THE DUNKL WEIGHT FUNCTION FOR REPRESENTATIONS OF RATIONAL CHEREDNIK ALGEBRAS

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1. Signature Characters and the Jantzen Filtration

1.1. Jantzen Filtrations on Standard Modules. Let $W$ be a finite complex reflection group with reflection representation $\mathfrak{h}$. Let $S \subset W$ be the set of complex reflections in $W$, and let $p$ be the $\mathbb{C}$-vector space of $W$-invariant functions $c : S \to \mathbb{C}$. For any $c \in p$, let $c^\dagger \in p$ be defined by $c^\dagger(s) = \overline{c(s^{-1})}$. Refer to any $c \in p$ satisfying $c = c^\dagger$ as real, and let $p_\mathbb{R} \subseteq p$ be the $\mathbb{R}$-vector space $p_\mathbb{R} := \{c \in p : c = c^\dagger\}$.

Recall that $p$ is the parameter space for the rational Cherednik algebras attached to $(W, \mathfrak{h})$; we denote the rational Cherednik algebra attached to $(W, \mathfrak{h})$ and parameter $c \in p$ by $H_c(W, \mathfrak{h})$. For each irreducible representation $\tau \in \text{Irr}(W)$, we have the associated standard module

$$\Delta_c(\tau) := H_c(W, \mathfrak{h}) \otimes_{\mathbb{C} W \rtimes S\mathfrak{h}} \tau$$

over $H_c(W, \mathfrak{h})$. As a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-vector space, $\Delta_c(\tau)$ is naturally identified with $S\mathfrak{h}^* \otimes \tau$ independently of $c$.

Fix a $W$-invariant positive definite Hermitian form $\langle \cdot, \cdot \rangle_\tau$ on $\tau$ (we agree that Hermitian forms are conjugate-linear in the second factor). Such a form is uniquely determined up to $\mathbb{R}^{>0}$-scaling. Similarly, fix a $W$-invariant positive-definite Hermitian form $\langle \cdot, \cdot \rangle_\mathfrak{h}$ on $\mathfrak{h}$. This determines the conjugate-linear isomorphism $T : \mathfrak{h} \to \mathfrak{h}^*$ given by

$$T(y)(x) := \langle x, y \rangle_\mathfrak{h}.$$
When the parameter $c \in \mathfrak{p}$ is real, i.e. $c = c^\dagger$, the standard module $\Delta_c(\tau)$ admits a unique $W$-invariant Hermitian form $\beta_{c,\tau}$ [ES, Proposition 2.2] such that the contravariance condition
\[
\beta_{c,\tau}(yv, v') = \beta_{c,\tau}(v, T(y)v)
\]
holds for all $v, v' \in \Delta_c(\tau)$ and $y \in \mathfrak{h}$ and that coincides with $(\cdot, \cdot)_\tau$ in degree 0.

As above, regarding the standard modules $\Delta_c(\tau)$ for various $c$ as the same $\mathbb{Z}_{\geq 0}$-graded vector space $\Delta(\tau)$, we may view the forms $\beta_{c,\tau}$ on $\Delta(\tau)$ as an algebraic family of Hermitian forms $\beta_{c,\tau}[d]$, parameterized by $c \in \mathfrak{p}_\mathbb{R}$ in a polynomial manner, on each finite-dimensional graded component $\Delta(\tau)[d]$. In particular, we are naturally led to consider Jantzen filtrations, as follows.

Let $c_0, c_1 \in \mathfrak{p}_\mathbb{R}$ be real parameters such that there exists $\delta > 0$ such that $\beta_{c(t),\tau}$ is nondegenerate for all $c(t) := c_0 + tc_1$ with $t \in (-\delta, \delta) \setminus \{0\}$. The finite-dimensional $\mathbb{C}$-vector spaces $\Delta(\tau)[d]$ for $n \geq 0$ along with the polynomial families of Hermitian forms $\beta_{c(t),\tau}[d]$ satisfy the conditions of [V, Definition 3.1]. In particular, for each $d \geq 0$ define the finite descending filtration
\[
\Delta(\tau)[d] = \Delta(\tau)[d]^{\geq 0} \supset \Delta(\tau)[d]^{\geq 1} \supset \cdots \supset \Delta(\tau)[d]^{\geq N} = 0
\]
on $\Delta(\tau)[d]$ as follows (to simplify the notation, we do not include the choice of $c_0, c_1$ in the notation for the filtration, but of course this filtration and its properties are heavily dependent on this choice). Let $\Delta(\tau)[d]^{\geq k}$ consist of those vectors $v \in \Delta(\tau)[d]$ such that there is some $\epsilon > 0$ and an analytic function $f_v : (-\epsilon, \epsilon) \to \Delta(\tau)[d]$ satisfying $f_v(0) = v$ and such that the analytic function
\[
t \mapsto \beta_{c(t),\tau}[d](f_v(t), v')
\]
vanishes at least to order $k$ for all $v' \in \Delta_c(\tau)[d]$ (clearly, one may equivalently consider only polynomial functions $f_v$). For each $k \geq 0$, define the Hermitian form $\beta_{c_0,\tau}[d]^{\geq k}$ on $\Delta(\tau)[d]^{\geq k}$ by
\[
\beta_{c_0,\tau}[d]^{\geq k}(v, v') = \lim_{t \to 0} \frac{1}{t} \beta_{c(t),\tau}[d](f_v(t), f_{v'}(t)),
\]
where $v, v' \in \Delta(\tau)[d]^{\geq k}$ and $f_v, f_{v'}$ are any analytic functions as above (this limit does not depend on the choice of such $f_v, f_{v'}$). We then have the following theorem:

**Theorem 1.1.1.** (Jantzen [J, 5.1], Vogan [V, Theorem 3.2]) The radical of the Hermitian form $\beta_{c_0,\tau}[d]^{\geq k}$ is precisely $\Delta(\tau)[d]^{\geq k+1}$.

In particular, the Hermitian form $\beta_{c_0,\tau}[d]^{\geq k}$ descends to a nondegenerate Hermitian form on the filtration subquotient $\Delta(\tau)[k] := \Delta(\tau)^{\geq k}/\Delta(\tau)^{\geq k+1}$. Denote this induced nondegenerate Hermitian form by $\beta_{c_0,\tau}[d][k]$. For all $k, d \geq 0$, let $(p_d^{(k)}, q_d^{(k)})$ denote the signature of the Hermitian form $\beta_{c_0,\tau}[d][k]$, i.e. $p_d^{(k)}$ (resp. $q_d^{(k)}$) is the dimension of a maximal positive definite (resp., negative definite) subspace of $\Delta(\tau)[k]$. Note that for any fixed $d \geq 0$ we have $p_d^{(k)} = q_d^{(k)} = 0$ for all sufficiently large $k$.

**Proposition 1.1.2.** (Vogan [V, Proposition 3.3])

(a) For all small positive $t$ (i.e. $t \in (0, \delta)$) and any $d \geq 0$, the signature of the nondegenerate Hermitian form $\beta_{c(t),\tau}[d]$ on $\Delta(\tau)[d]$ is
\[
\left( \sum_{k \geq 0} p_d^{(k)}, \sum_{k \geq 0} q_d^{(k)} \right).
\]
(b) Similarly, for all small negative $t$ (i.e. $t \in (-\delta, 0)$) and any $d \geq 0$, the signature of $\beta_{c(t), \tau}[d]$ is
\[
\left( \sum_{k \text{ even}} p_d^{(k)} + \sum_{k \text{ odd}} q_d^{(k)} ; \sum_{k \text{ odd}} p_d^{(k)} + \sum_{k \text{ even}} q_d^{(k)} \right)
\]
Define the descending filtration
\[
\Delta(\tau) = \Delta(\tau)^{\geq 0} \supset \Delta(\tau)^{\geq 1} \supset \cdots
\]
by
\[
\Delta(\tau)^{\geq k} := \bigoplus_{d \geq 0} \Delta(\tau)[d]^{\geq k} \subset \bigoplus_{d \geq 0} \Delta(\tau)[d] = \Delta(\tau).
\]

**Lemma 1.1.3.** The filtration of $\Delta_{c_0}(\tau)$ by the subspaces $\Delta(\tau)^{\geq k}$ is a filtration by $H_{c_0}(W, \mathfrak{h})$-submodules. We have $\Delta(\tau)^{\geq k} = 0$ for sufficiently large $k$.

**Proof.** Let $v \in \Delta(\tau)[d]^{\geq k}$, and let $f_v : (-\epsilon, \epsilon) \to \Delta(\tau)[d]$ be as in the definition of the filtration, exhibiting that $v \in \Delta(\tau)[d]^{\geq k}$. Then for any homogeneous $h \in H_{c_0}(W, \mathfrak{h})$ of degree $d'$, viewing $h \in H_{c(t)}(W, \mathfrak{h})$ for all $t \in \mathbb{R}$ via the PBW basis, the path $hf_s$ exhibits $hv$ as an element of $\Delta(\tau)[d + d']^{\geq k}$. In particular, $\Delta(\tau)^{\geq k}$ is a $H_{c_0}(W, \mathfrak{h})$-submodule of $\Delta(\tau)$. By the finite-length property of $H_{c_0}(W, \mathfrak{h})$-modules in category $\mathcal{O}_{c_0}(W, \mathfrak{h})$, it follows that the filtration $\Delta(\tau)^{\geq k}$ stabilizes in $k$. For any fixed $d$ we have $\Delta(\tau)[d]^{\geq k} = 0$ for any $k$ sufficiently large, and it follows that $\Delta(\tau)^{\geq k} = 0$ for all sufficiently large $k$. \qed

We refer to the filtration of $\Delta_{c_0}(\tau)$ appearing in Lemma 1.1.3 as the Jantzen filtration of $\Delta_{c_0}(\tau)$. Note that this Jantzen filtration depends on the choice of additional parameter $c_1 \in \mathfrak{p}_R$ determining the direction for the deformation.

### 1.2. Hermitian Duals

In this section we will introduce Hermitian duals in the setting of rational Cherednik algebras, analogous to the Hermitian duals considered by Vogan [V] in the Lie-theoretic setting. First we will briefly recall contragredient duals. Let $c \in \mathfrak{p}$ be any parameter for the rational Cherednik algebra attached to $(W, \mathfrak{h})$. Let $\bar{s} \in \mathfrak{p}$ be the parameter defined by $\bar{s}(s) = c(s^{-1})$. As explained in [EM, Section 3.11], there is a natural isomorphism
\[
\gamma : H_c(W, \mathfrak{h})^{\text{opp}} \to H_c(W, \mathfrak{h}^*)
\]
acting trivially on $\mathfrak{h}$ and $\mathfrak{h}^*$ and sending $w \mapsto w^{-1}$ for all $w \in W$. For any $M \in \mathcal{O}_c(W, \mathfrak{h})$, the restricted dual $M^\dagger := \bigoplus_{z \in \mathbb{C}} M_z^*$ is naturally a $H_c(W, \mathfrak{h})^{\text{opp}}$-module; by transfer of structure along $\gamma$, we regard $M^\dagger$ as a $H_{\bar{s}}(W, \mathfrak{h}^*)$-module. We have:

**Proposition 1.2.1.** ([EM, Proposition 3.32]) The assignment $M \mapsto M^\dagger$ determines a $\mathbb{C}$-linear equivalence of categories $^\dagger : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_{\bar{s}}(W, \mathfrak{h}^*)^{\text{opp}}$.

Given a $\mathbb{C}$-algebra $A$, let $\overline{A}$ denote the $\mathbb{C}$-algebra that is equal to $A$ as a ring and equal to the complex conjugate vector space of $A$. In other words, the identity map $\text{Id} : A \to \overline{A}$ is an isomorphism of rings and satisfies $z\text{Id}(a) = \text{Id}(za)$. Clearly $\overline{A}$ is a $\mathbb{C}$-algebra, with unit $\overline{1}$ satisfying $\overline{1}(z) = \eta(z)$ for all $z \in \mathbb{C}$, where $\eta : \mathbb{C} \to A$ is the unit map for the $\mathbb{C}$-algebra $A$. Similarly, for any $A$-module $M$ the complex conjugate vector space $\overline{M}$ is naturally an $\overline{A}$-module, and this clearly defines an conjugate-linear equivalence of categories $A\text{-mod} \to \overline{A}\text{-mod}$.

The complex conjugate of a rational Cherednik algebra is again a rational Cherednik algebra. More precisely:
Lemma 1.2.2. Fix a nondegenerate $W$-invariant Hermitian form $(\cdot, \cdot)$ on $\mathfrak{h}$, and let $T : \mathfrak{h} \to \mathfrak{h}^*$ be the conjugate-linear isomorphism introduced in Section 1.1. Then the mappings $y \mapsto Ty$ for $y \in \mathfrak{h}$, $x \mapsto T^{-1}x$ for $x \in \mathfrak{h}^*$, and $w \mapsto w$ for $w \in W$ extend uniquely to an isomorphism of $\mathbb{C}$-algebras
\[
\omega : H_c(W, \mathfrak{h}^*) \to \overline{H_c(W, \mathfrak{h})}.
\]
Proof. Regarded as a map $\mathfrak{h} \to \mathfrak{h}^*$, $T$ is an isomorphism of complex representations of $W$, and similarly for $T^{-1} : \mathfrak{h}^* \to \mathfrak{h}$. It follows that the assignments in the lemma extend uniquely to a $\mathbb{C}$-linear isomorphism $\mathbb{C}W \ltimes T(\mathfrak{h}^* \oplus \mathfrak{h}) \to \mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ which determines a map $\mathbb{C}W \ltimes T(\mathfrak{h}^* \oplus \mathfrak{h}) \to \overline{H_c(W, \mathfrak{h})}$. As $\mathfrak{h}$ is sent to $\mathfrak{h}^*$ and $\mathfrak{h}^*$ to $\mathfrak{h}$ under this map, the commutators $[x, x']$ and $[y, y']$ for $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$ are sent to 0 by this map. Let $(\cdot, \cdot)$ denote the natural pairing of $\mathfrak{h}$ with $\mathfrak{h}^*$. By definition of $T$, we have, for any $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$,
\[
\langle Ty, T^{-1}x \rangle = \langle T^{-1}x, y \rangle_{\mathfrak{h}} = \langle y, T^{-1}x \rangle_{\mathfrak{h}} = \langle x, y \rangle.
\]
In particular, under the map $\mathbb{C}W \ltimes T(\mathfrak{h}^* \oplus \mathfrak{h}) \to \overline{H_c(W, \mathfrak{h})}$, for any $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$ the image of the element
\[
[x, y] - \langle x, y \rangle \sum_{s \in S} \bar{c}_s \langle x, \alpha_s^\vee \rangle \langle y, \alpha_s \rangle s
\]
in $\overline{H_c(W, \mathfrak{h})}$ is
\[
[T^{-1}x, Ty] - \langle x, y \rangle \sum_{s \in S} c^s \langle x, \alpha_s^\vee \rangle \langle y, \alpha_s \rangle s
\]
\[
= [T^{-1}x, Ty] - \langle T^{-1}x, Ty \rangle + \sum_{s \in S} c^s \langle T^{-1}x, T\alpha_s^\vee \rangle \langle Ty, T^{-1}x \rangle \langle x, \alpha_s \rangle s.
\]
As $T\alpha_s^\vee \in \mathfrak{h}^*$ and $T\alpha_s \in \mathfrak{h}$ are eigenvectors for $s$ with nontrivial eigenvalues and $\langle T\alpha_s^\vee, T^{-1}\alpha_s \rangle = \frac{1}{\langle \alpha_s, \alpha_s^\vee \rangle} = \frac{1}{2} = 2$, the expression above is 0 in $H_c(W, \mathfrak{h})$. It follows that there is an induced map of $\mathbb{C}$-algebras $H_c(W, \mathfrak{h}^*) \to \overline{H_c(W, \mathfrak{h})}$, and clearly this map is an isomorphism. \hfill \Box

Let $\sigma : H_c(W, \mathfrak{h})^{\text{opp}} \to H_c(W, \mathfrak{h})$ be the isomorphism of $\mathbb{C}$-algebras obtained by composing $\gamma$ and $\omega$. For any $M \in \mathcal{O}_c(W, \mathfrak{h})$, the Hermitian dual $M^h := \overline{M^{\dagger}}$ is naturally a $\overline{H_c(W, \mathfrak{h})^{\text{opp}}}$-module, and by transfer of structure along $\sigma$ we regard $M^h$ as a $H_c(W, \mathfrak{h})$-module. Clearly $M^h \in \mathcal{O}_c(W, \mathfrak{h})$, so we have:

Lemma 1.2.3. The assignment $M \mapsto M^h$ defines a conjugate-linear equivalence of categories
\[
h : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_c(W, \mathfrak{h})^{\text{opp}}.
\]

We will use the following lemma later:

Lemma 1.2.4. Suppose the parameter $c \in \mathfrak{p}$ is real, i.e. $c = c^\dagger$. Let $M \in \mathcal{O}_c(W, \mathfrak{h})$ be equipped with a nondegenerate $W$-invariant contravariant Hermitian form $\beta$. Then the assignment
\[
m \mapsto \beta(\cdot, m)
\]
defines an isomorphism $M \cong M^h$ of $H_c(W, \mathfrak{h})$-modules.

Proof. That the map in question is a map of $H_c(W, \mathfrak{h})$-modules follows from the observation that the statement that $\beta$ is $W$-invariant and contravariant means precisely that $\beta(hv, v') = \beta(v, \sigma(h)v')$ for all $v, v' \in M$ and $h \in H_c(W, \mathfrak{h}) = H_c(W, \mathfrak{h})$. That the map is an isomorphism follows from the nondegeneracy of $\beta$. \hfill \Box
1.3. Rationality of Signature Characters. Let us now briefly recall the definition of characters and signature characters. Let $M \in \mathcal{O}_c(W, \mathfrak{h})$. Then the character of $M$, denoted $\text{ch}(M)$, is the formal series

$$\text{ch}(M)(w, t) = \sum_{z \in \mathbb{C}} t^z \text{Tr}_{M_z}(w), \quad w \in W.$$ 

In particular, taking $w = 1$, one obtains the Hilbert series for $M$:

$$\text{ch}(M)(1, t) = \sum_{z \in \mathbb{C}} t^z \dim M_z.$$ 

When $M$ is a lowest weight module of lowest weight $\lambda$, we define the shifted character $\text{ch}_0(M)$ by

$$\text{ch}_0(M) = t^{-h_c(\lambda)} \text{ch}(M).$$

For such $M$ we have $\text{ch}_0(M)(w, t) = (1 - t)^{-d} p_w(M)$, where $d$ is the dimension of support of $M$ and where, for each $w \in W$, $p_w(M)$ is a polynomial in $t$ with integer coefficients with $p_1(M)(1) \neq 0$.

Now suppose $M$ is equipped with a graded Hermitian form $\beta$. Then we similarly define the signature character $\text{sch}(M, \beta)$ as the formal series

$$\text{sch}(M, \beta) = \sum_{z \in \mathbb{C}} t^z \text{sign}(\beta_z)$$

where, for each $z \in \mathbb{C}$, $\text{sign}(\beta_z)$ denotes the signature of the restriction $\beta_z$ of the form $\beta$ to the weight space $M_z$, i.e. the signed difference between the dimension of a maximal positive definite subspace of $M_z$ and the dimension of a maximal negative definite subspace of $M_z$. When $M$ is a lowest weight module of lowest weight $\lambda$ and the parameter $c$ is real, we will often write $\text{sch}(M)$ rather than $\text{sch}(M, \beta)$, where it is implicit that we have chosen the Hermitian form $\beta$ to be $W$-invariant, contravariant, and positive definite in the lowest weight space; such a form exists and is unique up to scaling by a positive real number, and the resulting signature character does not depend on this scaling factor. In this setting, we also define the shifted signature character $\text{sch}_0(M)$ by

$$\text{sch}_0(M) := t^{-h_c(\lambda)} \text{sch}(M).$$

**Proposition 1.3.1.** For any irreducible complex representation $\lambda \in \text{Irr}(W)$ and real parameter $c \in p_{\mathbb{R}}$, the shifted signature character $\text{sch}_0(L_c(\lambda))$ is of the form

$$\text{sch}_0(L_c(\lambda)) = (1 - t)^{-d} p_{L_c(\lambda)}(t)$$

for some polynomial $p_{L_c(\lambda)}(t)$ with integer coefficients, where $d = \dim \text{Supp}(L_c(\lambda))$.

The proof of Proposition 1.3.1 is an inductive argument relying on the following lemmas:

**Lemma 1.3.2.** Let $\lambda \in \text{Irr}(W)$, and let $c_0, c_1 \in p_{\mathbb{R}}$ be real parameters so that the Jantzen filtration of $\Delta_{c_0}(\lambda)$ is defined. Let $c(s) = c_0 + sc_1$. For sufficiently small $s > 0$, we have

$$\text{sch}_0(\Delta_{c(s)}(\lambda)) = \text{sch}_0(\Delta_{c(-s)}(\lambda)) + 2t^{-h_{c_0}(\lambda)} \sum_{k \text{ odd}} \text{sch}(\Delta_{c_0}(\lambda)(k), \beta_{c_0, \lambda}(k))$$

and

$$\text{sch}_0(L_{c_0}(\lambda)) = \text{sch}_0(\Delta_{c(s)}(\lambda)) - t^{-h_{c_0}(\lambda)} \sum_{k \geq 1} \text{sch}(\Delta_{c_0}(\lambda)(k), \beta_{c_0, \lambda}(k)).$$
Proof. This is an immediate consequence of Proposition 1.1.2 and the definition of signature characters. \hfill \square

The following lemma (and its proof) is a reformulation for rational Cherednik algebras of Vogan’s [V, Lemma 3.9].

**Lemma 1.3.3.** Suppose \( c = c^1 \) and \( M \in \mathcal{O}_c(W, \mathfrak{h}) \) admits a \( W \)-invariant contravariant nondegenerate Hermitian form \( \beta \). Suppose \( M = \sum_i^n L_i \) in \( K^0(\mathcal{O}_c(W, \mathfrak{h})) \) for some simple modules \( L_1, \ldots, L_n \). Then there are \( \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\} \) such that

\[
\text{sch}(M, \beta) = \sum_{i=1}^n \epsilon_i \text{sch}(L_i).
\]

**Proof.** The proof is by induction on the length of \( M \). The case \( M = 0 \) is trivial, so suppose \( M \neq 0 \) and let \( L \subset M \) be a simple submodule. If the restriction of \( \beta \) to \( L \) is nondegenerate, then \( M = L \oplus L^\perp \) and \( L^\perp \) is an \( H_c(W, \mathfrak{h}) \)-submodule of \( M \) on which \( \beta \) is nondegenerate. As all \( W \)-invariant contravariant Hermitian forms on \( L \) are proportional, we have \( \text{sch}(L, \beta|_L) = \pm \text{sch}(L) \), and the claim follows by induction.

If the restriction of \( \beta \) to \( L \) is degenerate, it is 0. The inclusion \( L \subset M \) gives rise to a surjection \( M^h \to L^h \). By Lemma 1.2.4, we have an isomorphism \( M \to M^h \), \( m \mapsto \beta(\cdot, m) \).

The resulting composition \( \varphi : M \to L^h \) satisfies

\[
\varphi(m)(l) = \beta(l, m)
\]

for all \( l \in L \) and \( m \in M \). As \( \beta|_L = 0 \), we have \( L^\perp = \ker \varphi \supset L \), and in particular the form \( \beta \) descends to a nondegenerate \( W \)-invariant contravariant Hermitian form on the subquotient \( N := \ker \varphi / L \). Now we have that \( L \cong L^h \) so

\[
M = N + L + L^h = N + 2L
\]

in \( K^0(\mathcal{O}_c(W, \mathfrak{h})) \). Furthermore it follows from [V, Sublemma 3.18] that \( \text{sch}(M) = \text{sch}(N) \) and hence \( \text{sch}(M) = \text{sch}(N) + \text{sch}(L) - \text{sch}(L) \), and the claim follows by induction. \hfill \square

**Proof of Proposition 1.3.1.** Let \( c \in \mathfrak{p} \mathbb{R} \). There are finitely many \( s \in [0, 2] \), say \( \{s_1, \ldots, s_N\} \), such that \( \mathcal{O}_{sc}(W, \mathfrak{h}) \) is not semisimple. Furthermore, at \( s = 0 \) we have both that \( \mathcal{O}_{sc}(W, \mathfrak{h}) \) is semisimple and that every simple module \( L_0(\lambda) = \Delta_0(\lambda) \) is unitary. In particular, for all \( \lambda \) we have \( \text{sch}_0(L_0(\lambda)) = (1 - t)^{-l} \dim \lambda \) where \( l = \dim \mathfrak{h} \), so the proposition holds for \( c = 0 \). Furthermore, note that the signature character \( \text{sch}_0(\Delta_{c(s)}(\lambda)) \) does not depend on \( s \) for those \( s \) in a fixed interval \( (s_i, s_{i+1}) \). So, by induction, we may assume that the proposition holds for all \( sc \) with \( s \in [0, 1] \) and we need then only prove that it holds for all \( sc \) with \( s \in [1, 1 + \delta) \) for some \( \delta > 0 \).

For any \( \lambda \in \text{Irr}(W) \) such that \( L_{c_0}(\lambda) = \Delta_{c_0}(\lambda) \), the signature character \( \text{sch}_0(\Delta_{(1+s)c_0}(\lambda)) \) does not depend on \( s \) for \( |s| \) sufficiently small. In particular, the proposition holds for those \( \lambda \in \text{Irr}(W) \) minimal in any highest weight ordering \( \leq_{c_0} \) for \( \mathcal{O}_{c_0}(W, \mathfrak{h}) \). By induction, we may then assume that the proposition holds for those lowest weights \( \mu \in \text{Irr}(W) \) strictly lower than \( \lambda \) with respect to \( \leq_{c_0} \). Taking \( c_1 := c_0 \) in Lemma 1.3.2, it then follows from Lemmas 1.3.2 and 1.3.3 that \( \text{sch}_0(L_{c_0}(\lambda)) \) and all \( \text{sch}_0(\Delta_{(1+s)c_0}(\lambda)) \) for sufficiently small \( s > 0 \) are of the form \( (1 - t)^{-l} p(t) \) for some polynomial \( p(t) \) with integer coefficients. As the growth of the absolute value of the coefficients of the series \( \text{sch}_0(L_{c_0}(\lambda)) \) is bounded by the growth of the coefficients of \( \text{ch}_0(L_{c_0}(\lambda)) \), it follows that \( \text{sch}_0(L_{c_0}(\lambda)) \) is of the form \( (1 - t)^{-d} p(t) \) for some polynomial \( p \) with integer coefficients, as needed. \hfill \square
1.4. Asymptotic Signatures.

Lemma 1.4.1. Let \( s(t) = \sum_{n=0}^{\infty} s_nt^n, d(t) = \sum_{n=0}^{\infty} d_nt^n \in \mathbb{Z}[t] \) be power series absolutely convergent for \(|t| < 1\). Assume also that there is a power series \( f(t) = \sum_{n=0}^{\infty} f_nt^n \in \mathbb{Z}[t] \), absolutely convergent for \(|t| < 1\), satisfying \( f_N \geq 0 \) and \( \lim_{N \to \infty} f_N/(\sum_{n \leq N} f_n) = 0 \) and polynomials \( p(t), q(t) \in \mathbb{Z}[t] \) such that \( q(1) \neq 0 \) and \( s(t) = p(t)f(t) \) and \( d(t) = q(t)f(t) \). Then the limits

\[
\lim_{t \to 1^-} \frac{s(t)}{d(t)}, \quad \lim_{N \to \infty} \frac{\sum_{n \leq N} s_n}{\sum_{n \leq N} d_n}
\]

exist and both equal \( p(1)/q(1) \).

Proof. As \( s(t)/d(t) = p(t)/q(t) \) when \(|t| < 1\) and \( q(t) \neq 0 \), it is immediate that the first limit exists and equals \( p(1)/q(1) \). Write \( p(t) = \sum_n p_nt^n \) and \( q(t) = \sum_n q_nt^n \). For \( N >> 0 \), there are \( x_N, y_N \in \mathbb{R} \) satisfying \(|x_N| \leq (\sum_n |p_n|)(\sum_{-\deg(p) \leq n \leq N} f_n) \) and \(|y_N| \leq (\sum_n |q_n|)(\sum_{\deg(q) \leq n \leq N} f_n) \) and such that

\[
\frac{\sum_{n \leq N} s_n}{\sum_{n \leq N} d_n} = \frac{p(1)\sum_{n \leq N} f_n + x_N}{q(1)f_N + y_N} = \frac{p(1) + x_N/\sum_{n \leq N} f_n}{q(1) + y_N/\sum_{n \leq N} f_n}.
\]

From the above bounds for \( x_N \) and \( y_N \) and the assumption that \( \lim_{N \to \infty} f_N/(\sum_{n \leq N} f_n) = 0 \) it follows that \( \lim_{N \to \infty} x_N/\sum_{n \leq N} f_n = \lim_{N \to \infty} y_N/\sum_{n \leq N} f_n = 0 \). As \( q(1) \neq 0 \), we have

\[
\lim_{N \to \infty} \frac{\sum_{n \leq N} s_n}{\sum_{n \leq N} d_n} = \frac{p(1)}{q(1)}
\]
as needed. \( \square \)

Lemma 1.4.2. Let \( c \in \mathbb{R} \) be a real parameter and let \( \lambda \in \text{Irr}(W) \), so that the irreducible lowest weight module \( L_c(\lambda) \) admits a \( W \)-invariant contravariant nondegenerate Hermitian form \( \beta_{c,\lambda} \), normalized to be positive definite on \( \lambda \). For any \( n \geq 0 \), let \( \beta_{c,\lambda}^{\leq n} \) denote the restriction of \( \beta \) to the space \( L_c(\lambda)^{\leq n} := \bigoplus_{k \leq n} L_c(\lambda)[k] \). Then the limits

\[
\lim_{n \to \infty} \frac{\text{sign}(\beta_{c,\lambda}^{\leq n})}{\dim L_c(\lambda)^{\leq n}}, \quad \lim_{t \to 1^-} \frac{\text{sch}_0(L_c(\lambda))}{\text{ch}_0(L_c(\lambda)(1,t)}
\]

exist and are equal to the same rational number \( a_{c,\lambda} \in [-1,1] \). If \( d := \dim \text{Supp}(L_c(\lambda)) > 0 \), then the limit

\[
\lim_{n \to \infty} \frac{\text{sign}(\beta_{c,\lambda}[n])}{\dim L_c(\lambda)[n]}
\]

also exists and equals \( a_{c,\lambda} \).

Proof. In the case \( d = 0 \) the claim is clear, so we may assume \( d > 0 \). Using Proposition 1.3.1, write \( \text{ch}_0(L_c(\lambda))(1,t) = \sum_{n=0}^{\infty} d_nt^n = (1 - t)^{-d}p(t) \) and \( \text{sch}_0(L_c(\lambda)) = \sum_{n=0}^{\infty} s_nt^n = (1 - t)^{-d}q(t) \), where \( p(t) \) and \( q(t) \) are polynomials with integer coefficients satisfying \( p(1) \neq 0 \), where we use the series expansion about 0 for \((1 - t)^{-d} \). In particular, Lemma 1.4.1 applies, and the first two limits in the lemma statement exist and equal \( q(1)/p(1) \in \mathbb{Q} \). As we clearly have \(|\text{sign}(\beta_{c,\lambda}^{\leq n})| \leq \dim L_c(\lambda)^{\leq n}\), we also have \( q(1)/p(1) \in [-1,1] \), giving the first claim. The final claim follows similarly, noting that for \( n \gg 0 \) we have that \( \text{sign}(\beta_{c,\lambda}[n]) \) and \( \dim L_c(\lambda)[n] \) are polynomials in \( n \) of degree at most \( d \) and with coefficients of \( n^d \) given by \( q(1)/(d-1)! \) and \( p(1)/(d-1)! \), respectively. \( \square \)
Definition 1.4.3. For $c \in \mathfrak{p}_\mathbb{R}$, the rational number $a_{c,\tau} \in [-1, 1]$ appearing in Lemma 1.4.2 is the asymptotic signature of $L_c(\lambda)$. If $a_{c,\lambda} = \pm 1$, we say $L_c(\lambda)$ is quasi-unitary.

1.5. Isotypic Signature Characters. Any $M \in \mathcal{O}_c(W, \mathfrak{h})$ decomposes as a direct sum $M = \bigoplus_{\pi \in \text{Irr}(W)} M^\pi$, where $M^\pi$ is the $\pi$-isotypic subspace of $M$. As the grading element $h \in H_c(W, \mathfrak{h})$ commutes with $W$, these isotypic subspaces $M^\pi$ are graded subspaces of $M$. If $\beta$ is a $W$-invariant Hermitian form on $M$, it follows that the central idempotent $e_\pi \in \mathbb{C}W$ is self-adjoint with respect to $\beta$, and in particular the $M^\pi$ are orthogonal with respect to $\beta$. When such a $\beta$ is graded, we define the $\pi$-isotypic signature character $\text{sch}^\pi(M, \beta) := \sum_{\sigma \in \Sigma} \text{sign}(M^\sigma, \beta)t^\sigma$. Clearly, we have $\text{sch}(M, \beta) = \sum_{\pi \in \text{Irr}(W)} \text{sch}^\pi(M, \beta)$. When $M$ is lowest weight, we define $\text{sch}_0^\pi(M) \in \mathbb{Z}[t]$ completely analogously to $\text{sch}_0(M)$. Similarly, let $\text{ch}^\pi(M)$ denote the graded dimension of $M^\pi$, and when $M$ is lowest weight let $\text{ch}_0^\pi(M) \in \mathbb{Z}^\geq_0[t] = t^{-h_c(\lambda)}\text{ch}^\pi(M)$.

The identification of any standard module $\Delta_c(\lambda)$ with $S\mathfrak{h}^* \otimes \lambda$ used in Section 1.1 respects the $W$-action. In particular, in the setting considered in that section, each isotypic subspace $\Delta_c(\lambda, \pi)$ is equipped with a Jantzen filtration $\{\Delta_c(\lambda, \pi, k)\}_{k \geq 0}$, and we have $\Delta_c(\lambda, \pi) \geq k = \bigoplus_{\pi \in \text{Irr}(W)} \Delta_c(\lambda, \pi, k) \geq k$. Furthermore, the wall-crossing formulas in Lemma 1.3.2 and the decomposition in Lemma 1.3.3 of an arbitrary signature character $\text{sch}(M, \beta)$ in terms of signature characters $\text{sch}(L_i)$ of irreducible representations, and their proofs, have direct analogues for the isotypic signatures characters $\text{sch}^\pi$.

Let $l = \dim \mathfrak{h}$, let $\{d_i\}_{i=1}^l$ denote the fundamental degrees of $W$, and let $\mathbb{C}[\mathfrak{h}]^{\text{co}W} := \mathbb{C}[\mathfrak{h}] / \mathbb{C}[\mathfrak{h}]((\mathbb{C}[\mathfrak{h}]^W)^*)$ be the coinvariant algebra. Recall that $\mathbb{C}[\mathfrak{h}]^{\text{co}W}$ is graded and is isomorphic to $\mathbb{C}W$ as a $\mathbb{C}W$-module. For any $\mu, \pi \in \text{Irr}(W)$, let $\theta_\mu \in \mathbb{Z}^{\geq_0}[t]$ denote the graded dimension of the $\pi$-isotypic subspace of $\mu \otimes \mathbb{C}[\mathfrak{h}]^{\text{co}W}$.

Lemma 1.5.1. For any $\mu, \pi \in \text{Irr}(W)$, we have:

1. $\theta^\tau_\mu(1) = (\dim \mu)(\dim \pi)^2$

2. $\text{ch}_0^\pi(\Delta_0(\mu)) = \text{sch}_0^\pi(\Delta_0(\mu)) = \theta^\pi_\mu / \prod_{i=1}^l (1 - t^{d_i})$

3. $\lim_{n \to \infty} \frac{\dim \Delta_0(\mu, \pi, \leq n)}{\dim \Delta_0(\mu, \leq n)} = \lim_{t \to 1^-} \frac{\text{ch}_0^\pi(\Delta_0(\mu))(1, t)}{\text{ch}_0^\pi(\Delta_0(\mu))(1, t)} = \frac{(\dim \pi)^2}{|W|}$

Proof. Clearly $\theta^\tau_\mu(1)$ is the dimension of the $\pi$-isotypic subspace of $\mu \otimes \mathbb{C}[\mathfrak{h}]^{\text{co}W}$. As $\mathbb{C}[\mathfrak{h}]^{\text{co}W} \cong \mathbb{C}W$ as a $\mathbb{C}W$-module and as $\mu \otimes \mathbb{C}W \cong \mathbb{C}W \otimes \mathbb{C}W^{\dim \mu}$, the first statement follows. The second statement follows from the fact that $\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^{\text{co}W} \otimes \mathbb{C}[\mathfrak{h}]^W$ as a graded $\mathbb{C}W$-module. The third statement follows from the first two statements and from Lemma 1.4.1, using the facts that $\text{ch}_0(\Delta_\lambda(t))(1, t) = (\dim \mu)P_W(t) / \prod_{i=1}^l (1 - t^{d_i})$, where $P_W(t)$ is the Poincaré polynomial of $W$, and $P_W(1) = |W|$. \hfill \Box

Note that the signs $\varepsilon_\pi$ appearing in the $\pi$-isotypic version of Lemma 1.3.3 do not depend on the irreducible representation $\pi$. In particular, it follows from Lemma 1.5.1 and the proof of Proposition 1.3.1 that we have:

Lemma 1.5.2. For any $c \in \mathfrak{p}_\mathbb{R}$, there exists a collection of polynomials $\{n_c^{\lambda, \mu} \in \mathbb{Z}[t] : \lambda, \mu \in \text{Irr}(W)\}$ such that for every $\lambda, \mu \in \text{Irr}(W)$ we have

$$\text{sch}_0^\pi(L_c(\lambda)) = \sum_{\mu \in \text{Irr}(W)} n_c^{\lambda, \mu} \text{sch}_0^\pi(\Delta_0(\mu))$$
\[ \prod_{i=1}^{l} (1 - t^{d_i})^{-1} \sum_{\mu \in \text{Irr}(W)} n_{c}^{\lambda, \mu} \theta_{\mu}^{\pi}. \]

In particular, \( \text{sch}_{0}^\pi(L_{c}(\lambda)) \) is a rational function.

**Corollary 1.5.3.** In the setting of Lemma 1.5.2, the rational function \( \text{sch}_{0}^\pi(L_{c}(\lambda)) \) has a pole of order at most \( d := \dim \text{Supp}(L_{c}(\lambda)) \) at \( t = 1 \).

**Proof.** By Lemma 1.5.2, we see that \( \text{sch}_{0}^\pi(L_{c}(\lambda)) \) is absolutely convergent on the interval \((-1, 1)\), and by a comparison of coefficients we see that the rational function

\[ \text{sch}_{0}^\pi(L_{c}(\lambda))/\text{ch}_{0}(L_{c}(\lambda))(1, t) \]

takes values in \([-1, 1]\) on the interval \([0, 1]\). As \( \text{ch}_{0}(L_{c}(\lambda))(1, t) \) has a pole of order \( d \) at \( t = 1 \), the claim follows.

The following proposition allows the asymptotic signature \( a_{c, \lambda} \) of \( L_{c}(\lambda) \) to be computed in the spherical component when \( L_{c}(\lambda) \) has full support. Considering the case when \( L_{c}(\lambda) \) is finite-dimensional, note that this need not be true when \( L_{c}(\lambda) \) has proper support.

**Proposition 1.5.4.** Let \( c \in \mathfrak{p}_{\mathbb{R}}, \lambda \in \text{Irr}(W) \), and suppose \( L_{c}(\lambda) \) has full support. Then for all \( \pi \in \text{Irr}(W) \) the limit

\[ \lim_{n \to \infty} \frac{\text{sign}(\beta_{c, \lambda}^{\pi, \leq n})}{\dim L_{c}(\lambda)^{\pi, \leq n}} \]

exists and equals \( a_{c, \lambda} \). In particular, this limit is independent of \( \pi \).

**Proof.** As we have

\[ a_{c, \lambda} = \lim_{n \to \infty} \frac{\text{sign}(\beta_{c, \lambda}^{\leq n})}{\dim L_{c}(\lambda)^{\leq n}} = \lim_{n \to \infty} \frac{\sum_{\pi \in \text{Irr}(W)} \text{sign}(\beta_{c, \lambda}^{\pi, \leq n})}{\sum_{\pi \in \text{Irr}(W)} \dim L_{c}(\lambda)^{\pi, \leq n}} \]

it suffices to show that the limit in the proposition statement exists and is independent of \( \pi \). As \( L_{c}(\lambda) \) has full support, it follows from Lemma 1.5.1(3) that for any \( \pi \in \text{Irr}(W) \) we have

\[ \lim_{n \to \infty} \frac{\dim L_{c}(\lambda)^{\pi, \leq n}}{\dim L_{c}(\lambda)^{\leq n}} = \lim_{t \to 1^{-}} \frac{\text{ch}_{0}^{\pi}(L_{c}(\lambda))}{\text{ch}_{0}(L_{c}(\lambda))(1, t)} = \frac{(\dim \pi)^2}{|W|}. \]

By Lemmas 1.4.1 and 1.5.2, the limit in the lemma statement exists and we have

\[ \lim_{n \to \infty} \frac{\text{sign}(\beta_{c, \lambda}^{\pi, \leq n})}{\dim L_{c}(\lambda)^{\pi, \leq n}} = \lim_{t \to 1^{-}} \frac{\text{sch}_{0}^{\pi}(L_{c}(\lambda))}{\text{ch}_{0}(L_{c}(\lambda))(1, t)} \]

\[ = \lim_{t \to 1^{-}} \frac{\text{sch}_{0}^{\pi}(L_{c}(\lambda)) \cdot \text{ch}_{0}(L_{c}(\lambda))(1, t)}{\text{ch}_{0}(L_{c}(\lambda))(1, t) \cdot \text{ch}_{0}(L_{c}(\lambda))(1, t)} \]

\[ = \frac{\sum_{\mu \in \text{Irr}(W)} n_{c}^{\lambda, \mu}(1) \theta_{\mu}^{\pi}(1)}{(\dim \lambda) P_{W}(1) \cdot (\dim \pi)^2} \]

\[ = \frac{\sum_{\mu \in \text{Irr}(W)} n_{c}^{\lambda, \mu}(1)(\dim \mu)}{\dim \lambda}, \]

which is visibly independent of \( \pi \), as needed.

\[ \square \]
2. The Dunkl Weight Function

2.1. The Gaussian Inner Product. Assume $W$ is a finite real reflection group with real reflection representation $\mathfrak{h}_{\mathbb{R}}$, and let $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$ denote the complexified reflection representation. Let $c \in \mathfrak{p}_{\mathbb{R}}$; in particular, $c$ is real-valued and $c = \overline{c} = c^\dagger$. Let $y_1, \ldots, y_l$ be a basis of $\mathfrak{h}_{\mathbb{R}}$, and let $x_1, \ldots, x_l$ be the corresponding dual basis of $\mathfrak{h}^*_{\mathbb{R}}$. In this setting, $H_c(W, \mathfrak{h})$ contains an internal $\mathfrak{s}\mathfrak{l}_2$-triple:

$$\mathfrak{h} = \sum_i x_i y_i + \frac{1}{2} \dim \mathfrak{h} - \sum_{s \in S} c_s s = \sum_i (x_i y_i + y_i x_i)/2, \quad e := -\frac{1}{2} \sum_i x_i^2 \quad f := \frac{1}{2} \sum_i y_i^2.$$

As $f$ acts locally nilpotently on any module $M$ in $\mathcal{O}_c(W, \mathfrak{h})$, we can consider the operator $\exp(f)$ on $M$. In particular, following [1] and [ES], when $M$ is lowest weight with lowest weight $\lambda \in \text{Irr}(W)$ with $W$-invariant contravariant Hermitian form $\beta_{\mathfrak{c},\lambda}$, positive definite in the lowest weight space, we define the Gaussian inner product $\gamma_{\mathfrak{c},\lambda}$ on $M$:

**Definition 2.1.1.** ([ES, Definition 4.5]) The Gaussian inner product $\gamma_{\mathfrak{c},\lambda}$ on $M$ is the Hermitian form defined by

$$\gamma_{\mathfrak{c},\lambda}(v, v') = \beta_{\mathfrak{c},\lambda}(\exp(f)v, \exp(f)v').$$

For nonnegative integers $n \geq 0$, let $\gamma_{\mathfrak{c},\lambda}^{\leq n}$ denote the restriction of the form $\gamma_{\mathfrak{c},\lambda}$ to $M^{\leq n}$. As $\exp(f)$ preserves $M^{\leq n}$, we have $\text{sign}(\beta_{\mathfrak{c},\lambda}^{\leq n}) = \text{sign}(\gamma_{\mathfrak{c},\lambda}^{\leq n})$ for all $n$. In particular, the asymptotic signature $a_{\mathfrak{c},\lambda}$ can be equally well computed using the form $\gamma_{\mathfrak{c},\lambda}$.

We will need the following facts about $\gamma_{\mathfrak{c},\lambda}$ from [ES]:

**Proposition 2.1.2.** ([ES, Proposition 4.6])

(i) The form $\gamma_{\mathfrak{c},\lambda}$ satisfies: $\gamma_{\mathfrak{c},\lambda}(xv, v') = \gamma_{\mathfrak{c},\lambda}(v, xv')$ for all $x \in \mathfrak{h}_{\mathbb{R}}^*$. (ii) Up to scaling, $\gamma_{\mathfrak{c},\lambda}$ is the unique $W$-invariant Hermitian form on $M$ satisfying $\gamma_{\mathfrak{c},\lambda}((-y + Ty)v, v') = \gamma_{\mathfrak{c},\lambda}(v, yv')$ for all $y \in \mathfrak{h}_{\mathbb{R}}$.

2.2. Weight Function. Suppose there is a tempered distribution $K_{\mathfrak{c},\lambda}$ on $\mathfrak{h}_{\mathbb{R}}$ with values in $\text{Herm}(\lambda)$ representing $\gamma_{\mathfrak{c},\lambda}$ in the sense that

$$\gamma_{\mathfrak{c},\lambda}(P, Q) = \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^\dagger K_{\mathfrak{c},\lambda}(x) P(x)e^{-|x|^2/2}dx \quad \text{for } P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda.$$

As the space $\mathbb{R}[\mathfrak{h}_{\mathbb{R}}]e^{-|x|^2/2}$ is dense in the space $\mathcal{S}'(\mathfrak{h}_{\mathbb{R}})$ of tempered distributions on $\mathfrak{h}_{\mathbb{R}}$, it follows that such $K_{\mathfrak{c},\lambda}$ is unique if it exists. It follows from this uniqueness and the $W$-invariance of $\gamma_{\mathfrak{c},\lambda}$ that $K_{\mathfrak{c},\lambda}$ is $W$-invariant in the sense that

$$K_{\mathfrak{c},\lambda}(wx) = wK(x)w^{-1}$$

for all $w \in W$.

**Lemma 2.2.1.** Let $\Delta := \prod_{s \in S} \alpha_s$, and suppose $K_{\mathfrak{c},\lambda} \in \mathcal{S}'(\mathfrak{h}_{\mathbb{R}}) \otimes \text{Herm}(\lambda)$ is a tempered distribution representing $\gamma_{\mathfrak{c},\lambda}$ as above. For each $y \in \mathfrak{h}_{\mathbb{R}}$ we have

$$\Delta \partial_y K_{\mathfrak{c},\lambda} + \sum_{s \in S} c_s \alpha_s(y) \frac{\Delta}{\alpha_s}(sK_{\mathfrak{c},\lambda} + K_{\mathfrak{c},\lambda}s) = 0.$$
in the sense of tempered distributions. In particular, the restriction of $K_{c,\lambda}$ to a distribution on $\mathfrak{h}_{\mathbb{R},\text{reg}}$ satisfies the differential equation

\begin{equation}
\partial_y K_{c,\lambda} + \sum_{s \in S} \frac{c_s \alpha_s(y)}{\alpha_s} (sK_{c,\lambda} + K_{c,\lambda}s) = 0
\end{equation}

for all $y \in \mathfrak{h}_{\mathbb{R}}$.

Proof. Recall that the action of any $P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C} \lambda$ is given by the Dunkl operator

$$y \mapsto \partial_y \otimes 1 - \sum_{s \in S} \frac{c_s \alpha_s(y)}{\alpha_s} (1 - s) \otimes s.$$ 

As $Ty_i = x_i$, the property (ii) in Proposition 2.1.2 implies that for any $P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$ we have (replacing $Q$ by $\Delta Q$)

$$\int_{\mathfrak{h}_{\mathbb{R}}} \Delta Q(x)^{\dagger} K_{c,\lambda}(\partial_i \otimes 1 - \sum_{s \in S} \frac{c_s \alpha_s(y_i)}{\alpha_s} (1 - s) \otimes s)(P(x)) e^{-|x|^2/2} dx$$

$$= \int_{\mathfrak{h}_{\mathbb{R}}} (x_i - \partial_i \otimes 1 + \sum_{s \in S} \frac{c_s \alpha_s(y_i)}{\alpha_s} (1 - s) \otimes s)(\Delta Q(x)^{\dagger} K_{c,\lambda} P(x) e^{-|x|^2/2} dx$$

where $\partial_i := \partial_{y_i}$. As $\partial_i e^{-|x|^2/2} = -x_i e^{-|x|^2/2}$, the equality above can be rewritten as

$$0 = \int_{\mathfrak{h}_{\mathbb{R}}} Q(x)^{\dagger} (\Delta \partial_i K_{c,\lambda}) P(x) d\mu(x)$$

$$+ \int_{\mathfrak{h}_{\mathbb{R}}} \sum_{s \in S} \frac{c_s \alpha_s(y_i)}{\alpha_s} \left( (1 - s)(\Delta Q(x)^{\dagger} K_{c,\lambda} P(x) + \Delta Q(x)^{\dagger} K_{c,\lambda}(1 - s)(P(x))) d\mu(x)$$

where $d\mu(x) := e^{-|x|^2/2} dx$. It follows from the $W$-invariance of $K_{c,\lambda}$ and the fact that $\alpha_s(sx) = -\alpha_s(x)$ that we have

$$\int_{\mathfrak{h}_{\mathbb{R}}} \sum_{s \in S} \frac{c_s \alpha_s(y_i)}{\alpha_s} (s \otimes s)(\Delta Q(x)^{\dagger} K_{c,\lambda} P(x) + \Delta Q(x)^{\dagger} K_{c,\lambda}(s \otimes s)(P(x))) d\mu(x) = 0.$$ 

As $P, Q$ were arbitrary, it follows from the computations above that the distribution

$$\Delta \partial_i K_{c,\lambda} + \sum_{s \in S} \frac{c_s \alpha_s(y_i)}{\alpha_s} \left( sK_{c,\lambda} + K_{c,\lambda}s \right)$$

is identically 0, as it vanishes on the dense subspace $\mathbb{R}[\mathfrak{h}_{\mathbb{R}}] e^{-|x|^2/2} \subset \mathcal{S}(\mathfrak{h}_{\mathbb{R}})$, giving the first statement. As $\Delta$ is non-vanishing on $\mathfrak{h}_{\mathbb{R},\text{reg}}$, the second claim follows as well.

Definition 2.2.2. For any $s \in S$, let $b_s$ denote the operator $\alpha_s^{-1}(1 - s)$ on functions on $\mathfrak{h}_{\mathbb{R}}$.

Lemma 2.2.3. The operator $b_s$ preserves the spaces of polynomials, analytic functions, smooth functions, smooth compactly support functions, and Schwartz functions on $\mathfrak{h}_{\mathbb{R}}$. In particular, $b_s$ is a continuous operator on $\mathcal{S}(\mathfrak{h}_{\mathbb{R}})$ and $C^\infty_c(\mathfrak{h}_{\mathbb{R}})$.

Proof. That $b_s$ preserves polynomials and analytic functions is clear. Considering Taylor approximations, it is easy to see that $b_s$ preserves smooth functions and is a continuous operator on $\mathcal{S}(\mathfrak{h}_{\mathbb{R}})$ and on $C^\infty_c(\mathfrak{h}_{\mathbb{R}})$. \qed
It follows that the Dunkl operator

\[ D_y = \partial_y \otimes 1 - \sum_{s \in S} \frac{c_s \alpha_s(y)}{\alpha_s} (1 - s) \otimes s \]

defines a continuous operator on \( S(\mathfrak{h}_\mathbb{R}) \otimes \lambda \).

**Lemma 2.2.4.** Let \( K \) be as in Lemma 2.2.1. Then, for every \( y \in \mathfrak{h}_\mathbb{R} \) and \( \varphi \in S(\mathfrak{h}_\mathbb{R}) \otimes \lambda \) we have \( K(D_y \varphi) = 0 \). Conversely, any nonzero \( K \in S(\mathfrak{h}_\mathbb{R}) \otimes \text{Herm}(\lambda) \) that satisfies the above equation and is \( W \)-equivariant represents the form \( \gamma_{c,\lambda} \) up to scaling.

**Proof.** A direct computation shows that the vanishing of \( K(D_y \varphi) \) for all \( y \in \mathfrak{h}_\mathbb{R} \) and \( \varphi \in \mathbb{R}[\mathfrak{h}_\mathbb{R}]e^{-|x|^2/2} \otimes \lambda \) is precisely equivalent to the adjointness condition \( y^\dagger = T^{-1}(y) - y \) for the form \( \gamma_{c,\lambda} \). By the continuity of \( K \circ D_y \) and density of \( \mathbb{R}[\mathfrak{h}_\mathbb{R}]e^{-|x|^2/2} \) in \( S(\mathfrak{h}_\mathbb{R}) \), this is in turn equivalent to the vanishing of \( K(D_y \varphi) \) for all \( y \in \mathfrak{h}_\mathbb{R} \) and \( \varphi \in S(\mathfrak{h}_\mathbb{R}) \otimes \lambda \). The converse statement follows from the uniqueness statement in [ES, Proposition 4.6]. \( \square \)

**Lemma 2.2.5.** Let \( \lambda \in \text{Irr}(W) \), and let \( K \) be a \( \text{End}(\lambda) \)-valued analytic function on \( \mathfrak{h}_{\mathbb{R},\text{reg}} \) satisfying the differential equation

\[ \partial_y K + \sum_{s \in S} \frac{c_s \alpha_s(y)}{\alpha_s} (sK + Ks) = 0 \]

for each \( y \in \mathfrak{h}_\mathbb{R} \). Then \( K \) is homogeneous of degree \(-2\chi_\lambda(\sum_{s \in S} c_s s) / \dim \lambda \).

**Proof.** For any \( v \in \mathfrak{h}_{\mathbb{R},\text{reg}} \) we have for \( t > 0 \)

\[ \frac{d}{dt} K(tv) = \partial_y K(tv) = -\sum_{s \in S} \frac{c_s \alpha_s(v)}{\alpha_s(tv)} (sK(tv) + K(tv)s) \]

\[ = -\frac{1}{t} \sum_{s \in S} c_s (sK(tv) + K(tv)s) - \frac{1}{t} \frac{2\chi_\lambda(\sum_{s \in S} c_s s)}{\dim \lambda} K(tv), \]

where the last equality holds because the central element \( \sum_{s \in S} c_s s \) acts on \( \text{End}(\lambda) \) on both the right and left by the scalar \( \chi_\lambda(\sum_{s \in S} c_s s) / \dim \lambda \). In particular, we have \( K(tv) = t^{-2\chi_\lambda(\sum_{s \in S} c_s s) / \dim \lambda} K(v) \) for all \( t > 0 \). \( \square \)

**Lemma 2.2.6.** Let \( c \in \mathfrak{p}_\mathbb{R} \), let \( \lambda \in \text{Irr}(W) \), let \( K_0 \in \text{End}(\lambda) \), and let \( x_0 \in \mathfrak{h}_{\mathbb{R},\text{reg}} \). Then there is a unique analytic function \( K : \mathfrak{h}_{\mathbb{R},\text{reg}} \to \text{End}(\lambda) \) satisfying

1. \( K(x_0) = K_0 \)
2. \( K(wx) = wK(x)w^{-1} \) for all \( w \in W \)
3. \( \partial_y K + \sum_{s \in S} \frac{c_s \alpha_s(y)}{\alpha_s} (sK + Ks) = 0 \) for all \( y \in \mathfrak{h}_\mathbb{R} \).

In particular, on the connected component \( \mathcal{C} \) of \( \mathfrak{h}_{\mathbb{R},\text{reg}} \) containing \( x_0 \) the function \( K \) is given by

\[ K(x) = f(x) M(x) K_0 (f(x) M(x))^\dagger \]

where \( M(x) \) is the monodromy of the KZ connection

\[ \nabla_{\text{KZ}} := d - \sum_{s \in S} c_s \frac{d \alpha_s}{\alpha_s} (1 - s) \]
on the trivial bundle $\lambda \times h_{\mathbb{R},\text{reg}} \to h_{\mathbb{R},\text{reg}}$ from $x_0$ to $x$ in $C$ and where $f(x)$ is the (scalar) monodromy of the scalar-valued flat connection

$$d + \sum_{s \in S} c_s \frac{d\alpha_s}{\alpha_s}$$
onumber

on the trivial bundle $\mathbb{R} \times h_{\mathbb{R},\text{reg}} \to h_{\mathbb{R},\text{reg}}$ from $x_0$ to $x$ in $C$. In particular, if $K_0$ represents a Hermitian form on $\lambda$ then $K(x)$ represents an equivalent Hermitian form for all $x \in h_{\mathbb{R},\text{reg}}$.

Proof. Any $K : h_{\mathbb{R},\text{reg}} \to \text{End}(\lambda)$ satisfying the conditions (1)-(3) must be unique, so we need only check that the unique $W$-equivariant function $K(x)$ given by

$$K(x) = f(x)M(x)K_0(f(x)M(x))^†$$

on $C$ satisfies those conditions. Clearly such $K$ satisfies (1) and (2), and (3) follows by a straightforward calculation. □

We will assume that the roots $\alpha_s \in h_{\mathbb{R}}^*$ have been chosen to form a system of positive roots. Let $\alpha_1, ..., \alpha_l$ be the associated simple roots, and let $C := \{x \in h_{\mathbb{R}} : \alpha_i(x) > 0 \text{ for } i = 1, ..., l\}$ be the associated fundamental Weyl chamber.

**Lemma 2.2.7.** Let $\lambda \in \text{Irr}(W)$ and $c \in p_{\mathbb{R}}$. Any analytic function $K : h_{\mathbb{R},\text{reg}} \to \text{End}(\lambda)$ satisfying the system of differential equations

$$\partial_y K + \sum_{s \in S} \frac{c_s \alpha_s(y)}{\alpha_s}(sK + Ks) = 0 \quad \text{for all } y \in h_{\mathbb{R}}$$

is a locally $L^1$ function if $|c_s|$ is sufficiently small for all $s \in S$.

Proof. Let $K : h_{\mathbb{R},\text{reg}} \to \text{End}(\lambda)$ satisfy the differential equation above. To see that $K$ is locally integrable it suffices to check that $K$ is locally integrable on $C \subset h_{\mathbb{R},\text{reg}}$, the positive Weyl chamber associated to the positive roots $\alpha_s$. Let $\omega_1^\vee, ..., \omega_l^\vee \in \overline{C}$ be the fundamental dominant coweights, so that $\alpha_i(\omega_j^\vee) = \delta_{ij}$. As $K$ is homogeneous by Lemma 2.2.5, it suffices to show that $K$ is integrable over the region

$$R := \{x \in h_{\mathbb{R},\text{reg}} : \alpha_i(x) \in (0, 1), i = 1, ..., l\} \subset C.$$

For a matrix $A = (a_{ij}) \in \text{End}(\lambda)$, let $||A|| := (\sum_{ij} a_{ij}^2)^{1/2}$ denote the matrix norm (the choice of basis involved is unimportant). Given a point $v \in C$, we have, for those $t > 0$ such that $\alpha_i(v - t\omega_j^\vee) > 0$,

$$\frac{d}{dt}||K(v - t\omega_j^\vee)|| \leq ||\frac{d}{dt}K(v - t\omega_j^\vee)|| = \left|\left| \sum_{s \in S} \frac{c_s \alpha_s(\omega_j^\vee)}{\alpha_s(v - t\omega_j^\vee)} (sK(v - t\omega_j^\vee) + K(v - t\omega_j^\vee)s) \right|\right|$$

$$\leq \sum_{s \in S} \frac{2||c_s||s||\alpha_s(\omega_j^\vee)||}{\alpha_s(v - t\omega_j^\vee)} ||K(v - t\omega_j^\vee)||.$$

By integration, we see that

$$||K(v - t\omega_j^\vee)|| \leq ||K(v)|| \prod_{s \in S} \left( \frac{\alpha_s(v)}{\alpha_s(v - t\omega_j^\vee)} \right)^{2||c_s||s}.$$

With $\rho^\vee := \sum \omega_i^\vee \in C$, we therefore have

$$||K(x)|| \leq ||K(\rho^\vee)|| \prod_{s \in S}(\alpha_s(\rho^\vee)/\alpha_s(x))^{2||c_s||s} \tag{2.2}$$
for all $x \in R$, and the claim follows immediately from this estimate. \qed

We need to impose further conditions on such an analytic function $K$ in order for it to represent the form $\gamma_{c,\lambda}$. In [D2, Section 5] and [D1, Equation 9], Dunkl derived certain conditions on such $K$ near the boundaries of the Weyl chambers. We will now establish similar conditions in terms of the nature of the singularities of $K$ along the reflection hyperplanes, and we will relate these conditions to the invariance of Hermitian forms on representations of the Hecke algebra.

**Definition 2.2.8.** For each simple reflection $s_i$, let
\[ C_i := \{ x \in h_R : \alpha_i(x) = 0, \alpha_j(x) > 0 \text{ for } j \neq i \} \]
be the open codimension-1 face of $C$ determined by $\alpha_i$.

Let $\lambda \in \text{Irr}(W)$ and let $c \in p$ be such that $c_s \notin \frac{1}{2}\ZZ \setminus \{0\}$ for all $s \in S$. Fix a simple reflection $s_i$, and choose coordinates $z_1, \ldots, z_l \in h_R^* \subset h^*$ for $h$ with $z_1 = \alpha_i$ and $z_j(\alpha_i^{\vee}) = 0$ for $j > 1$. Consider the modified $KZ$ connection
\[ \nabla'_{KZ} := d + \sum_{s \in S} c_s \frac{d\alpha_s}{\alpha_s} s \]
on the trivial vector bundle on $h_{reg}$ with fiber $\lambda$. The connection $\nabla'_{KZ}$ is flat and therefore has a unique local extension of any initial value in the fiber to a homomorphic flat section. Furthermore, considering the restriction of $\nabla'_{KZ}$ in the $z_1$ direction near a point $x_0 \in C_i$ lying on the reflection hyperplane $\ker(\alpha_i)$, it follows from the standard theory of ordinary differential equations with regular singularities (see, e.g. [W, Theorem 5.5]) that there is a holomorphic function $P_i(z)$, defined in a complex analytic $z_1$-disc about $x_0$ and satisfying $P_i(x_0) = \text{Id}$, such that the function
\[ z \mapsto P_i(z)z_1^{-c_i s_i} \]
gives a fundamental $\text{End}_C(\lambda)$-valued (multivalued) solution to the restriction of the system $\nabla'_{KZ}$ to a small punctured $z_1$-disc about $x_0$. We may extend $P_i(z)$ in the $z_2, \ldots, z_l$ directions by taking solutions of the restricted connection
\[ \nabla''_{KZ} := d + \sum_{s \in S \setminus \{s_i\}} c_s \frac{d\alpha_s}{\alpha_s} s \]
along affine hyperplanes parallel to $\ker(\alpha_i)$ to produce a single valued function $P_i(z)$, holomorphic in $z_2, \ldots, z_l$, defined in a complex analytic neighborhood of $D$ of $C \cup C_i \cup s_i(C)$ (note that the restricted connection above is regular along $C_i$ itself). The independence of the function $z_1^{-c_i s_i}$ on the variables $z_2, \ldots, z_l$ and the uniqueness of extensions of solutions of $\nabla'_{KZ}$ then implies that the such that the function
\[ N_i(z) := P_i(z)z_1^{-c_i s_i} \]
gives a fundamental $\text{End}_C(\lambda)$-valued (multivalued) solution to the system $\nabla'_{KZ}$ on the region $D_{reg} := D \cap h_{reg}$. In particular, $P_i(z)$ is holomorphic on $D_{reg}$. Continuous dependence of solutions to ordinary differential equations on parameters and initial conditions implies that $P_i(z)$ is continuous on all of $D$. In particular, viewing $P_i(z)$ as a function on $D_{reg}$ holomorphic in $z_1$, the singularities of $P_i(z)$ along $\ker(\alpha_i)$ are removable. It follows that $P_i(z)$ is holomorphic in $z_1$ on all of $D$. As $P_i(z)$ is holomorphic in each $z_j$ for $j > 1$, it follows by Hartogs' theorem that $P_i(z)$ is holomorphic on all of $D$. 


We next derive further properties of $P_i(z)$ arising from $W$-equivariance of the connection $\nabla''_{KZ}$ and the initial condition $P_i(x_0) = \text{Id}$. In particular, we see that $s_i N_i(s_i z)s_i$ is another multivalued fundamental solution of $\nabla'_{KZ}$ on $D_{\text{reg}}$. As $s_i N_i(s_i z)s_i = s_i P_i(s_i z)s_i(-z_1)^{-c_i s_i} = s_i P_i(s_i z)s_i z_1^{-c_i s_i} e^{-\pi i c_i s_i}$, it follows that $s_i P_i(s_i z)s_i z_1^{-c_i s_i}$ is such a solution as well. By the proof of [W, Theorem 5.5], in particular its use of [W, Theorem 4.1], and the assumption that $c_i \notin \frac{1}{2} \mathbb{Z}\backslash \{0\}$, any function $\tilde{P}_i(z)$ holomorphic in a small complex analytic $z_1$-disc about $x_0$ such that $\tilde{P}_i(x_0) = \text{Id}$ and the function $\tilde{P}_i(z) z_1^{-c_i s_i}$ is a solution to the system $\nabla'_{KZ}$ on the punctured $z_1$-disc must coincide with $P_i(z)$ along the entire disc. It follows that $s_i P_i(s_i z)s_i = P_i(z)$ for $z$ in a small complex analytic $z_1$-disc about $x_0$. Taking $D$ to be $s_i$-stable, it then follows by continuation in the variables $z_2, ..., z_l$ using the system $\nabla''_{KZ}$ that $s_i P_i(s_i z)s_i = P_i(z)$ for all $z \in D$.

We summarize the conclusions of the above discussion in the following lemma:

**Lemma 2.2.9.** Let $\lambda \in \text{Irr}(W)$, let $c \in \mathfrak{p}$ be such that $c_i \notin \frac{1}{2} \mathbb{Z}\backslash \{0\}$ for a given simple reflection $s_i \in S$, and let $z_1, ..., z_l$ be coordinates for $\mathfrak{h}$ as in the discussion above. There is a $s_i$-stable complex analytic neighborhood $D$, independent of $c$ and circled with respect to the coordinate $z_1$, of $\mathcal{C} \cup \mathcal{C} \cup s_i(\mathcal{C})$ in $\mathfrak{h}$ and a holomorphic $GL_C(\lambda)$-valued function $P_i(z)$ on $D$ such that

1. $P_i(z) \alpha_i(z)^{-c_i s_i}$ is a $\text{End}_C(\lambda)$-valued multivalued holomorphic fundamental solution to the modified $KZ$ system

$$\nabla''_{KZ} := d + \sum_{s \in S} c_s \frac{d \alpha_s}{\alpha_s} s$$

on the domain $D_{\text{reg}} := D \cap \mathfrak{h}_{\text{reg}}$

2. $s_i P_i(s_i z)s_i = P_i(z)$ for all $z \in D$. In particular, $P_i$ takes values in $\text{End}_{s_i}(\lambda)$ along $\text{ker}(\alpha_i) \cap D$.

3. $P_i(z)$ is a solution of the system

$$\nabla''_{KZ} := d + \sum_{s \in S \setminus \{s_i\}} c_s \frac{d \alpha_s}{\alpha_s} s$$

along $\text{ker}(\alpha_i) \cap D$.

Furthermore, any such $P_i$ is determined by its value at any point $x_0 \in \mathcal{C}$, and for any constant invertible $A \in \text{Aut}_{s_i}(\lambda)$ the function $P_i(z)A$ is another such function. In particular, any two such functions $P_i(z), \tilde{P}_i(z)$ are related by an equality $\tilde{P}_i(z) = P_i(z)A$ for a unique such $A$.

Now, let $K : \mathcal{C} \to \text{End}_C(\lambda)$ be a real analytic function satisfying the differential equation

$$dK + \sum_{s \in S} c_s \frac{d \alpha_s}{\alpha_s} (sK + Ks) = 0$$

for all $y \in \mathfrak{h}_\mathbb{R}$. Choose a simple reflection $s_i$, and fix a function $P_i(z)$ and domain $D$ as in Lemma 2.2.9. By the proof of Lemma 2.2.6, there is a matrix $K_i \in \text{End}_C(\lambda)$, clearly uniquely determined given $P_i(z)$, such that the multivalued $\text{End}_C(\lambda)$-valued holomorphic function

$$z \mapsto P_i(z) \alpha_i(z)^{-c_i s_i} K_i(P_i(z) \alpha_i(z)^{-c_i s_i})$$

is a solution of the system (2.3) on $D$ and coincides with $K$ on $\mathcal{C}$. Here, as before, the Hermitian transpose $\dagger$ on $\text{End}_C(\lambda)$ is defined with respect to the chosen $W$-invariant non-degenerate Hermitian form on $\lambda$. As $P_i(z)$ is determined only up to right multiplication by
some $A \in \text{Aut}_{s_i}(\lambda)$, here $K_i$ is determined by $K$ only by the action of $\text{Aut}_{s_i}(\lambda)$ on $\text{End}_{C}(\lambda)$ by $A.M = A\overline{M}A^T$. As $s_i^{-1} = s_i$, $s_i$-invariance of $A$ implies that of $A^\dagger$, and we see that $s_i$-invariance of $K_i$ is a property of $K$, not depending on the choice of $P_i(z)$. In particular, we make the following definition:

**Definition 2.2.10.** In the setting of the previous paragraph, we say that the function $K$ is asymptotically $W$-invariant if $K_i \in \text{End}_{s_i}(\lambda)$ for all simple reflections $s_i$. 

**Theorem 2.2.11** (Existence of Dunkl Weight Function for Small $c$). Let $\lambda \in \text{Irr}(W)$ and let $c \in \mathfrak{p}_\mathbb{R}$ be a real parameter with each $|c_s|$ sufficiently small. Let $K : \mathfrak{h}_{\mathbb{R},\text{reg}} \to \text{Herm}(\lambda)$ be a nonzero $W$-equivariant real analytic function satisfying the system (2.3). Then the following are equivalent:

(a) $K$ represents the form $\gamma_{c,\lambda}$ up to rescaling by a nonzero real number.

(b) For any point $x \in \mathfrak{h}_{\mathbb{R},\text{reg}}$, $K(x)$ determines a $B_W$-invariant nondegenerate Hermitian form on $KZ_x(\Delta_c(\lambda))$, where $B_W := \pi_1(\text{reg}/W, x)$ is the braid group and $KZ_x$ denotes the $KZ$ functor $KZ_x : \mathcal{O}_c(W, \mathfrak{h}) \to H_q(W)/\text{mod}_{f.d.}$ obtained by taking the monodromy representation at $x$.

(c) $K|_x$ is asymptotically $W$-invariant in the sense of Definition 2.2.10.

Furthermore, the space of such $K$ satisfying (a)-(c) forms a one-dimensional real vector space.

**Proof.** We will first show that (a) and (c) are equivalent. As the function $K$ is locally integrable by Lemma 2.2.7 and homogeneous by Lemma 2.2.5 it determines a tempered distribution with values in $\text{Herm}(\lambda)$. Therefore, we may define a Hermitian form $\gamma_K$ on $\Delta_c(\lambda)$ by the formula

$$\gamma_K(P, Q) = \int_{\mathfrak{h}_\mathbb{R}} Q(x)^\dagger K(x)P(x)e^{-|x|^2/2}dx$$

for all $P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$. As $K$ is nonzero and $W$-equivariant, the same is true for $\gamma_K$ by the density of $\mathbb{R}[\mathfrak{h}_{\mathbb{R}}]e^{-|x|^2/2} \subset \mathcal{S}(\mathfrak{h}_{\mathbb{R}})$. It follows from [ES, Proposition 4.6] that $\gamma_K$ is a nonzero real multiple of $\gamma_{c,\lambda}$ if and only if $\gamma_K$ is contravariant, i.e. if and only if $\gamma_K$ satisfies the identity

$$\gamma_K((-y+Ty)P, Q) = \gamma_K(P, yQ)$$

for all $y \in \mathfrak{h}_\mathbb{R}$. This condition is equivalent to the vanishing of the integral

$$\int_{\mathfrak{h}_\mathbb{R}} ((Ty-y)(Q)^\dagger KP - Q^\dagger K(yP)e^{-|x|^2/2}dx = 0.$$

As the operator $b_s = \alpha_s^{-1}(1 - s)$ is a continuous operator on $\mathcal{S}(\mathfrak{h}_{\mathbb{R}})$, it follows that the Dunkl operator $D_y = \partial_y - \sum_{s \in S} c_s \alpha_s(y)b_s \otimes s$ is a continuous operator on $\mathcal{S}(\mathfrak{h}_{\mathbb{R}}) \otimes \lambda$ as well. Note furthermore that we have for $y \in \mathfrak{h}_\mathbb{R}$ and $P \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$ that $D_y(Pe^{-|x|^2/2}) = (D_y - T(y))(P)e^{-|x|^2/2}$. Without loss of generality we may take $Q$ to be a constant function, hence annihilated by $y \in \mathfrak{h}_\mathbb{R}$, and it follows from the density of $\mathbb{R}[\mathfrak{h}_{\mathbb{R}}]e^{-|x|^2/2} \subset \mathcal{S}(\mathfrak{h}_{\mathbb{R}})$ that the vanishing condition above is equivalent to the vanishing of the $\lambda$-valued integral

$$\int_{\mathfrak{h}_\mathbb{R}} K(D_y \varphi)dx$$

all $\varphi \in \mathcal{S}(\mathfrak{h}_{\mathbb{R}}) \otimes \lambda$ and $y \in \mathfrak{h}_\mathbb{R}$. Following Dunkl, for any $\epsilon > 0$ define the region

$$\Omega_\epsilon := \{x \in \mathfrak{h}_\mathbb{R} : |\alpha_s(x)| > \epsilon \text{ for all } s \in S\} \subset \mathfrak{h}_{\mathbb{R},\text{reg}}.$$
For any $\varphi \in S(\mathfrak{h}_R) \otimes \lambda$ the function $K(D_y \varphi)$ is integrable, so we have

$$\lim_{\epsilon \to 0} \int_{\Omega\epsilon} K(D_y \varphi) dx = \int_{\mathfrak{h}_R} K(D_y \varphi) dx$$

by dominated convergence. The region $\Omega\epsilon$ avoids the singularities of $D_y$, and so, following Dunkl again, a direct calculation using the $W$-equivariance of $K$, the $W$-invariance of $\Omega\epsilon$, and the system of differential equations $K$ satisfies shows that

$$\int_{\Omega\epsilon} K(D_y \varphi) dx = \int_{\Omega\epsilon} \partial_y (K \varphi) dx.$$

Furthermore, as the Schwartz function $D_y \varphi$ depends linearly on $y$, we see that $\gamma_K$ is contravariant if and only if

$$(2.4) \quad \lim_{\epsilon \to 0} \int_{\Omega\epsilon} \partial_y (K \varphi) = 0$$

for a spanning set of $y \in \mathfrak{h}_R$ and dense set of $\varphi \in S(\mathfrak{h}_R)$.

Fix a simple reflection $s_i$. Computing this integral as an iterated integral with inner integrals chosen as in [D2, Section 5] and choosing test functions $\varphi$ to be bump functions supported on small balls centered at a point $x_0 \in C_i$ and $s_i$-invariant, we see that the vanishing of the limit

$$\lim_{\epsilon \to 0} \int_{\Omega_{\epsilon i}} \partial_{\alpha_i^\vee} (K \varphi) dx$$

implies that

$$\lim_{\epsilon \to 0} \int_{\Omega_{\epsilon i}} (K - s_i K s_i)(x + \epsilon \alpha_i^\vee) \psi(x) dx = \lim_{\epsilon \to 0} \int_{\Omega_{\epsilon i}} (K(x + \epsilon \alpha_i^\vee) - K(x - \epsilon \alpha_i^\vee)) \psi(x) dx = 0$$

for all $\psi \in C_c^\infty(C_i)$, by the $W$-equivariance of $K$. Choosing coordinates $z_1, ..., z_l$ adapted to the wall $C_i$ as in the discussion preceding Definition 2.2.10, we have that $K$ is given on $C_i$ by the formula

$$K(z) = P_i(z) z_{\epsilon_i s_i}^{-\epsilon_i} K_i z_{\epsilon_i s_i}^{-\epsilon_i} P_i(z)^\dagger$$

with $P_i(z)$ and $K_i$ as above. As $P_i(z) = Id$ for $z \in C_i$, it follows that the commutator $[s_i, P_i(z)]$ is analytic in $z$ and vanishes along $C_i$. In particular, there is an analytic function $Q_i(z)$ on $C \subset C_i \cup s_i(C)$ with values in $\text{End}_{C_i}(\lambda)$ such that $[s_i, P_i(z)] = z_1 Q_i(z)$. For $\sigma_1, \sigma_2 \in \{\pm 1\}$ let $K_{i\sigma_1,\sigma_2} \in \text{End}_C(\lambda)$ be the projection of $K_i$ to the $(\sigma_1, \sigma_2)$ simultaneous eigenspace of the commuting operators $\lambda_{s_i}$ and $\rho_{s_i}$ on $\text{End}_{C_i}(\lambda)$ of left and right, respectively, multiplication by $s_i$. We have $K_i = K_{i,1,1} + K_{i,-1,-1} + K_{i,-1,1} + K_{i,1,-1}$, and a straightforward computation shows

$$(2.5) \quad (K - s_i K s_i)(z) = 2P_i(z)(K_{i,1,1} + K_{i,-1,-1}) P_i(z)^\dagger + z_1^{-2\epsilon_i} R_i(z)$$

for $z \in C_i$, where $R_i(z) \in \text{End}_C(\lambda)$ is analytic on $C$ and extends continuously to $C \cup C_i$. It follows that we have

$$\lim_{\epsilon \to 0} \int_{\Omega_{\epsilon i}} (K - s_i K s_i)(x + \epsilon \alpha_i^\vee) \psi(x) dx = \int_{C_i} 2(K_{i,1,1} + K_{i,-1,-1}) \psi(x) dx$$

for all $\psi \in C_c^\infty(C_i)$. For this integral to vanish for all such $\psi$, we must have that $K_{i,1,-1} + K_{i,-1,1} = 0$, and hence that $K_i = K_{i,1,1} + K_{i,-1,-1}$ commutes with $s_i$, so $K|_{C_i}$ is asymptotically $W$-invariant. In particular, we see that (a) implies (c).
Conversely, if (c) holds, we have an equality analogous to (2.5) about every reflection hyperplane \( \ker \alpha_s \), and in particular the function \( K - s_i K s_i \) extends continuously to all of \( \mathcal{C} \cup \mathcal{C}_i \cup s_i(\mathcal{C}) \) with value 0 along \( \mathcal{C}_i \). Let \( y \in \mathfrak{h}_{\mathbb{R}, \text{reg}} \) and \( \varphi \in C_\infty^0(\mathfrak{h}_{\mathbb{R}}) \). By the \( W \)-equivariance of \( K \) and the inner integrals used by Dunkl recalled in the previous paragraph, to show the vanishing of the limit 2.4 it suffices to show that the limit
\[
(2.6) \quad \lim_{\epsilon \to 0} \int_{\mathcal{C}_{i,\epsilon}} (K - s_i K s_i)(x + \epsilon \alpha_i^\vee / 2)\varphi(x) dx
\]
vanishes, where
\[
C_{i,\epsilon} := \{ x \in \mathfrak{h}_{\mathbb{R}} : \alpha_i(x) = 0, \alpha_j(x) > \epsilon \text{ for } j \neq i \} \subset \mathcal{C}_i.
\]
For each \( \epsilon > 0 \) define the function \( f_\epsilon \) on \( \mathcal{C}_i \) by
\[
f_\epsilon(x) := \chi_{C_{i,\epsilon}}(x)(K - s_i K s_i)(x + \epsilon \alpha_i^\vee / 2),
\]
where \( \chi_{C_{i,\epsilon}} \) is the indicator function of \( C_{i,\epsilon} \). The functions \( f_\epsilon \) converge pointwise to 0 as \( \epsilon \) tends to 0, so to show that the limit (2.6) vanishes we need to justify exchanging the order of the limit and integration. In particular, by the Vitali convergence theorem and the compactness of \( \text{Supp}(\varphi) \), it suffices to show that the \( f_\epsilon \) are uniformly integrable on sets of the form \( X \cap \mathcal{C}_i \) where \( X \subset \mathfrak{h}_{\mathbb{R}} \) is a compact subset. To this end, fix a compact subset \( X \subset \mathfrak{h}_{\mathbb{R}} \) and define the polynomial \( \Delta_i := \prod_{j \neq i} \alpha_j \). From the estimate (2.2) in Lemma 2.2.7 and the fact that \( K - s_i K s_i \) extends continuously to \( \mathcal{C} \cup \mathcal{C}_i \) with value 0 along \( \mathcal{C}_i \) it follows that there is a constant \( \mu > 0 \), independent of \( c \), such that \( \Delta_i^{-\mu \sum_{s \in S} |c_s|} (K - s_i K s_i) \) extends continuously to all of \( \mathcal{C} \) with value 0 along all of the boundary walls. Let
\[
M := \max_{x \in \mathcal{C} \cap X} \| \Delta_i^{-\mu \sum_{s \in S} |c_s|} (K - s_i K s_i) \|
\]
so that we have
\[
\| (K - s_i K s_i)(x) \| \leq M \Delta_i^{-\mu \sum_{s \in S} |c_s|}(x)
\]
for all \( x \in \mathcal{C} \cap X \). In particular, to show uniform integrability of the functions \( \{ f_\epsilon \}_{\epsilon > 0} \) over \( X \cap \mathcal{C}_i \) it suffices to show uniform integrability of the family of functions
\[
g_\epsilon(x) := \chi_{C_{i,\epsilon}}(x) \Delta_i^{-\mu \sum_{s \in S} |c_s|}(x + \epsilon \alpha_i^\vee / 2)
\]
for \( \epsilon > 0 \). For \( x \in C_{i,\epsilon} \) we have \( \Delta_i(x + \epsilon \alpha_i^\vee / 2) = \Delta_i(x + \epsilon(\alpha_i^\vee/2 - \omega_i)) \), and \( x + \epsilon(\alpha_i^\vee/2 - \omega_i) \in C_i \). In particular, uniform integrability of the \( g_\epsilon \) on sets \( X \cap \mathcal{C}_i \) with \( X \subset \mathfrak{h}_{\mathbb{R}} \) compact follows from the integrability of \( \Delta_i^{-\mu \sum_{s \in S} |c_s|} \) over such sets, which in turn holds for sufficiently small \( |c_s| \). This completes the proof that (c) implies (a).

Now we will show the equivalence of (b) and (c). Fix a point \( x_0 \in \mathfrak{h}_{\mathbb{R}, \text{reg}} \), and identify \( K(x_0) \) with a Hermitian form on \( KZ_{x_0}(\Delta_c(\lambda)) \), where we make the natural identification of \( KZ_{x_0}(\Delta_c(\lambda)) \) with a \( C \)-vector space and, as throughout, identify Hermitian forms on the latter with \( \text{End}_C(\lambda) \) via the preferred \( W \)-invariant positive definite Hermitian form on \( \lambda \). For simplicity, and without loss of generality, we take \( x_0 \in \mathcal{C} \). Consider the \( W \)-equivariant flat connection (over \( \mathbb{R} \))
\[
\nabla_{\mathbb{R}} := d + \sum_{s \in S} c_s \left( \frac{d\lambda_s}{\alpha_s} \lambda_s + \frac{d\rho_s}{\alpha_s} \rho_s \right)
\]
on the trivial real vector bundle over \( \mathfrak{h}_{\text{reg}} \) with fiber \( \text{End}_C(\lambda) \), where \( \lambda_s \) and \( \rho_s \) denote left and right, respectively, multiplication by \( s \). Given \( x_1 \in \mathfrak{h}_{\text{reg}} \) and \( A \in \text{End}_C(\lambda) \), the local flat
section of the connection $\nabla_R$ with value $A$ at $x_1$ is given by

$$x \mapsto f(x)M(x)AM(x)^\dagger \overline{f(x)} = |f(x)|^2M(x)AM(x)^\dagger,$$

where $M$ is the local $\text{End}_C(\lambda)$-valued fundamental solution to the KZ connection

$$\nabla_{KZ} = d - \sum_{s \in S} c_s \frac{d_{\lambda_s}}{\lambda_s}(1 - s),$$

whose monodromy defines the braid group action on $KZ_x(\Delta_c(\lambda))$, with value $\text{Id}$ at $x_1$, and where $f$ is the scalar valued function introduced in the proof of Lemma 2.2.6. As the parameter $c$ is real, the scalar monodromy of $f$ about the reflection hyperplanes $\ker(\alpha_i)$ in $\mathfrak{h}_{\text{reg}}/W$ takes values on the unit circle, and in particular $|f|^2$ is single-valued on $\mathfrak{h}_{\text{reg}}/W$. It follows that action $T_i.A := T_iAT_i^\dagger$ of the braid group $B_W$ on the space of Hermitian forms on $\lambda = KZ_{x_0}(\Delta_c(\lambda))$ is given by the monodromy of the connection $\nabla_R$ over $\mathfrak{h}_{\text{reg}}/W$. In particular, the form $K(x_0)$ is $B_W$-invariant if and only if there is a global single-valued flat section for $\nabla_R$ over $\mathfrak{h}_{\text{reg}}/W$ with value $K(x_0)$ at $x_0$, which in turn can be checked at each of the reflection hyperplanes $\ker(\alpha_i)$ for simple reflections $s_i$. Fix a simple $s_i$, and let $P_i(z)$ be as in Lemma 2.2.9. As the function $K$ satisfies the system $\nabla_R$ along $C$, it follows from the discussion above and that following Lemma 2.2.9 that there is a unique $K_i \in \text{End}_C(\lambda)$ such that the (potentially multi-valued) continuation of $K$ via $\nabla_R$ near $\ker(\alpha_i)$ takes the form

$$z \mapsto P_i(z)|\alpha_i(z)|^{-c_i}K_i;|\alpha_i(z)|^{-c_i}P_i(z)^\dagger.$$

Let $K_i = K_i^{1,1} + K_i^{1,-1} + K_i^{-1,1} + K_i^{-1,-1}$ be the decomposition of $K$ into simultaneous eigenvectors for $\lambda_{s_i}$ and $\rho_{s_i}$ as before. Then, the function above takes the form

$$P_i(z) \left( |\alpha_i(z)|^{-2c_i}K_i^{1,1} + \left( \frac{\alpha_i(z)}{\alpha_i(z)} \right)^{c_i}K_i^{1,-1} + \left( \frac{\alpha_i(z)}{\alpha_i(z)} \right)^{-c_i}K_i^{-1,1} + |\alpha_i(z)|^{2c_i}K_i^{-1,-1} \right) P_i^\dagger(z).$$

For small nonzero $c_i$, this function is single valued along a loop around the hyperplane $\ker(\alpha_i)$ in $\mathfrak{h}_{\text{reg}}$ if and only if $K_i^{-1,1} = K_i^{1,-1} = 0$, i.e. so that $K_i$ commutes with $s_i$. As $P_i(z)$ is $s_i$-equivariant, this condition is then also equivalent to the triviality of the monodromy over a loop around $\ker(\alpha_i)$ in $\mathfrak{h}_{\text{reg}}/W$ as well. In particular, we see that (b) and (c) are equivalent, as needed.

Finally, we will prove the claim that the space of $B_W$-invariant Hermitian forms on $KZ_{x_0}(\Delta_c(\lambda))$ is 1-dimensional over $\mathbb{R}$. Recall that the parameter $c$ is such that $\mathcal{O}_c(W, \mathfrak{h})$ is semisimple, and hence so is $H_q(W)$, where $q(s_i) = q_i := e^{2\pi i c_i}$. Given a finite-dimensional representation $V$ of $B_W$ over $\mathbb{C}$, we may form its Hermitian dual $V^h := V^\ast$, a representation of $B_W$; as usual, the element $T_i \in B_W$ acts by $T_i.f := f \circ T_i^{-1}$. When the action of $B_W$ on $V$ factors through the Hecke algebra $H_q(W)$, so that each generator $T_i$ satisfies the quadratic relation $(T_i - 1)(T_i + q_i) = 0$, it follows that $T_i$ satisfies the quadratic relation $(T_i - 1)(T_i + \overline{q_i}^{-1}) = 0$ in the representation $V^h$. As $c_i \in \mathbb{R}$, we have $\overline{q_i}^{-1} = q_i$, and it follows that $V^h$ is a representation of $H_q(W)$ as well. When $V$ is irreducible then $V^h$ is irreducible as well. Furthermore, by [DGHM, Theorem 3.4, Remark 3.1], any basis element $T_w \in H_q(W)$ with $w \in W$ of minimal length in its conjugacy class acts in any irreducible representation $V$ of $H_q(W)$ with eigenvalues of the form $\zeta \prod f_i^\lambda$ where $\zeta$ is a root of unity and $f_i \in \mathbb{Q}$. For such $w$, it follows that $T_w$ act in $V$ and $V^h$ by operators of the same trace. In particular, the character tables of $V$ and $V^h$ coincide, so $V \cong V^h$ as representations of $H_q(W)$. As in Lemma 1.2.4, isomorphisms $V \cong V^h$ of $H_q(W)$-modules can be identified with $B_W$-invariant
sesquilinear forms $\beta : V \times V \to \mathbb{C}$, with an isomorphism $\varphi : V \cong V^h$ corresponding to the $B_W$-invariant sesquilinear form $\beta_\varphi$ given by $\beta_\varphi(x, y) = \varphi(y)(x)$. Given such a form $\beta$, one may form its Hermitian transpose $\beta^t$ by the formula $\beta^t(x, y) = \overline{\beta(y, x)}$. As the space of such $\beta$ is 1-dimensional over $\mathbb{C}$, given a nonzero $B_W$-invariant sesquilinear form $\beta$ there is a nonzero scalar $z \in \mathbb{C}$ such that $\beta^t = z\beta$. It follows that $\overline{z}^{-1}\beta$ defines a $B_W$-invariant Hermitian form on $V$, and that the space of such forms is 1-dimensional over $\mathbb{R}$. The claim now follows, taking $V = KZ_{x_0}(\Delta_c(\lambda))$. \qed

2.3. Analytic Continuation on the Regular Locus.

**Lemma 2.3.1.** Let $\lambda \in \text{Irr}(W)$ and fix a simple reflection $s_i \in S$. Fix a point $x_0 \in C_i$, and for each $c \in \mathfrak{p}$ such that $c_i \notin \frac{1}{2}\mathbb{Z}\setminus\{0\}$ let $P_{i,c}(z) : D \to GL(\lambda)$ be the holomorphic function from Lemma 2.2.9 satisfying $P_{i,c}(x_0) = Id$. Then $P_{i,c}(z)$ is a holomorphic function of $(c, z) \in \{ c \in \mathfrak{p} : c_i \notin \frac{1}{2}\mathbb{Z}\setminus\{0\} \} \times D$. Furthermore, there is an entire function $q(c_i)$, nonvanishing for $c_i \notin \frac{1}{2}\mathbb{Z}\setminus\{0\}$, such that $q(c_i)P_{i,c}(z)$ is holomorphic on all of $\mathfrak{p} \times D$.

**Proof.** As $P_{i,c}(z)$ is holomorphic in $z$ by definition, by Hartog’s theorem it suffices to show holomorphicity in $c$ for all fixed $z \in D$. By the analytic dependence on parameters and initial conditions the of solutions of the systems $\nabla'_{KZ}$ and $\nabla''_{KZ}$ appearing in Lemma 2.2.9, it suffices to show that $P_{i,c}(z)$ is analytic in $c$ for any one point $z \in D \cap \mathfrak{h}_{\text{reg}}$. This will follow from analysis of the system $\nabla'_{KZ}$ along the lines of that appearing in [W, Section 3], as follows.

Choose a linear coordinate system $z_1, ..., z_l \in \mathfrak{h}^*$ with $z_1 = \alpha_i$ and $z_j(\alpha_j^\vee) = 0$ for $j > 1$. Near the point $x_0$, the system $\nabla'_{KZ}$ restricted to the $z_1$ direction takes the form

\begin{equation}
(2.7)
\frac{dz_1}{d\zeta} = A_c(z_1)F_c(z_1)
\end{equation}

where $A_c(z_1)$ is jointly analytic in $c$ and $z_1$ for $|z_1|$ sufficiently small, independently of $c$, and satisfies $A_c(0) = -c_is_i$. Fix a point $c^0 \in \mathfrak{p}$ with $c_i^0 \notin \frac{1}{2}\mathbb{Z}\setminus\{0\}$, and let $A_c(z_1) = \sum_{n \geq 0} a_{c,n}z_1^n$ be the series expansion for $A_c$ in $z_1$, where each $a_{c,n}$ is an analytic function of $c$. Similarly, let $P_{i,c}(z) = \sum_{n \geq 0} p_{c,n}z_1^n$ be the series expansion in $z_1$ for $P_{i,c}$ at $x_0$. We have $p_{c,0} = Id$, and as $P_{i,c}(z)z_1^{-c_is_i}$ satisfies the system (2.7) it follows that

\begin{equation}
(2.8)
(n + c_is_i)p_{c,n} - p_{c,n-1}c_is_i = \sum_{r=0}^{n-1} a_{c,n-r}p_{c,r}
\end{equation}

for all $n > 0$. As $c_i \notin \frac{1}{2}\mathbb{Z}\setminus\{0\}$ it follows that $n + c_is_i$ and $-c_is_i$ have no eigenvalues in common, and in particular the recurrence (2.8) uniquely determines $p_{c,n}$ for all $n > 0$. Specifically, we have

\begin{equation}
(2.9)
p_{c,n} = (\lambda_{n+c_is_i} - \rho_{c_is_i})^{-1}\sum_{r=0}^{n-1} a_{c,n-r}p_{c,r}.
\end{equation}

Let $B \subset \mathfrak{p}$ be a small ball around $c^0$ with closure not intersecting $\{ c \in \mathfrak{p} : c_i \notin \frac{1}{2}\mathbb{Z}\setminus\{0\} \}$. It follows that there is a uniform bound $C > 0$ such that $||(\lambda_{n+c_is_i} - \rho_{c_is_i})^{-1}|| \leq C$ for all $n > 0$ and for all $c \in B$, and therefore we have

\begin{equation}
(2.10)
||p_{c,n}|| \leq C\sum_{r=0}^{n-1} ||a_{c,n-r}|| ||p_{c,r}||.
\end{equation}
Now, let $\varphi_c(z_1) = \sum_{n \geq 1} \|a_{c,n}\| z_1^n$. The series $\varphi_c(z_1)$ is convergent in an interval of $z_1$ about 0 independent of $c$ and satisfies $|\varphi_c(z_1)| \leq C'$ for $c \in B$ and $z_1$ in this interval, as $A_c$ is analytic in $c$ and $z$ and regular for $z$ away from the hyperplanes $\ker(s)$ for $s \in S \setminus \{s_1\}$. Next, define $\hat{a}_c(z_1) = (1 - C\varphi_c(z_1))^{-1}\|\Id\|$. Note that $\hat{a}_c(z_1)$ is analytic in $z_1$ in an interval $I$ about 0, independent of $c$, because $\varphi_c(0) = 0$. Let $\hat{a}_c(z_1) = \sum_{n \geq 0} \hat{a}_{c,n} z_1^n$ be the series expansion of $\hat{a}_c$ with respect to $z_1$. It follows from the definition of $\hat{a}_c$ that we have $\hat{a}_c(z_1) = C\varphi_c(z_1)\hat{a}(z_1) + \|\Id\|$, and comparison with the inequalities in (2.10) implies that we have

\[(2.11) \quad \|p_{c,n}\| \leq \hat{a}_{c,n}\]

for all $n \geq 0$. As each $p_{c,n}$ is analytic in $c \in B$ by equation (2.9), to show that $P_{i,c}(z) = \sum_{n \geq 0} p_{c,n} z_1^n$ is analytic in $c$ for small $|z_1|$ it suffices to show that this series expansion converges uniformly with respect to $c \in B$. In particular, by the inequality (2.11) it suffices to show that the series expansion $\sum_{n \geq 0} \hat{a}_{c,n} z_1^n$ for $\hat{a}_c(z_1)$ converges uniformly with respect to $c \in B$ for $z_1$ in a neighborhood of 0. To this end, note that the expansion $\varphi_c(z_1) = \sum_{n \geq 1} \|a_{c,n}\| z_1^n$ and the definition $\hat{a}_c(z_1) = (1 - C\varphi_c(z_1))^{-1}\|\Id\|$ implies that

\[
\hat{a}_{c,k} = \sum_{m_1 + \ldots + m_n = k, m_i \in \mathbb{Z}^+} C^n \prod_{i=1}^n \|a_{c,m_i}\|.
\]

Without loss of generality, we may replace $C$ by max\{C,1\}. Now, let $r > 0$ be a positive number such that $A_c(z)$ is holomorphic for $z$ in a neighborhood of the closed ball $B_r(x_0)$, and let $A_c^{(n)}(z_1) := \frac{\partial^n}{\partial z_1^n} A_c(z_1)$. The Cauchy estimate then gives

\[
\|a_{c,n}\| = \frac{1}{n!} |A_c^{(n)}(0)|| \leq \frac{1}{n!} r^n \max |z_1| = r \|A_c(z_1)|| \leq \frac{C''}{r^n}
\]

where $C'' := \max_{|z_1|=r,c \in B} |A_c(z_1)||$. Replacing $C''$ by max\{C''$,r$\}$ we may assume that $C''/r \geq 1$. From the above equation for $\hat{a}_{c,k}$ we therefore have the bound

\[
\hat{a}_{c,k} \leq (CC''/r)^k \# \left\{ (m_1, \ldots, m_n) : m_i \in \mathbb{Z}^+, \sum_i m_i = k \right\} < (2CC''/r)^k.
\]

It now follows that for $|z_1| < r/2CC'$ the convergence of the series $\sum_{n \geq 0} \hat{a}_{c,n} z_1^n$ is uniform with respect to $c \in B$. The same is therefore true for $P_{i,c}(z_1) = \sum_{n \geq 0} p_{c,n} z_1^n$, which is therefore analytic in $c \in B$. We conclude that $P_{i,c}(z)$ is holomorphic \{c \in \mathfrak{p} : c \notin \frac{1}{2}\mathbb{Z}\setminus\{0\}\} \times D$.

It remains to show the existence of an entire function $q(c)$, nonvanishing for $c \notin \frac{1}{2}\mathbb{Z}\setminus\{0\}$, such that $q(c)P_{i,c}(z)$ is holomorphic on all of $\mathfrak{p} \times D$. By the Weierstrass theorem on the existence of entire functions on $\mathbb{C}$ with specified zeros, we need only show that for each $k \in \frac{1}{2}\mathbb{Z}\setminus\{0\}$ there is an integer $N_k \geq 0$ such that $(c_i - k)^{N_k} P_{i,c}(z)$ is holomorphic in $(c,z) \in \{c \in \mathfrak{p} : c_i \in B_{1/4}(k)\} \times D'$ for every polydisc $D'$ contained in $D$ and centered about a point $x \in \ker(a_s) \cap D$. Write $D' = \{z_1 \in \mathbb{C} : |z_1| < R\} \times D''$ for some polydisc $D'' \subset \mathbb{C}^{l-1}$, and for a point $z \in D$ write $z = (z_1, z')$. Let $c^0 \in \mathfrak{p}$ be the unique parameter with $c^0(s_i) = k$ and $c^0(s) = 0$ for all $s \in S$ not conjugate to $s_i$. Then $P_{i,c}(z)$ has a Laurent series expansion in the domain $\{c \in \mathfrak{p} : c_i \in B_{1/4}(k)\} \times D'$ centered at $c^0$, and it suffices to show that the set of integers $m \in \mathbb{Z}$ such that $c_i^m$ appears in this Laurent series in a term with nonzero coefficient is bounded from below. Let $P_{i,c}(z) = \sum_{n \geq 0} p_{c,z',n} z_1^n$ be the resulting expansion for $P_{i,c}(z)$ in $z_1$, where the $p_{c,z',n}$ are holomorphic in $(c, z') \in \{c \in \mathfrak{p} : c_i \in U_0 := B_{1/4}(k)\} \times D''$. We
need to show that there is an integer $N_k \geq 0$ such that $(c_i-k)^N_k p_{c,z',n}$ is holomorphic in $U := \{c \in \mathfrak{p} : c_i \in B_{1/4}(k)\} \times D''$ for all $n \geq 0$. In the case $n = 0$, note that $p_{c,z',0}$ is in fact holomorphic on all of $U$; this follows from the analytic dependence of solutions of the system $\nabla''_{KZ}$ on parameters $c$ and initial conditions and the fact that $P_{i,c}(z)$ satisfies the system $\nabla''_{KZ}$ along $\ker(\alpha) \cap D$ and has value $P_{i,c}(x_0) = \text{Id}$ for all $c$. For $n > 0$, note that the functions $p_{c,z',n}$ satisfy the recursion (2.9). The function $(\lambda_{n+c_\ast s_i} - \rho_{c_\ast s_i})^{-1}$ of $c_i$ is holomorphic in $c_i \in B_{1/4}(k)$ for $n \neq |2k|$, and for $n = |2k|$ it is holomorphic in $c_i \in B_{1/4}(k) \setminus k$ with a pole at $c_i = k$. Let $N_k$ be the order of this pole. It follows from the recursion that $c_i - k)^N_k p_{c,z',n}$ is holomorphic in $(c, z') \in \{c \in \mathfrak{p} : c_i \in B_{1/4}(k)\} \times D''$ for all $n \geq 0$, completing the proof. □

**Lemma 2.3.2.** Let $\lambda \in \text{Irr}(W)$ and $x \in \mathcal{C}$. There exists a meromorphic function $B : \mathfrak{p} \to \text{End}_\mathcal{C}(\lambda)$ such that $B(0) = \text{Id}$ and such that for every $c \in \mathfrak{p}_\mathbb{R}$ where $B$ is analytic the endomorphism $B(c)$ defines a $B_W$-invariant Hermitian form on $KZ_x(\Delta_c(\lambda))$. (As usual, we identify $KZ_x(\Delta_c(\lambda))$ with $\lambda$ as a $\mathcal{C}$-vector space and we identify sesquilinear forms on $\lambda$ with $\text{End}_\mathcal{C}(\lambda)$ via the preferred $W$-invariant nondegenerate Hermitian form $(\cdot, \cdot)$ on $\lambda$.) Furthermore, such $B$ is uniquely determined up to multiplication by a scalar-valued meromorphic factor.

**Proof.** For each simple reflection $s_i \in S$, let $P_{i,c}(z)$ be as in Lemma 2.3.1 and let $N_{i,c}(z) = (P_{i,c}(z)\alpha_i(z))^{-c_\ast s_i}$. By Lemma 2.3.1, $P_{i,c}(z)$ is meromorphic in $c$, and hence $N_{i,c}(z)$ is also meromorphic in $c$. Consider the system of linear equations

$$\begin{align*}
[N_{i,c}(x)A N_{i,c}(x)^\dagger, s_i] = 0, \quad i = i, ..., l - 1
\end{align*}$$

in the variable $A \in \text{End}_\mathcal{C}(\lambda)$, with coefficients meromorphic in $c \in \mathfrak{p}$ and analytic near $0 \in \mathfrak{p}$. By Theorem 2.2.11, we see that when $c \in \mathfrak{p}$ is such that $|c_s|$ is sufficiently small for all $s \in S$, there is a one-dimensional space of solutions to (2.12). When $c = 0$, this one dimensional space contains $\text{Id} \in \text{End}_\mathcal{C}(\lambda)$. From the system (2.12), select a collection of linear functionals $\{f_c^{(j)}\}_{j=1}^{t^2-1}$ on $\text{End}_\mathcal{C}(\lambda)$, meromorphic in $c \in \mathfrak{p}$ and analytic near $0 \in \mathfrak{p}$, that form a complete set of relations for the system (2.12) for $c$ near $0$. Let $\text{Tr} : \text{End}_\mathcal{C}(\lambda) \to \mathbb{C}$ denote the trace. The system of linear equations

$$\begin{align*}
\begin{cases}
 f_c^{(j)}(A) = 0 & j = 1, ..., t^2 - 1, \\
 \text{Tr}(A) = l
\end{cases}
\end{align*}$$

depending meromorphically on $c \in \mathfrak{p}$, therefore has a unique solution $A_c \in \text{End}_\mathcal{C}(\lambda)$, with $A_c$ depending on $c$ meromorphically. When $c \in \mathfrak{p}_\mathbb{R}$ it is easy to see that $N_{i,c}(x) = N_{i,c}(x)$, and for such $c$ it follows that $A^\dagger = A_c$ by uniqueness of the solution $A_c$. In particular, $A_c$ takes values in Hermitian forms on $KZ_x(\Delta_c(\lambda))$ for $c \in \mathfrak{p}_\mathbb{R}$. Finally, for each $c \in \mathfrak{p}$ and simple reflection $s_i$, let $T_{i,c} \in \text{Aut}_\mathcal{C}(\lambda)$ denote the monodromy action of the braid group generator $T_i$ on $KZ_x(\Delta_c(\lambda)) =_{\text{e.s.}} \lambda$. As $T_{i,c}$ depends holomorphically on $c$ and as we have $T_{i,c} A_c T_{i,c}^\dagger = A_c$ for all $c \in \mathfrak{p}_\mathbb{R}$ with $|c_s|$ sufficiently small for all $s \in S$, it follows that $T_{i,c} A_c T_{i,c}^\dagger = A_c$ for all $c \in \mathfrak{p}_\mathbb{R}$ where $A_c$ is regular. Taking $B(c) = A_c$, this proves the existence of the function $B$. For the uniqueness of $B$ up to multiplication by a scalar meromorphic function, recall that the space of $B_W$-invariant Hermitian forms on $KZ_x(\Delta_c(\lambda))$ is 1-dimensional for all $c \in \mathfrak{p}_\mathbb{R}$, such that $\mathcal{O}_c(W, \mathfrak{h})$ is semisimple, and in particular in a ball about $0$ in $\mathfrak{p}_\mathbb{R}$. If $B', B'' : \mathfrak{p} \to \text{End}_\mathcal{C}(\lambda)$ are two meromorphic functions as in the lemma, any ratio $g_{ij} := B_{ij}' / B_{ij}''$ of corresponding matrix coefficients are scalar valued meromorphic functions, and these ratios must all agree in a ball about $0$ in $\mathfrak{p}_\mathbb{R}$. It follows that all such ratios $g_{ij}$ agree on all of $\mathfrak{p}$ and
Proof. The existence of the function integral representation:

\[ \int c \text{rameters of solutions of the system appearing in condition (3) on initial conditions } K \text{h polynomial growth on Lemma 2.2.7, the continuity of } K \text{ is an integer } N \text{ and an integer } l := l \text{ as needed. } \square \]

Fix a bounded open interval \( I \subset \mathbb{I}_R \) about 0 such that for all \( c \in I \) the function \( \Delta^{2N} K_c \) is continuous with polynomial growth on \( \mathfrak{h}_R \) and the restriction of the form \( \gamma_c \) to \( \Delta^N \Delta_c(\lambda) \) has the following integral representation:

\[ \gamma_c(\Delta^N P, \Delta^N Q) = f(c) \int_{\mathfrak{h}_R} Q^1 \Delta^{2N} K_c Pe^{-|x|^2/2} dx, \quad P, Q \in \Delta_c(\lambda). \]

The existence of the function \( f \) follows from the meromorphicity of \( B|_I \) and the Weierstrass theorem on the existence of meromorphic functions with prescribed zeros and poles. The existence and uniqueness of the function \( K \) then follows from the analytic dependence of solutions of the system appearing in condition (3) on initial conditions \( K_c(x_0) \) and parameters \( c_s \).

Finally, for any bounded open interval \( I \subset \mathbb{I}_R \) about 0 there is an analytic function \( f : I \to \mathbb{R} \) and an integer \( N \geq 0 \) such that for all \( c \in I \) the function \( \Delta^{2N} K_c \) is continuous with polynomial growth on \( \mathfrak{h}_R \) and the restriction of the form \( \gamma_c \) to \( \Delta^N \Delta_c(\lambda) \) has the following integral representation:

\[ \gamma_c(\Delta^N P, \Delta^N Q) = f(c) \int_{\mathfrak{h}_R} Q^1 \Delta^{2N} K_c Pe^{-|x|^2/2} dx, \quad P, Q \in \Delta_c(\lambda). \]

\[ \gamma_c(\Delta^N P, \Delta^N Q) := \int_{\mathfrak{h}_R} Q^1 \Delta^{2N} K_c Pe^{-|x|^2/2} dx. \]
For fixed $P, Q \in \Delta^N\Delta_c(\lambda)$, $\tilde{\gamma}_c(\Delta^N P, \Delta^N Q)$ is holomorphic in $c \in U$. Similarly, $\gamma_c(\Delta^N P, \Delta^N Q)$ is a polynomial in $c \in p_{\text{reg}}$, and we may therefore regard it as a polynomial in the complex variable $c \in p$. In particular, the quotient $\gamma_c(\Delta^N P, \Delta^N Q)/\tilde{\gamma}_c(\Delta^N P, \Delta^N Q)$ defines a meromorphic function $f_{P,Q}: U \to \mathbb{C}$. By Theorem 2.2.11, for $c$ in a neighborhood of $0$ in $I$ the forms $\gamma_c$ and $\tilde{\gamma}_c$ coincide up to a real scalar multiple. It follows that the functions $f_{P,Q}$ coincide in a neighborhood of $0$ in $I$ and therefore coincide with a single meromorphic function $f: U \to \mathbb{C}$ on all of $U$, real valued on $I$. Furthermore, for $c \in U$ the continuous function $\Delta^{2N}K_c$ of polynomial growth defines a nonzero tempered distribution. As $\mathbb{R}[\mathfrak{h}_{\mathbb{R}}] e^{-|x|^2/2}$ is dense in the Schwartz space $S(\mathfrak{h}_{\mathbb{R}})$, it follows that for each $c \in U$ it cannot be the case that $\tilde{\gamma}_c(\Delta^N P, \Delta^N Q) = 0$ for all $P, Q \in \Delta_c(\lambda)$. In particular, $f$ is holomorphic on $U$, and its restriction to $I$ gives the desired function.

**Corollary 2.3.4.** a) Let $q: S \to \mathbb{C}^\times$ be a $W$-invariant function satisfying $|q_s| = 1$ for all $s \in S$. Then every irreducible representation of the Hecke algebra $H_q(W)$ admits a nondegenerate $B_W$-invariant Hermitian form, unique up to $\mathbb{R}^\times$-scaling.

b) Let $c \in p_{\mathbb{R}}$ and let $x_0 \in \mathfrak{h}_{\mathbb{R},\text{reg}}$. If $\lambda \in \text{Irr}(W)$ is such that $KZ_{x_0}(L_c(\lambda))$ is nonzero and unitary, i.e. so that the nondegenerate $B_W$-invariant Hermitian form, appropriately scaled, is positive definite, then $L_c(\lambda)$ is quasi-unitary.

**Proof.** Any $q$ as in (a) is of the form $q_s = c^{2\pi ic_s}$ for some $c \in p_{\mathbb{R}}$. Fix such $c \in p$ and choose a point $x_0 \in \mathfrak{h}_{\mathbb{R},\text{reg}}$. Any irreducible representation of $H_q(W)$ is isomorphic to some $KZ_{x_0}(L_c(\lambda))$ for some $\lambda \in \text{Irr}(W)$ such that $\text{Supp}(L_c(\lambda)) = \mathfrak{h}$. Let $N_c(\lambda)$ be the maximal proper submodule of $\Delta_c(\lambda)$, so that $L_c(\lambda) = \Delta_c(\lambda)/N_c(\lambda)$. Let $K_c: \mathfrak{h}_{\mathbb{R},\text{reg}} \to \text{Herm}(\lambda)$ be as in Theorem 2.3.3, so that in particular $K_c(x_0)$ defines a nonzero $B_W$-invariant Hermitian form on $\lambda =_v s. KZ_{x_0}(\Delta_c(\lambda))$. As $KZ_{x_0}$ is exact, we have $KZ_{x_0}(L_c(\lambda)) \cong KZ_{x_0}(\Delta_c(\lambda))/(\Delta_c(\lambda)/KZ_{x_0}(N_c(\lambda)))$. As $KZ_{x_0}(L_c(\lambda))$ is irreducible, to show that it admits a nondegenerate $B_W$-invariant Hermitian form, unique up to $\mathbb{R}^\times$ scaling, it suffices to show that $KZ_{x_0}(N_c(\lambda))$ lies in the radical of $K_c(x_0)$. To this end, take a vector $v \in KZ_{x_0}(N_c(\lambda))$. Viewing $KZ_{x_0}(N_c(\lambda))$ as a submodule of $KZ_{x_0}(\Delta_c(\lambda))$ and identifying the latter with $\lambda$ as a vector space, let $\tilde{v} \in N_c(\lambda)$ be such that its value at $x_0$ is $v$. Let $N \geq 0$ and $f(c)$ be as in Theorem 2.3.3, so that we have

$$\gamma_c(\Delta^N P, \Delta^N Q) = f(c) \int_{\mathfrak{h}_{\mathbb{R}}} Q^\dagger \Delta^{2N} K_c P e^{-|x|^2/2} dx, \quad P, Q \in \Delta_c(\lambda)$$

for all $P, Q \in \Delta_c(\lambda)$. As $L_c(\lambda)$ is of full support and as $\gamma_c(\lambda)$ induces a nondegenerate form on $L_c(\lambda)$, it follows that there exist $P, Q \in \Delta_c(\lambda)$ such that $\gamma_c(\Delta^N P, \Delta^N Q) \neq 0$. In particular, it follows that $f(c) \neq 0$. As $\tilde{v} \in N_c(\lambda) = \text{Rad}(\gamma_c)$, it follows that $\gamma_c(\Delta^N v', \Delta^N p\tilde{v}) = 0$ for all $v' \in \lambda$ and $p \in \mathbb{R}[\mathfrak{h}_{\mathbb{R}}]$. This, together with the above integral representation and the nonvanishing of $f(c)$ implies that we have

$$\int_{\mathfrak{h}_{\mathbb{R}}} p \Delta^{2N} K_c \tilde{v} e^{-|x|^2/2} dx = 0$$

for all $p \in \mathbb{R}[\mathfrak{h}_{\mathbb{R}}]$. As $\Delta^{2N} K_c \tilde{v}$ defines a tempered distribution with values in $\lambda$, by the density of $\mathbb{R}[\mathfrak{h}_{\mathbb{R}}] e^{-|x|^2/2}$ in $S(\mathfrak{h}_{\mathbb{R}})$, it follows that $\Delta^{2N} K_c \tilde{v} = 0$ pointwise on $\mathfrak{h}_{\mathbb{R}}$. In particular, $K_c(x_0)\tilde{v} = 0$. As $v \in KZ_{x_0}(N_c(\lambda))$ was arbitrary, it follows that $KZ_{x_0}(N_c(\lambda)) \subset \text{Rad}(K_c(x_0))$. In particular, $K_c(x_0)$ descends to a nondegenerate $B_W$-invariant Hermitian form on $KZ_{x_0}(L_c(\lambda))$, as needed (and we have $KZ_{x_0}(N_c(\lambda)) = \text{Rad}(K_c(x_0))$), proving (a).
For (b), suppose furthermore that $K_c(x_0)$, appropriately scaled, defines a positive definite Hermitian form on $KZ_{x_0}(L_c(\lambda))$. This, the properties of $K_c$, and the argument in the previous paragraph imply that for all $x \in \mathfrak{h}_{\mathbb{R},\text{reg}}$ we have that $K_c(x)$ is positive semidefinite with radical equal to $KZ_x(N_c(\lambda))$. It follows from the integral representation of $\gamma_c$ that $\gamma_c$ is positive definite on $\Delta^NL_c(\lambda)$. As $L_c(\lambda)$ is of full support, it follows that

$$\lim_{n \to \infty} \frac{\dim(\Delta^NL_c(\lambda))^{\leq n}}{\dim L_c(\lambda)^{\leq n}} = 1.$$ 

In particular, it follows that $L_c(\lambda)$ is quasi-unitary. \hfill \Box

2.4. Extension to Distribution via Resolution of Singularities.

Definition 2.4.1. Call a square matrix $M$ non-resonant if no two eigenvalues of $M$ differ by a positive integer, and resonant otherwise.

Lemma 2.4.2. Fix positive integers $n, N \geq 0$ and $m \leq n$. Let $A = \sum_{i=1}^{n} A_i(z)dz_i$ be a meromorphic 1-form on $\mathbb{C}^n$ with values in $\text{Mat}_{N \times N}(\mathbb{C})$ such that $A_i(z)$ is holomorphic on $\mathbb{C}^n$ for $m < i \leq n$ and such that, for $1 \leq i \leq m$, $A_i(z) = A_{i,\text{res}}z_i^{-1} + A_{i,\text{reg}}(z)$ for some matrices $A_{i,\text{res}}$ and holomorphic functions $A_{i,\text{reg}}(z)$. Assume furthermore that each matrix $A_{i,\text{res}}, 1 \leq i \leq m$, is Hermitian and that $dA = [A,A] = 0$, so that the connection $\nabla_c = d + cA$ is flat for any $c \in \mathbb{C}$. Suppose $K_c : (\mathbb{C}^\times)^m \times \mathbb{C}^{n-m} \to \text{Mat}_{N \times N}(\mathbb{C})$ is a real analytic function depending real analytically on $c \in \mathbb{R}$ and satisfying the differential equation

$$dK_c + c(AK_c + K_cA) = 0$$

on $(\mathbb{C}^\times)^m \times \mathbb{C}^{n-m}$ as a real manifold. Then, for $c \in \mathbb{R}$ in the open set

$$U_A := \{ c \in \mathbb{C} : cA_{i,\text{res}} \text{ is non-resonant for all } 1 \leq i \leq m \},$$

the matrix coefficients of $K_c$ are linear combinations of functions of the form

$$|z_1|^{2\epsilon_{i_1}} \cdots |z_m|^{2\epsilon_{i_m}}f_c(z)$$

where, for $1 \leq i \leq m$, $\lambda_i$ is an eigenvalue of $A_{i,\text{res}}$, and where $f_c(z)$ is real analytic on all of $\mathbb{C}^n$ and real analytic in $c \in \mathbb{C}$.

Proof. If $c \in \mathbb{R}$ and $F$ is any local fundamental solution of the system $\nabla_c F = 0$, it is easy to see that solutions $K$ of the equation (2.13) are the functions $K(z) = F(z)MF(z)^\dagger$, where $M \in \text{Mat}_{N \times N}(\mathbb{C})$ is an arbitrary matrix and $\dagger$ denotes the Hermitian transpose. In particular, to analyze matrix coefficients of $K$ we will first study the solutions of $\nabla_c F = 0$.

Note that the case $m = 0$ is trivial. Assume $m > 0$. The arguments leading to Lemma 2.2.9 imply that, as long as $cA_{m,\text{res}}$ is non-resonant, there exists a (possibly multi-valued) $\text{Mat}_{N \times N}(\mathbb{C})$-valued holomorphic function $P_c^{(m)}(z)$ on $(\mathbb{C}^\times)^{m-1} \times \mathbb{C}^{n-m+1}$ such that $P_c^{(m)}(z)z_m^{cA_{m,\text{res}}}$ is a fundamental solution of the system $\nabla_c$. The proof of Lemma 2.3.1 shows that we may take $P_c^{(m)}(z)$ to be meromorphic in $c \in \mathbb{C}$ and holomorphic on the open set

$\{ c \in \mathbb{C} : cA_{m,\text{res}} \text{ is non-resonant} \}$. If $m > 1$, we may continue in this manner inductively. In that case, note that by construction $P_c^{(m)}(z)$ satisfies the system

$$d + \sum_{1 \leq i \leq n, i \neq m} A_i(z)dz_i$$

along each hypersurface in $(\mathbb{C}^\times)^{m-1} \times \mathbb{C}^{n-m+1}$ with constant $z_m$-coordinate. Similarly, the above system has a fundamental solution of the form $P_c^{(m-1)}(z)z_m^{cA_{m-1,\text{res}}}$ where $P_c^{(m-1)}(z)$ is
holomorphic (and possibly multi-valued) on \((\mathbb{C}^\times)^{m-2} \times \mathbb{C}^{n-m+2}\), meromorphic in \(c \in \mathbb{C}\) and regular in \(c\) over the set \(\{c \in \mathbb{C} : cA_{i,\text{res}} \text{ is non-resonant for } i = m - 1, m\}\), and satisfies the system

\[
d + \sum_{1 \leq i \leq m, i \neq m-1, m} A_i(z)dz_i
\]

along each subspace of \((\mathbb{C}^\times)^{m-2} \times \mathbb{C}^{n-m+2}\) in which \(z_{m-1}\) and \(z_m\) are constant. It follows that there is a function \(M_c^{(m)}(z_m)\), holomorphic and entire in \(z_m\) and meromorphic in \(c \in \mathbb{C}\) and regular in \(\{c \in \mathbb{C} : cA_{i,\text{res}} \text{ is non-resonant for } i = m - 1, m\}\) such that \(P_c^{(m)}(z)z_m^{cA_m,\text{res}} = F_c^{(m-1)}(z)z_{m-1,\text{res}} M_c^{(m)}(z_m)z_m^{cA_m,\text{res}}\). Continuing in this way by induction, we see that, for \(c \in U_A\), where \(U_A\) is the open subset of \(\mathbb{C}\) given by

\[
U_A := \{c \in \mathbb{C} : cA_{i,\text{res}} \text{ is non-resonant for all } 1 \leq i \leq m\},
\]

\(\nabla_c\) has a fundamental solution of the form

\[
F_c(z) = P_c^{(1)}(z)z_1^{cA_1,\text{res}} M_c^{(2)}(z_2, ..., z_m) ... z_{m-1,\text{res}} M_c^{(m)}(z_m)z_m^{cA_m,\text{res}}
\]

where \(P_c^{(1)}(z)\) and the \(M_c^{(k)}(z_k, ..., z_m)\) are entire in \(z\), meromorphic in \(c \in \mathbb{C}\), and holomorphic in \(c \in U_A\). As \(U_A\) is the complement in \(\mathbb{C}\) of a discrete subset (of \(\mathbb{R}\)), it follows from the Weierstrass factorization theorem that by multiplying through by an appropriate entire function in \(c\) we may assume that \(P_c^{(1)}(z)\) and the \(M_c^{(k)}\) are entire in \(c\).

Now consider the function \(K_c\) in the lemma statement. For \(c \in \mathbb{R} \cap U_A\) we must have \(K_c(z) = F_c(z)N_c(z)F_c(z)^\dagger\) for some matrix \(N_c(z) \in \text{Mat}_N(\mathbb{C})\). It follows that \(N_c(z)\) is analytic in all of \(c \in \mathbb{R}\) and that the equality \(K_c(z) = F_c(z)N_c(z)F_c(z)^\dagger\) holds for all \(c \in \mathbb{R}\). As the eigenvalues of the \(A_{i,\text{res}}\) are real, it follows that for \(c \in \mathbb{R}\) the matrix coefficients of \(K_c(z)\) are linear combinations of functions of the form

\[
z_1^{\lambda_1}z_1^{\lambda'_1} ... z_m^{\lambda_m}z_m^{\lambda'_m}f_c(z)
\]

with \(f_c(z)\) as in the lemma statement and \(\lambda_i, \lambda'_i \in \mathbb{R}\) eigenvalues of \(A_{i,\text{res}}\). As \(K_c(z)\) is single-valued on \((\mathbb{C}^\times)^m \times \mathbb{C}^{n-m}\), it follows for \(c \in \mathbb{R} \cap U_A\) that either \(f_c(z) = 0\) or \(\lambda_i = \lambda'_i\) for \(1 \leq i \leq m\); by analyticity the same must hold for all \(c \in \mathbb{R}\). We have \(z_1^{\lambda_1}z_1^{\lambda'_1} = |z_1|^{2\Re \lambda_i}\), completing the proof.

We can now show the existence of the Dunkl weight function \(K_c(x)\) as a tempered distribution on \(\mathfrak{h}_\mathbb{R}\) for all values of \(c \in \mathfrak{p}_\mathbb{R}\):

**Theorem 2.4.3.** Let \(W\) be a finite Coxeter group with reflection representation \(\mathfrak{h}\) and let \(\lambda \in \text{Irr}(W)\) be an irreducible representation. Then there is a family \(K_c(x) = K_{c,\lambda}(x)\), depending real analytically on \(c \in \mathfrak{p}_\mathbb{R}\), of tempered distributions on \(\mathfrak{h}_\mathbb{R}\) with values in \(\text{Herm}(\lambda)\) such that \(K_c(x)\) represents the Gaussian inner product \(\gamma_{c,\lambda}\) on \(\Delta_c(\lambda)\) in the sense that

\[
\gamma_{c,\lambda}(P, Q) = \int_{\mathbb{R}} Q(x)^\dagger K_{c,\lambda}P(x)e^{-|x|^2/2}dx
\]

for all \(P, Q \in \mathbb{C}[\mathfrak{h}] \otimes \lambda\).

**Proof.** As usual, we have fixed a positive-definite \(W\)-invariant Hermitian form on \(\lambda\) and we implicitly use this form to identify Hermitian (generally, sesquilinear) forms on \(\lambda\) with elements of \(\text{End}_\mathbb{C}(\lambda)\).

Let \(I_\mathbb{R} \subset \mathfrak{p}_\mathbb{R}\) be a 1-dimensional \(\mathbb{R}\)-linear subspace. As in Theorem 2.3.3, let \(K_c : \mathfrak{h}_{\mathbb{R},\text{reg}} \to \text{End}_\mathbb{C}(\lambda)\) be a nonvanishing real analytic function of \(x \in \mathfrak{h}_{\mathbb{R},\text{reg}}\), depending real analytically...
on \( c \in \mathbb{R} \), satisfying the \( W \)-equivariance condition \( K_c(x) = wK_c(wx) \) for all \( w \in W \) and the differential equation

\[
dK_c + \sum_{s \in S} c_s \frac{d\alpha_s}{\alpha_s} (K_c s + sK_c) = 0
\]
on \( \mathfrak{h}_{\mathbb{R}, \text{reg}} \), and representing the form \( \gamma_{c, \lambda} \), up to a meromorphic scalar factor \( f(c) \), on the subspace \( \Delta^N \Delta_c(\lambda) \) for \( N >> 0 \) depending on \( c \). To simplify the notation, we will identify \( \mathbb{I}_c \) with \( \mathbb{R} \); let \( A \) be the Dunkl differential 1-form on \( \mathfrak{h}_{\text{reg}} \) with values in \( \text{End}_c(\lambda) \) such that the differential equation above takes the form \( dK_c + c(AK_c + K_cA^\dagger) = 0 \) along \( \mathfrak{h}_{\mathbb{R}, \text{reg}} \). Viewing this equation as a differential equation on the complex manifold \( \mathfrak{h}_{\text{reg}} \), we may extend \( K_c \) to a (possibly multi-valued) holomorphic function \( K_c : \mathfrak{h}_{\text{reg}} \to \text{End}_c(\lambda) \) with the same properties; however, by the proof of Theorem 2.2.11 we see that \( K_c \) is single valued for \( |c| \) small, and therefore for all \( c \in \mathbb{R} \) by analyticity.

Let \( \pi : Y \to \mathfrak{h} \) be the De Concini-Procesi wonderful model [DP] for the hyperplane arrangement \( \bigcup_{s \in S} \ker(\alpha_s) \subset \mathfrak{h} \). Recall that \( Y \) is a smooth \( \mathbb{C} \)-variety with \( W \)-action, \( \pi \) is proper and \( W \)-equivariant, \( \pi \) restricts to an isomorphism \( \pi^{-1}(\mathfrak{h}_{\text{reg}}) \to \mathfrak{h}_{\text{reg}} \), and \( \pi^{-1}(\bigcup_{s \in S} \ker(\alpha_s)) \) is a normal crossings divisor. Furthermore, let \( \pi_{\mathbb{R}} : Y_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}} \) denote the real locus of \( \pi \), which shares the same properties as \( \pi \) but for the corresponding real hyperplane arrangement in \( \mathfrak{h}_{\mathbb{R}} \). Let \( Y_{\mathbb{R}, \text{reg}} = \pi^{-1}(\mathfrak{h}_{\text{reg}}) \) and let \( Y_{\mathbb{R}, \text{reg}} = Y_{\text{reg}} \cap Y_{\mathbb{R}} = \pi^{-1}(\mathfrak{h}_{\mathbb{R}, \text{reg}}) \).

Consider the regular-singular meromorphic 1-form \( \pi^*A \) on \( Y \). Clearly, \( \pi^*A \) is holomorphic on \( Y_{\text{reg}} \) and \( \pi^*K_c \) satisfies the differential equation \( d\pi^*K_c + c(\pi^*A\pi^*K_c + \pi^*K_c(\pi^*A))^\dagger = 0 \). Let \( D \) be a component of the divisor \( Y \setminus Y_{\text{reg}} \). Let \( d \in D \) be a generic point, let \( \gamma \) be a generic curve in \( Y \) intersecting \( D \) at the generic point \( d \), and let \( A(\pi(d)) \in \text{End}_c(\lambda) \) be the sum of the residues of \( A \) on the reflection hyperplanes passing through the generic point \( \pi(d) \) of \( \pi(D) \subset \mathfrak{h} \) (note that \( A(\pi(d)) \) depends only on the stabilizer \( W' = \text{Stab}_W(\pi(d)) \) and is identified with a Hermitian form on \( \lambda \)). It follows that the residue of \( \pi^*A \) on \( D \) is \( A(\pi(d)) \). Next, let \( b \in Y \) be arbitrary. As \( Y \setminus Y_{\text{reg}} \) is a normal crossings divisor, there exists a coordinate system \( z_1, ..., z_n \) for \( Y \) about \( b \) and an integer \( m \), \( 0 \leq m \leq n \) such that the divisor \( Y \setminus Y_{\text{reg}} \) is given by the equation \( z_1 \cdots z_m = 0 \). We then see that, in these coordinates, the 1-form \( \pi^*A \) and function \( \pi^*K_c \) are as in Lemma 2.4.2. In particular, in these local coordinates, the matrix coefficients of \( \pi^*K_c \) are linear combinations of functions of the form \( |z_1|^{2\lambda_1} \cdots |z_m|^{2\lambda_m} f_c(z_1, ..., z_n) \), where \( f_c(z) \) is holomorphic in \( z \) and real analytic in \( c \in \mathbb{R} \) and where, for \( 1 \leq i \leq m \), \( \lambda_i \) is an eigenvalue of the residue of \( \pi^*A \) along the component \( z_i = 0 \). Let \( x_1, ..., x_n \) denote the corresponding real coordinates. It is well-known that the function \( |x_1|^{2\lambda_1} \cdots |x_m|^{2\lambda_m} \) on the open set \( \{ x : x_i \neq 0, i = 1, ..., m \} \) extends canonically to a distribution meromorphic in \( c \). As the functions \( f_c \) are smooth and depend analytically on \( c \), it follows that the function \( \pi^*K_c|_{Y_{\mathbb{R}, \text{reg}}} \) extends canonically to a distribution \( \mu_c \) on \( \mathbb{R} \) depending analytically on \( c \in \mathbb{R} \) and with poles at a discrete set of parameters \( c \). Furthermore, it then follows from the Weierstrass factorization theorem that there exists an entire function \( q(c), c \in \mathbb{C} \) such that \( \mu_c \) is analytic and non-vanishing at all \( c \in \mathbb{R} \).

Now consider the pushforward \( \pi_{\mathbb{R}, \ast}\mu_c \), a distribution on \( \mathfrak{h}_{\mathbb{R}} \) depending real analytically on \( c \in \mathbb{R} \). By the construction above and the \( W \)-equivariance of \( K_c \) and \( \pi_{\mathbb{R}} \), it follows that \( \pi_{\mathbb{R}, \ast}\mu_c \) is \( W \)-equivariant for all \( c \in \mathbb{R} \). Furthermore, recall that for \( |c| \) small the function \( K_c \) is locally integrable and defines a distribution on \( \mathfrak{h}_{\mathbb{R}} \). By the construction of \( \mu_c \), we see that for \( |c| \) small the distribution \( \pi_{\mathbb{R}, \ast}\mu_c \) is given by integration against \( q(c)K_c \). By Lemma 2.2.5 it follows that there is a constant \( a \in \mathbb{R} \) such that \( \pi_{\mathbb{R}, \ast}\mu_a \) is homogeneous of degree \( ca \) for \( |c| \) small; as \( \pi_{\mathbb{R}, \ast}\mu_c \) is analytic in \( c \), it follows that \( \pi_{\mathbb{R}, \ast}\mu_c \) is homogeneous, and
hence tempered, for all $c \in \mathbb{R}$. Viewing $\pi_{\mathfrak{h}, \star, \mu_c}$ as an operator $S(\mathfrak{h}) \otimes \lambda \to \lambda$, it follows that for small $|c|$ we have $\pi_{\mathfrak{h}, \star, \mu_c} \circ D_y(c) = 0$ for all $y \in \mathfrak{h}$, where $D_y(c)$ is the Dunkl operator on $S(\mathfrak{h}) \otimes \lambda$ in the direction $y \in \mathfrak{h}$ and with parameter $c$. By analyticity again, it follows that $\pi_{\mathfrak{h}, \star, \mu_c} \circ D_y(c) = 0$ for all $y \in \mathfrak{h}$ and $c \in \mathbb{R}$. Next, it follows from Theorem 2.2.11 that for $|c|$ small we have $\pi_{\mathfrak{h}, \star, \mu_c}(e^{-|x|^2/2}) = r(c)\text{Id}$, where $r(c)$ is analytic and non-vanishing; it follows that $r$ extends uniquely to an analytic function on all of $\mathbb{R}$ such that $\pi_{\mathfrak{h}, \star, \mu_c}(e^{-|x|^2/2}) = r(c)\text{Id}$ holds for all $c \in \mathbb{R}$. It follows from Lemma 2.2.4 that for each $c \in \mathbb{R}$ we have that the distribution $\pi_{\mathfrak{h}, \star, \mu_c}$ represents the form $r(c)\gamma_{c, \lambda}$ on $\Delta_c(\lambda)$. As $\pi_{\mathfrak{h}, \star, \mu_c}$ is determined by its action on the dense subspace $\mathbb{R}[\mathfrak{h}]e^{-|x|^2/2}$ and is nonzero by construction, it follows that $r(c)$ is non-vanishing. The distribution $r(c)^{-1}\pi_{\mathfrak{h}, \star, \mu_c}$ therefore represents $\gamma_{c, \lambda}$. By an abuse of notation, let $K_c$ denote this distribution.

It remains to show that $K_c$ is analytic in $c \in \mathfrak{p}$ as a distribution, i.e. that $K_c(\varphi)$ is an analytic function of $c$ for all test functions $\varphi \in S(\mathfrak{h})$ (the construction above showed analytic dependence on $c$ along any 1-dimensional subspace $\mathfrak{l} \subset \mathfrak{p}$). However, it is easy to see that Lemma 2.4.2 and the above proof are easily adapted to the multi-parameter case $\dim \mathfrak{p} > 1$ and we see that $K_c$ is analytic in $c$.

Remark 2.4.4. The support of the distribution $K_c$ coincides with the set of real points of the support of $L_c(\lambda)$:

$$\text{Supp}(K_c) = \text{Supp}(L_c(\lambda))_{\mathbb{R}}.$$ 

This was shown by Etingof [E, Proposition 3.10] in the case $\lambda = \text{triv}$, and the proof generalizes to arbitrary $\lambda$ without modification.

References