \mathcal{M}^c(D_X)$ be a coherent D_X-module. Then M has a resolution
$$P^\bullet \longrightarrow M \longrightarrow 0$$
by locally free D_X-modules of finite rank.
Furthermore, there exists a finite resolution
$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$
by locally projective D_X-mods of finite rank.

Prop If R is a sheaf of rings on a topological space X, then $\mathrm{Mod}(R)$ has enough injectives and flats.

Let $D^b(D_X)$ denote the bounded derived category of all D_X-modules, let $D^b_{qcoh}(D_X)$ and $D^b_{coh}(D_X)$ denote the full subcats of those objects with $\mathcal{O}$-quasicoherent, D-coherent (respectively) cohomology. Let $\mathcal{M}_{qcoh}(D_X)$ and $\mathcal{M}_{coh}(D_X)$ denote the cats of $\mathcal{O}$-qcoh, D-coh (resp) D_X-modules.

Thm (1) Any object in $D^b(D_X)$ is represented by a bounded complex of flat D_X-modules. Any object of $D^b_{qcoh}(D_X)$ is represented by a bounded complex of locally projective D_X-modules belonging to $M_{qcoh}(D_X)$.

(2) The natural functors

$$D^b(M_{qcoh}(D_X)) \longrightarrow D^b_{qcoh}(D_X)$$
$$D^b(M_{coh}(D_X)) \longrightarrow D^b_{coh}(D_X)$$

are equivalences

(3) The natural functor

$$D^b(M_{hol}(D_X)) \longrightarrow D^b_{hol}(D_X)$$

is an equivalence of categories. In particular, For $M, N \in M_{hol}(D_X)$,

$$Ext^i_{M(D_X)}(M,N) \cong Ext^i_{M_{hol}(D_X)}(M,N).$$

Duality Functor

Define $\mathbb{D}: D^b(D_X) \longrightarrow D^b(D_X)$,

$$\mathbb{D}M^\bullet := R\,\underline{Hom}_{D_X}(M, D_X)[n] \otimes K_X^{-1} \qquad (n = \dim X)$$

where $\underline{Hom}$ is sheaf hom / D_X, $[n]$ is the shift functor, and K_X^{-1} is the side-changing sheaf.

Thm (1) $\mathbb{D}^2 \cong id$ on $D^b(M^l_{coh}(D_X))$

(2) For $M, N \in D^b(M^l_{coh}(D_X))$, we have

$$Hom(M, \mathbb{D}N) \cong Hom(N, \mathbb{D}M).$$

Inverse Image Functor

Let $\pi: X \longrightarrow Y$ be a morphism of smooth alg $\mathbb{C}$-vars. Then we define the inverse image functor

$\pi^!: D^b(D_Y) \longrightarrow D^b(D_X)$ by

$$\pi^!(M^\bullet) = D_{X\to Y} \otimes^L_{\pi^{-1}D_Y} \pi^{-1}M^\bullet[\dim X - \dim Y]$$

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$\pi^!$ sends $\mathcal{O}$-qcoh $\longrightarrow$ $\mathcal{O}$-qcoh, and holonomic $\longrightarrow$ holonomic.
It does not in general preserve D-coherence.

Direct Image Have functor $\pi_*: D^b(D_X) \longrightarrow D^b(D_Y)$,

$$\pi_* M^\bullet = R\pi_{*\text{sheaf}}\left(M \otimes^L_{D_X} D_{X\to Y}\right)$$

(twist by side changing sheaf to get story for left D-mods).

Properties
- Both $\pi^!, \pi_*$ preserve $\mathcal{O}$-quasicoherence
- If π smooth, $\pi^!$ preserves D-coherence
- If π projective, π_* preserves D-coherence
- If π projective, f_* is left adjoint to $f^!$ on D-coherent complexes, and f_* commutes with $\mathbb{D}$.

We want to define $\pi^* := \mathbb{D}\pi^!\mathbb{D}$ and
$\pi_! = \mathbb{D}\pi_*\mathbb{D}$, but the problem is $\pi^!(\text{coh})$ may not be coh, same for π_*.
For the holonomic case, these problems go away:

Thm The functors $\pi_*, \pi^!, \mathbb{D}$, and $\boxtimes$ preserve (cohomological) holonomicity. $\mathbb{D}$ preserves actual holonomicity.

Cor Can define $\pi^* := \mathbb{D}\pi^!\mathbb{D}$, $\pi_! := \mathbb{D}\pi_*\mathbb{D}$ between bounded derived cats of holonomic D-modules.

Fact This satisfies the "formalism of 6 functors"

Prop 1) π^* is left adjoint to π_*
2) $\pi^!$ is right adjoint to $\pi_!$

Classification of Irreducible Holonomic D-Modules:
Minimal Extension:

Thm (Existence of Minimal/Intermediate/Goresky-MacPherson-Deligne Extension)

Let X be irreducible, $U \subset X$ open subset. For every holonomic D_U-module N, there exists a unique holonomic D_X-module, denoted $j_{!*}N$ (if $j: U \hookrightarrow X$) such that

1) $j^! j_{!*} N \cong N$ (so $j_{!*}N$ is an extension of N)
2) $j_{!*}N$ has no subs or quotients supported on $X \setminus U$.

Prop N irred $\Rightarrow$ $j_{!*}N$ irred.

Pf All but possibly one composition factor of N must be supported on $X \setminus U$ (to see this, take $j^!$, which is just restriction and use (1)).

So if ≥ 2 comp factors, either one at top or one at bottom is supported on $X \setminus U$, contradiction.

So, any irred holonomic on U has a unique holonomic irred extension to X.

Idea of Construction

(this is general, get for any $\pi: X \to Y$ by factoring as open immersion, then projection, $\pi_! = \pi_*$ for proj)

- $\exists$ morphism $j_! N \longrightarrow j_* N$.

- $j_! N \in D^{\leq 0}_{hol}(X)$, but $j_* \in D^{\geq 0}_{hol}(X)$, so morphism $j_! N \longrightarrow j_* N$ factorizes as

$$j_! N \longrightarrow H^0(j_! N) \longrightarrow H^0(j_* N) \longrightarrow j_* N.$$

$j_{!*}N$ is the image here.

FACT $j_{!*}$ commutes with $\mathbb{D}$.

So now we can see how to classify/describe all irreducible holonomic D-modules.
Let M be an irreducible holonomic D_X-module.
Let $Y \subset X$ be its support, which is an irreducible closed subvariety of X. By Kashiwara's Thm, M is obtained as the pushforward of a holonomic D_Y-module, M_Y. M_Y is ~~ge~~ ~~an~~ local system on some open subset $U \subset Y$. So M_Y must be the minimal extension of this local system.

So, we see irred holonomic D_X-modules are given by starting with a ~~t~~ local system N on a locally closed subspace $Z \subset X$, taking the minimal extension of N to $\overline{Z}$, and then pushing forward to X.

The equiv relation on such pairs ~~(N~~ (Z, N) generated by $Z' \subset Z$, ~~N|~~ $N|_{Z'} = N' \Rightarrow (Z, N) \sim (Z', N')$ clearly coincides with the equiv relation given by iso of corresponding holonomics on X, so holonomic irred D-mods are parameterized by ~~such pairs~~ equivalence classes of such pairs.