BABY VERMA MODULES FOR RATIONAL CHEREDNIK ALGEBRAS

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Abstract. These are notes for a talk in the MIT-Northeastern Spring 2015 Geometric Representation Theory Seminar. The main source is [G02]. We discuss baby Verma modules for rational Cherednik algebras at $t = 0$.

Contents

1. Background

1.1. **Definitions.** Let $\mathfrak{h}$ be a finite dimensional $\mathbb{C}$-vector space and let $W \subset GL(\mathfrak{h})$ be a finite subgroup generated by the subset $S \subset W$ of complex reflections it contains. Let $c : S \to \mathbb{C}$ be a conjugation-invariant function. For $s \in S$ we denote $c_s := c(s)$. For each $s \in S$ choose eigenvectors $\alpha_s \in \mathfrak{h}^*$ and $\alpha_s^\vee \in \mathfrak{h}$ with nontrivial eigenvalues $\epsilon(s)^{-1}, \epsilon(s)$, respectively. Recall that for $t \in \mathbb{C}$ we have the associated *rational Cherednik algebra* $H_{t,c}(W, \mathfrak{h})$, denoted $H_{t,c}$ when W and $\mathfrak{h}$ are implied, which is defined as the quotient of

$$\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$$

Date: February 10, 2015.

by the relations

$$[x, x'] = 0 \qquad [y, y'] = 0 \qquad [y, x] = t(y, x) + \sum_{s \in S} (\epsilon(s) - 1) c_s \frac{(y, \alpha_s)(x, \alpha_s^\vee)}{(\alpha_s, \alpha_s^\vee)} s$$

for $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$. Note that this definition does not depend on the choice of α_s and $\alpha_s^\vee$. This algebra is naturally $\mathbb{Z}$-graded, setting $\deg W = 0$, $\deg \mathfrak{h}^* = 1$, and $\deg \mathfrak{h} = -1$. One may also view the parameters t, c as formal variables to obtain a universal Cherednik algebra H, of which $H_{t,c}$ is a specialization.

1.2. **PBW Theorem.** For any parameters t, c we have the natural $\mathbb{C}$-linear multiplication map

$$S\mathfrak{h}^* \otimes \mathbb{C}W \otimes S\mathfrak{h} \to H_{t,c}.$$

The PBW theorem for rational Cherednik algebras states that this map is a vector space isomorphism. This is very important.

1.3. $t = 0$ **vs.** $t \neq 0$**.** For any $a \in \mathbb{C}^\times$ we have $H_{t,c} \cong H_{at,ac}$ in an apparent way. Thus the theory of rational Cherednik algebras has a dichotomy with the cases $t = 0$ and $t \neq 0$ (the latter may as well be $t = 1$). As important special specializations, we have the isomorphisms

$$H_{0,0} \cong \mathbb{C}W \ltimes S(\mathfrak{h} \oplus \mathfrak{h}^*) \qquad H_{1,0} \cong \mathbb{C}W \ltimes D(\mathfrak{h})$$

where S denotes symmetric algebra and $D(\mathfrak{h})$ denotes the algebra of differential operators on $\mathfrak{h}$. These isomorphisms give some flavor of the distinctions between the theory for $t = 0$ and $t \neq 0$. In the case $t = 1$ one may define and study a certain category $\mathcal{O}_c$ of $H_{1,c}$-modules analogous to the BGG category $\mathcal{O}$ for semisimple Lie algebras. Today we focus instead on the case $t = 0$ and introduce and study a certain class of finite-dimensional representations of $H_{0,c}$ called the *baby Verma modules*.

2. Restricted Cherednik Algebras

2.1. **A Central Subalgebra.**

Proposition 1. *The natural embedding*

$$S\mathfrak{h}^W \otimes_{\mathbb{C}} S\mathfrak{h}^{*W} \to H_{0,c}$$

by multiplication factors through the center $Z_c := Z(H_{0,c})$.

Proof. This was seen last week as an immediate consequence of the Dunkl operator embedding. □

Let $A \subset Z_c$ denote the image of this embedding.

2.2. **Coinvariant Algebras.** The *coinvariant algebra* for the action of W on $\mathfrak{h}$ is the quotient

$$S\mathfrak{h}^{coW} := S\mathfrak{h}/S\mathfrak{h}^W_+ S\mathfrak{h}$$

where $S\mathfrak{h}^W_+$ is the augmentation ideal of the invariants $S\mathfrak{h}^W$. This is a $\mathbb{Z}$-graded algebra. It also has the structure of a W-module inherited from the W-action on $S\mathfrak{h}$ since $S\mathfrak{h}^W_+ S\mathfrak{h}$ is a W-stable ideal.

Proposition 2. $S\mathfrak{h}^{coW}$ *and* $S\mathfrak{h}^{*coW}$ *afford the regular representation of* W*.*

Proof. Let $\pi : \mathfrak{h} \to \mathfrak{h}/W$ denote the projection. Then $\pi_*\mathcal{O}_\mathfrak{h}$ is a coherent sheaf with W-action, and $S\mathfrak{h}^{*coW}$ is its fiber at 0. By Chevalley's theorem, $S\mathfrak{h}^{*W}$ is a polynomial algebra on $\dim \mathfrak{h}$ homogeneous elements of $S\mathfrak{h}^*$, so by a theorem of Serre $S\mathfrak{h}^*$ is a free module over $S\mathfrak{h}^{*W}$. For $v \in \mathfrak{h}^{reg}$ this the fiber at $\pi(v)$ is $\mathbb{C}[Wv] \cong \mathbb{C}W$ as W-modules. But the multiplicity of the irreducible representation L of W in the fiber at a point $\overline{u} \in V/W$ is the fiber dimension at $\overline{u}$ of the coherent sheaf $\mathrm{Hom}_W(L, \pi_*\mathcal{O}_\mathfrak{h})$, which is hence upper-semicontinuous. But if $m_L(x)$ is this multiplicity of L at x, since $\pi_*\mathcal{O}_\mathfrak{h}$ is free of rank $|W|$ we see $\sum_L (\dim L) m_L(x) = |W|$ and so that the m_L are continuous, hence constant. It follows that the zero fiber is the regular representation too, as needed. □

So we see $S\mathfrak{h}^{*coW}$ is a graded version of the regular representation of W. This allows us to define a related family of polynomials, the *fake degrees* of W. In particular, if T is an irreducible W-representation, and $T[i]$ denotes its shift to degree i, we have the polynomial

$$f_T(t) := \sum_{i \in \mathbb{Z}} (S\mathfrak{h}^{*coW} : T[i]) t^i$$

where the notation $(S\mathfrak{h}^{*coW} : T[i])$ means the multiplicity of $T[i]$ in $S\mathfrak{h}^{*coW}$ in degree i. Note $f_T(1) = \dim T$. These have been computed for all finite Coxeter groups, where we have no preference for $\mathfrak{h}$ vs. $\mathfrak{h}^*$, and for many complex reflection groups as well.

2.3. **Restricted Cherednik Algebras.** A is a $\mathbb{Z}$-graded central subalgebra of $H_{0,c}$. Viewing $A = S\mathfrak{h}^W \otimes S\mathfrak{h}^{*W}$, let A_+ denote the ideal of A consisting of elements without constant term. Then we can form the *restricted Cherednik algebra* as the quotient

$$\overline{H_c} := \frac{H_{0,c}}{A_+ H_{0,c}}.$$

As A is $\mathbb{Z}$-graded this inherits a $\mathbb{Z}$-grading from $H_{0,c}$. It follows immediately from the PBW theorem that we have an isomorphism of vector spaces given by multiplication

$$S\mathfrak{h}^{coW} \otimes \mathbb{C}W \otimes S\mathfrak{h}^{*coW} \to \overline{H_c}$$

which we view as a PBW theorem for restricted Cherednik algebras. In particular we see $\dim \overline{H_c} = |W|^3$.

Some motivation for considering this algebra is the following. $H_{0,c}$ is a countable-dimensional algebra so by Schur's lemma its center acts on any irreducible representation through some central character, corresponding to a closed point in the Calogero-Moser space $\mathrm{Spec}(Z_c)$. In particular, since $H_{0,c}$ is finite over its center Z_c, it follows that any irreducible representation of $H_{0,c}$ is finite-dimensional. By considering representations of the algebra $\overline{H_c}$ we are specifying that we only want to consider representations whose central characters lie above $0 \in \mathfrak{h}^*/W \times \mathfrak{h}/W$ with respect to the map

$$\mathrm{Spec}(Z_c) \to \mathrm{Spec}(S\mathfrak{h}^W \otimes S\mathfrak{h}^{*W}).$$

These are the most important central characters to consider, as for central characters above other points in $\mathfrak{h}^*/W \times \mathfrak{h}/W$ one can reduce to the representation theory of $\overline{H_c}$ for some parabolic subgroup $W' \subset W$.

3. Baby Verma Modules for $\overline{H_c}$

In the presence of the PBW theorem for restricted Cherednik algebras, it is natural to define an analogue of Verma modules in this setting. Let Λ denote the set of isomorphism classes of irreducible $\mathbb{C}$-representations of W. Let $\overline{H_c^-} = \mathbb{C}W \ltimes S\mathfrak{h}^{coW}$, a subalgebra of $\overline{H_c}$ of dimension $|W|^2$. We have a natural map of algebras $\overline{H_c^-} \to \mathbb{C}W$, $f \otimes w \mapsto f(0)w$, and via this map we may view any W-module as a $\overline{H_c^-}$-module. For $S \in \Lambda$, let $M(S)$, the *baby Verma module associated to* S, be the induced module

$$M(S) := \overline{H_c} \otimes_{\overline{H_c^-}} S.$$

Placing S in degree 0, $M(S)$ is then non-negatively graded with $M(S)^0 = S$. As a graded $S\mathfrak{h}^{*coW} \rtimes \mathbb{C}W$-module we have

$$M(S) = S\mathfrak{h}^{*coW} \otimes_{\mathbb{C}} S$$

and hence $\dim M(S) = |W| \dim S$ and in the Grothendieck group of graded W-modules we have

$$[M(S)] = \sum_{T \in \Lambda} f_T(t)[T \otimes S]$$

where $f_T(t)$ is the fake degree of W associated to T defined earlier.

Let $\overline{H_c} - \text{mod}$ denote the category of $\overline{H_c}$-modules, $\overline{H_c} - \text{mod}_{\mathbb{Z}}$ denote the category of $\mathbb{Z}$-graded $\overline{H_c}$-modules with graded $\overline{H_c}$-maps, and let $F : \overline{H_c} - \text{mod}_{\mathbb{Z}} \to \overline{H_c} - \text{mod}$ denote the forgetful functor. We view $M(S)$ as an object in $\overline{H_c} - \text{mod}_{\mathbb{Z}}$ as explained above.

Proposition 3. *Let $S, T \in \Lambda$. Then we have*

(1) The baby Verma $M(S)$ has a simple head. We denote it by $L(S)$.

(2) $M(S)[i]$ is isomorphic to $M(T)[j]$ if and only if $S = T$ and $i = j$.

(3) $\{L(S)[i] : S \in \Lambda, i \in \mathbb{Z}\}$ forms a complete set of pairwise non-isomorphic simple objects in $\overline{H_c} - \text{mod}_{\mathbb{Z}}$.

(4) $F(L(S))$ is a simple $\overline{H_c}$-module and $\{F(L(S)) : S \in \Lambda\}$ is a complete set of pairwise non-isomorphic simple $\overline{H_c}$-modules.

(5) If $P(S)$ is the projective cover of $L(S)$, then $F(P(S))$ is the projective cover of $F(L(S))$.

Proof. (1) Any vector of $M(S)$ in degree 0 generates $M(S)$, so a proper graded submodule is positively graded. Thus $M(S)$ has a unique maximal proper graded submodule, so a unique irreducible graded quotient.

(2) If $M(S)[i] \cong M(T)[j]$ then clearly $i = j$ as otherwise they are not supported in the same degrees. But then any isomorphism as graded $\overline{H_c}$-modules is an isomorphism as graded W-modules, and by inspecting lowest degrees we see $S = T$.

(3) Identical analysis to the above shows the modules in question are pairwise non-isomorphic. By Frobenius reciprocity any nonzero $N \in \overline{H_c} - \text{mod}_{\mathbb{Z}}$ admits a nonzero map from some $M(S)[i]$, so every simple $L \in \overline{H_c} - \text{mod}_{\mathbb{Z}}$ is isomorphic to some $L(S)[i]$.

(4) To see $F(L(S))$ is simple it suffices to check that $F(M(S))$ has a unique maximal proper submodule, equal to its unique maximal proper graded submodule. For any vector $v \in M(S)$, let $v = \sum_{i \geq 0} v_i$ be its decomposition into graded components. If $v_0 \neq 0$, then for each $i > 0$ there exists $a_i \in \overline{H_c}^i$ such that $v_i = a_i v_0$. It follows by induction on the number of nonzero homogeneous components that $v_0 \in \overline{H_c} v$,

and hence $\overline{H_c}v = M(S)$ as $M(S)$ is generated by any nonzero vector of degree 0. Thus any proper $\overline{H_c}$-submodule of $F(M(S))$ has nonzero graded components only in positive degree, so $F(M(S))$ has a unique maximal proper submodule. A similar argument shows that the submodule generated by all homogeneous components of vectors from this module is again proper, so this maximal proper submodule is graded and we see $F(L(S))$ is simple. To see that every simple is isomorphic to some $F(L(S))$, note that if N is any finite-dimensional $\overline{H_c}$-module then the space

$$\{n \in N : \mathfrak{h}n = 0\}$$

is nonzero ($S\mathfrak{h}_+$ is nilpotent in $\overline{H_c}$) and W-stable. So we can find a copy of some $S \in \Lambda$ in this space, and hence N admits a nonzero $\overline{H_c}$-homomorphism from $M(S)$ by Frobenius reciprocity. It follows that any simple is isomorphic to some $F(L(S))$.

(5) Projective objects in $\overline{H_c} - \mathrm{mod}_{\mathbb{Z}}$ are direct summands of direct sums of shifts of $\overline{H_c}$, and hence F maps projective objects to projective objects. Certainly $F(P(S))$ admits a surjective map to $F(L(S))$, so we need only check that $F(P(S))$ is indecomposable. For this, note $\mathrm{Hom}_{\overline{H_c}}(F(P(S)), F(L(S)))$ is naturally $\mathbb{Z}$-graded. If it were not isomorphic to $\mathbb{C}[0]$ as a $\mathbb{Z}$-graded $\mathbb{C}$-vector space then $P(S)$ would admit a nonzero graded homomorphism to some simple object $L(S)[i]$ of $\overline{H_c} - \mathrm{mod}_{\mathbb{Z}}$ with $i \neq 0$. So we see $\mathrm{Hom}_{\overline{H_c}}(F(P(S)), F(L(S))) = \mathbb{C}[0]$ and similarly $\mathrm{Hom}_{\overline{H_c}}(F(P(S)), F(L(T))) = 0$ for $T \neq S$. It follows that $F(P(S))$ is indecomposable. □

4. Decomposition of $\overline{H_c}$

4.1. **The morphism Υ.** Recall that we have the inclusion $A \to Z_c$ of the algebra $A := S\mathfrak{h}^W \otimes S\mathfrak{h}^{*W}$ into the center $Z_c := Z(H_{0,c})$. This induces a map on spectra

$$\Upsilon : X_c = \mathrm{Spec}(Z_c) \to \mathfrak{h}^*/W \times \mathfrak{h}/W = \mathrm{Spec}(S\mathfrak{h}^W \otimes S\mathfrak{h}^{*W})$$

where X_c is the Calogero-Moser space we saw last week. We will be concerned with the schematic fiber $\Upsilon^*(0)$ above 0. We have

$$\Upsilon^*(0) = \mathrm{Spec}(Z_c/A_+Z_c)$$

and as Z_c is finite over A we see Z_c/A_+Z_c is a finite-dimensional algebra, $\Upsilon^*(0)$ is a finite discrete space. We denote this underlying space by $\Upsilon^{-1}(0)$. We denote the local ring of $\Upsilon^*(0)$ at $M \in \Upsilon^{-1}(0)$ by $\mathcal{O}_M$, and it is given by

$$\mathcal{O}_M = (Z_c)_M/A_+(Z_c)_M.$$

We refer to the $\mathrm{Spec}(\mathcal{O}_M)$ as the (schematic)*components* of $\Upsilon^*(0)$. In particular, we see

$$\frac{Z_c}{A_+Z_c} = \prod_{M \in \Upsilon^{-1}(0)} \mathcal{O}_M.$$

4.2. **$\mathcal{O}_M$ is naturally $\mathbb{Z}$-graded.** Z_c inherits a grading from $H_{0,c}$, and $A \subset Z_c$ is a graded subalgebra, and so it follows that $\Upsilon : X_c \to \mathfrak{h}^*/W \times \mathfrak{h}/W$ is a $\mathbb{C}^*$-equivariant morphism. In particular, since $0 \in \mathfrak{h}^*/W \times \mathfrak{h}/W$ is a fixed point for the $\mathbb{C}^*$-action, it follows that $\Upsilon^*(0)$ inherits a $\mathbb{C}^*$-action. Since $\Upsilon^{-1}(0)$ is a discrete space, this action fixes each point, and hence the components $\mathrm{Spec}(\mathcal{O}_M)$ inherit a $\mathbb{C}^*$-action, and hence $\mathcal{O}_M$ inherits a $\mathbb{Z}$-grading from $H_{0,c}$ in this way.

4.3. **Blocks of $\overline{H_c}$.** We have the natural map

$$\frac{Z_c}{A_+ Z_c} = \prod_{M \in \Upsilon^{-1}(0)} \mathcal{O}_M \to \overline{H_c} = \frac{H_{0,c}}{A_+ H_{0,c}}.$$

In particular, since Z_c is central in $H_{0,c}$, the idempotents on the left side map to a set of commuting idempotents with sum 1 in $\overline{H_c}$ on the right side. In fact this map is injective. This follows from the observation that Z_c is a summand of the A-module $H_{0,c}$, which follows from the corresponding statement for $c = 0$, which was proven last week, and a standard argument involving filtered deformations. This gives rise to a corresponding direct sum decomposition of the algebra $\overline{H_c}$:

$$\overline{H_c} = \bigoplus_{M \in \Upsilon^{-1}(0)} \mathcal{B}_M.$$

It is proved by Brown and Gordon in [BG01] that these summands $\mathcal{B}_M$ are indecomposable algebras. We refer to the $\mathcal{B}_M$ as the *blocks* of the restricted Cherednik algebra $\overline{H_c}$. From last week, we know that if $M \in \mathrm{Spec}(Z_c)$ is a smooth point that

$$\mathcal{B}_M \cong \mathrm{Mat}_{|W|}(\mathcal{O}_M).$$

4.4. **The map Θ.** Recall that for any $S \in \Lambda$ irreducible representation of W, we have the associated baby Verma module $M(S)$ for $\overline{H_c}$. This module has a simple head, so is indecomposable, so in particular is a nontrivial module for a unique block $\mathcal{B}_M$. This defines a map

$$\Theta : \Lambda \to \Upsilon^{-1}(0).$$

Any simple module of $\mathcal{B}_M$ is a simple module of $\overline{H_c}$ via the projection $\overline{H_c} \to \mathcal{B}_M$, and we have seen already that the simple modules of $\overline{H_c}$ are precisely the simple quotients of the baby Verma modules, so we conclude Θ is surjective. When $M \in \mathrm{Spec}(Z_c)$ is a smooth point, we have $\mathcal{B}_M \cong \mathrm{Mat}_{|W|}(\mathcal{O}(M))$, so in particular $\mathcal{B}_M$ is Morita equivalent to the local ring $\mathcal{O}_M$ and hence has a unique simple module, and so in this case we have $M = \Theta(S)$ for a unique $S \in \Lambda$. In particular, if $\mathrm{Spec}(Z_c)$ is smooth, Θ is a bijection.

4.5. **Poincare polynomial of $\mathcal{O}_M$.** Recall that the local ring $\mathcal{O}_M$ is $\mathbb{Z}$-graded and finite-dimensional. It is therefore natural to ask about its Poincare polynomial

$$P_M(t) := \sum_{i \in \mathbb{Z}} \dim \mathcal{O}_M^i t^i.$$

This is computed via the following theorem of Gordon. We will write p_S for $p_{\Theta(S)}$.

Theorem 4. *Suppose $M \in \Upsilon^{-1}(0)$ is a smooth point of Spec(Z_c). Then $M = \Theta(S)$ for a unique simple W-module $S \in \Lambda$. If b_S denotes the smallest power of t appearing in the associated fake degree $f_S(t)$, and similarly for b_{S^*}, then we have*

$$p_S(t) = t^{b_S - b_{S^*}} f_S(t) f_{S^*}(t^{-1}).$$

In particular, if W is a finite Coxeter group so that $S \cong S^*$, we have

$$p_S(t) = f_S(t) f_S(t^{-1})$$

5. The Symmetric Group Case

We now specialize to the case of $W = S_n$ and nonzero parameter $c \neq 0$. In this case $\mathrm{Spec}(Z_c)$ is smooth, so the previous theorem applies to all $M \in \Upsilon^{-1}(0)$.

Recall in this case the irreducible representations of S_n are labeled in a natural way by the partitions $\lambda \vdash n$ of n. We will denote the irreducible representation of S_n corresponding to λ by S_λ. Stembridge [S89] proved the following formula for the fake degree f_{S_λ} in terms of the principal specialization of the Schur function s_λ:

$$f_{S_\lambda}(t) = (1-t)\cdots(1-t^n)s_\lambda(1,t,t^2,...).$$

In case this looks like a proper power series to you, don't worry: from Stanley [S99] we have the following combinatorial description of this expression. In particular, if T is a standard Young tableau of shape λ, then we define its *descent set* $D(T)$ to be the set of all $i \in \{1,...,n\}$ such that i appears in a row lower than the row containing $i+1$. We then define the *major index* $\mathrm{maj}(T)$ by

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

We then have the formula

$$f_{S_\lambda}(t) = (1-t)\cdots(1-t^n)s_\lambda(1,t,t^2,...) = \sum_T t^{\mathrm{maj}(T)}$$

where the sum is over all standard Young tableaux T of shape λ. Clearly this is a polynomial, and we have a combinatorial description of the coinvariant algebra $S\mathfrak{h}^{coS_n}$ as a graded S_n-module. We thus have the description of the Poincare polynomial $p_{S_\lambda}(t)$ of $\mathcal{O}_{\Theta(S_\lambda)}$ in terms of specializations of Schur functions:

$$p_{S_\lambda}(t) = \prod_{i=1}^{n}(1-t^i)(1-t^{-i})s_\lambda(1,t,t^2,...)s_\lambda(1,t^{-1},t^{-2},..).$$

In terms of the Kostka polynomials

$$K_\lambda(t) := (1-t)\cdots(1-t^n)\prod_{u\in\lambda}(1-t^{h_u(\lambda)})^{-1} \in \mathbb{Z}[t]$$

where $h_\lambda(u)$ is the hook length of u in λ, and the statistic

$$b(\lambda) := \sum_{i\geq 1}(i-1)\lambda_i$$

we have

$$(1-t)\cdots(1-t^n)s_\lambda(1,t,t^2,...) = t^{b(\lambda)}K_\lambda(t).$$

In particular, we see

$$p_{S_\lambda}(t) = K_\lambda(t)K_\lambda(t^{-1}).$$

6. References

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