LECTURE I  The symplectic topologist as a dynamicist

Paul Seidel, MIT

Motivated by recent work of
Cineli–Ginzburg–Gürel, Shelukhin
(all errors and omissions due to my ignorance: I am not a specialist)
Symplectic linear algebra

$\text{Sp}(2n) \subset \text{SL}(2n, \mathbb{R})$ linear maps which preserve

$$\omega_{\mathbb{R}^{2n}} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$$

i.e. $A \in \text{Sp}(2n)$ satisfies

$$\langle v, Jw \rangle = \langle Av, AJw \rangle$$

where $J = i$ on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

Eigenvalues come in blocks

1) $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$

2) $e^{i\Theta} \quad e^{-i\Theta}$

Intuitively, only 2) shows a nonzero “net amount of rotation”

To measure that, we pass to the universal cover:

$$\pi_1(\text{Sp}(2n)) \cong \pi_1(\text{U}(n)) = \mathbb{Z}$$

So

$$\text{U}(n) \hookrightarrow \text{Sp}(2n) \quad \text{U}(n) \hookrightarrow \widetilde{\text{Sp}}(2n)$$
Rotation number. There is a unique homogeneous quasimorphism \( \rho \):
\[
\begin{array}{ccc}
U(n) & \xrightarrow{\det} & S^1 \\
\uparrow & \searrow \text{det} & \uparrow e^{-\pi i \Theta} \\
\tilde{U}(n) & \xrightarrow{\det} & \mathbb{R} \\
\downarrow & \searrow \rho & \\
\tilde{Sp}(2n) & \to & \mathbb{R} \\
\end{array}
\]

homogeneous: \( \rho(\tilde{A}^k) = k \rho(\tilde{A}) \)

quasimorphism:
\[
| \rho(\tilde{A}) \rho(\tilde{B}) - \rho(\tilde{A} \tilde{B}) | < \text{Const.}
\]

This is a key notion in studying iterations \( k \mapsto \tilde{A}^k \)

Conley-Zehnder index. Take
\[
\text{Sp}(2n)^* = \{ A \in \text{Sp}(2n) : 1 \text{ is not an eigenvalue} \}
\]

Then there is a quasimorphism
\[
\begin{array}{ccc}
\text{Sp}(2n)^* & \xrightarrow{\text{sign}(\det(4-I-A))} & \{ \pm 1 \} \\
\uparrow & \searrow (-1)^m & \\
\tilde{\text{Sp}}(2n)^* & \xrightarrow{\mu} & \mathbb{Z} \\
\end{array}
\]

what the index \( \mu(\tilde{A}) \) counts:
when going from \( \tilde{A} \) to \( \tilde{A} \tilde{B} \) in \( \tilde{Sp}(2n) \), how often do the eigenvalues cross \( 1 \) (with signs).
Examples:

- \( \tilde{A} = \begin{pmatrix} e^t & e^{-t} \\ e^{-t} & e^t \end{pmatrix} \), \( \mu(\tilde{A}) = 1 \)

- \( \tilde{A} = \exp(tJH) \), \( H \) nondegenerate symmetric and \( t > 0 \) small

- \( \mu(\tilde{A}) = \text{Morse index of } H = \text{number of negative eigenvalues} \)

- \( \tilde{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \)

- \( \mu(\tilde{A}) = -2 \left\lfloor \frac{\theta}{2\pi} \right\rfloor \)

- \( \mu(\tilde{A}^{-1}) = 2n - \mu(\tilde{A}) \)

\( \mu \) is not homogeneous, but we can instead use the homogeneity of \( \rho \) and the fact

\[
\mu(\tilde{A}) = \rho(\tilde{A}) + \text{correction in } (0, 2n)
\]

For instance, as \( k \to \infty \),

\[
\begin{cases}
\mu(\tilde{A}^k) \to +\infty ; \text{ or} \\
\mu(\tilde{A}^k) \in (0, 2n) \text{ for all } k ; \text{ or} \\
\mu(\tilde{A}^k) \to -\infty
\end{cases}
\]

This reduces to

\[
\begin{cases}
\rho(\tilde{A}) > 0 \\
\rho(\tilde{A}) = 0 \\
\rho(\tilde{A}) < 0
\end{cases}
\]
Hamiltonian diffeomorphisms

Imagine a time-dependent system in classical mechanics,

\[ H = H_t(p, q), \quad t \in \mathbb{R}, \quad (p, q) \in \mathbb{R}^{2n} \]

Equations of motion:

\[ \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \]

Example:

\[ H_t(p, q) = \frac{1}{2} \| p \|^2 + V_t(q) \]

\[ \dot{p} = -\nabla V_t, \quad \dot{q} = p \Rightarrow \ddot{q} = -\nabla V_t \]

Suppose \( H_t = H_{t+1} \). Then it makes sense to look for \( k \)-periodic trajectories (\( k = 1, 2, 3, \ldots \))

Global analogue: \( (M^{2n}, \omega_M) \) is a compact symplectic manifold \( \omega_M \in \Omega^2(M) \) closed, locally \( \omega_M = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n \)

Every \( H \in C(\mathbb{M}, \mathbb{R}) \) gives rise to a Hamiltonian vector field \( X_H \).

Time-dependent \( H_t \Rightarrow X_t \).

Solving

\[ x = X_t(x) \]

gives a family \( (\phi_t) \) of diffeomorphisms of \( M \). Assume \( H_t = H_{t+1} \)

\[ \Rightarrow \phi_{t+k} = \phi_t \circ \phi_k \quad (k \in \mathbb{Z}) \]

We set \( \phi = \phi_1 \), and look for \( k \)-periodic points of \( \phi \).
Nondegeneracy: The aim is to make the local structure of fixed/periodic points simple.

\[ \varphi(x) = x \] nondegenerate fixed point:

1. not an eigenvalue of \( \text{D} \varphi_x \)
2. \( \varphi_x \in \text{Sp}(2n) \)

\[ \varphi^k(x) = x \] nondegenerate periodic point:

no root of unity is an eigenvalue of \( \text{D} \varphi^k_x \)

\( \Rightarrow x \) is a nondegenerate fixed point of \( \varphi, \varphi^2, \varphi^3, \ldots \)

A "generic" \( \varphi \) has only nondegenerate periodic points.

Example: \( H \in C^\infty(M, \mathbb{R}) \) a Morse function (time-independent) and small. Remember the local formula:

\[ \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \]

If \( x = (p, q) \) is a critical point of \( H \), it is a fixed point of \( \varphi \) (nondegenerate, and there are no other fixed points).

Thm ("Arnold conjecture"): If \( \varphi \) has nondegenerate fixed points,

\[ \# \text{Fix}(\varphi) \geq \sum_i \dim H^i(M; \mathbb{F}) \]

for any field \( \mathbb{F} \) (long story; recent breakthrough by Abouzaid-Blumberg for \( \text{char}(\mathbb{F}) > 0 \)
From now on, always assume $\varphi$ has nondegenerate periodic points.

**Question** ("Nondegenerate Conley conjecture") Are there necessarily infinitely many periodic points?

- **Yes** For $M = T^{2n} = \mathbb{R}^n/\mathbb{Z}^{2n}$ with $\omega_M = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ (Conley-Zehnder 1984)

- **No** For $M = S^2$, $\varphi$ an irrational rotation
  (or more generally, any $M$ which has a Hamiltonian torus action with isolated fixed points)

**Topological data:**

$$[\omega_M] \in H^2(M; \mathbb{R})$$
$$c_1(M) = c_1(TM) \in H^2(M; \mathbb{Z})$$

**Answer is yes:**

- $c_1(M) = 0$ (Salamon-Zehnder 1992)
- $c_1(M) = -[\omega_M]$ (Ginzburg-Gürel 2010)
- ...

Take $M = \mathbb{C}P^2$ blown up at $m \leq 8$ points, $c_1(M) = [\omega_M]$ (del Pezzo)

$m = 3$ **No**, they admit torus action
$m > 3$ **Unknown**, the symplectic topology changes dramatically at $m = 5$ (algebra-geometric moduli)
Hamiltonian Floer cohomology

Take the twisted free loop space

\[ \mathcal{L}_\varphi = \{ x : \mathbb{R} \rightarrow M, \ x(t) = \varphi(x(t+1)) \} \]

It carries the (multivalued) action functional à la Hamilton-Jacobi

\[ dA(x) \overline{\xi} = - \int_0^1 \omega_M(x', \overline{\xi}) \, dt \]

Critical points have \( x' = 0 \iff x \) is constant at a fixed point of \( \varphi \). Nondegenerate fixed points \( \iff A \) is Morse.

Floer cohomology is (formally) a kind of Morse theory for \( A \).

One defines a chain complex

\[ CF^*(\varphi) = \bigoplus K \cdot x \]

(\( K = \) coefficient field, as in ordinary cohomology, let's say \( \text{char}(K) = 0 \); \( x \) are the fixed points), which with a suitable differential defines Floer cohomology \( HF^*(\varphi) \).

**Theorem** \( HF^*(\varphi) \cong H^*(M; K) \)

(cannotical given \( \varphi = \varphi_1 \leftrightarrow \varphi_0 = \text{id}_M \))

The Arnol'd conjecture follows directly from that.
Conley conjecture for $c_1(M) = 0$

In this case, if $\varphi(x) = x$, $D\varphi_x$ has a preferred lift to $Sp(2n)$.

The Conley-Zehnder index

$$\mu(\varphi, x) = \mu(\tilde{D}\varphi_x) \in \mathbb{Z}$$

defines the degree of $x \in CF^*(\varphi)$ (if $c_1(M) = 0$, $HF^*(\varphi)$ is only graded mod 2).

Suppose that $\varphi$ only has finitely many periodic points. After passing to an iterate, these all become fixed points. We look at the Floer complex for $\varphi^k$.

\[ CF^*(\varphi^k) = \bigoplus \mathbb{R} \mathbb{K} x \]

Fixed point of $\varphi$ in degree $\mu(\varphi^k, x)$

For $k \gg 0$, the iteration rule says that $\mu(\varphi^k, x) \not\in \{0, 2n\}$.

But that is a contradiction to

$HF^*(\varphi^k) \cong H^*(M; \mathbb{K}) \neq 0$, $\ast = 0, 2n$

Situation for $M = S^2$, $\varphi$ rotation

\[ \theta \in (0, 2\pi) \quad \theta \in (2\pi, 4) \]
Floer cohomology as a TQFT (topological quantum field theory)

Remember \( CF^*(\varphi) = \bigoplus \mathbb{R} \times X \) for fixed points \( \varphi(x) = x \), thought of as constant loops in \( L_\varphi \). The differential is defined using “gradient flow lines” \( \mathbb{R} \rightarrow L_\varphi \), or maps

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\mathbf{u} : \mathbb{R} \times \mathbb{R} & \rightarrow M \\
\mathbf{u}(s,t) = \varphi(\mathbf{u}(s,t+1)) \\
\overline{\partial} \mathbf{u} = \frac{\partial \mathbf{u}}{\partial s} + J_t(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial t} = 0
\end{array}
\right.
\end{align*}
\]

where \( J_t \) is an almost complex structure on \( M \) (necessarily \( t \)-dependent, \( J_t = \varphi_* J_{t+1} \))

Formal structure: think of this as maps from a cylinder, with a “seam” marking \( \varphi \)-periodicity

\[
\mathbb{R} \times S^1 \xrightarrow{\varphi} \mathbb{R} \times S^1
\]

differential for \( HF^*(\varphi) \)

One can use other Riemann surfaces as well:

- “pair-of-pants” product
  \[ HF^*(\varphi) \otimes HF^*(\varphi) \rightarrow HF^*(\varphi^2) \]

- \( p \)-fold product
  \[ HF^*(\varphi) \otimes \cdots \rightarrow HF^*(\varphi^p) \]
The proof of the Conley conjecture in the case $[\omega_M] = -c_1(M)$ uses the product structure. Slightly more precisely,

- Assume only finitely many periodic orbits;
- Show that then, the $p$-fold pair-of-pants power of any class in $HF^*(\varphi)$ becomes zero, $p \gg 0$;
- Contradiction! Because we know what the pair-of-pants product corresponds to in $H^*(M, \Omega K) \cong HF^*(\varphi)$: the (small) quantum product.

**Small quantum product** is a deformation of the cup product by contributions from $J$-holomorphic maps $\mathbb{CP}^1 \to M$.

Ordinary cup product is the intersection of cycles; non-local contribution of a holomorphic map.

Example: $H^*(S^2) = \Omega K \cdot 1 \oplus \Omega K \cdot a$

- classical product: $a \cdot a = 0$
- quantum product: $a \ast a = 1$

Unfortunately, no known relation between $\ast$ and Conley conjecture!
(Quantum) Steenrod operations

From now on, we work with a coefficient field $\mathbb{K}$, $\text{char}(\mathbb{K}) = p > 0$.

Recall group cohomology of the cyclic group,

$$H^*_\mathbb{Z}_p(\text{point}; \mathbb{K}) = H^*_\mathbb{Z}_p(\text{contractible}; \mathbb{K}) = \mathbb{K}[[t]] \oplus \mathbb{K}[[t]] \Theta$$

$|t| = 2$, $|\Theta| = 1$

is one-dimensional in each degree (for $p = 2$, this is $H^*_*(\text{RP}_\infty; \mathbb{Z}_2)$).

The classical Steenrod operation is

$$\text{St}: H^i(M; \mathbb{K}) \rightarrow H^p_\mathbb{Z}_p(M; \mathbb{K}) = (H^*_\mathbb{K}(M; \mathbb{K}) \otimes H^*_\mathbb{Z}_p(\text{point}; \mathbb{K}))_p$$

It exploits the symmetry of the ordinary cup-product

$$\text{St}(x) = x \cup \cdots \cup x + \text{terms with } t \text{ or } \Theta$$

on the other hand,

$$\text{St}(x) = (\text{nonzero constant}) \cdot x + t$$

+ cohomology classes of higher degree

Therefore, $\text{St}$ is injective. There is (under certain assumptions on $M$) a quantum counterpart $Q\text{St}$.

Example: $M = S^2$, $a \in H^2(S^2; \mathbb{K})$

classical $\text{St}(a) = (\text{nonzero const.})$ at quantum $p = 2$, $Q\text{St}(a) = at + 1$
Quantum Steenrod and iteration
Remember that Floer cohomology is defined using the loop space \( \mathcal{L}_\varphi \).
For an iterate \( \varphi^p \), we have

\[ \mathcal{L}_{\varphi^p} = \{ x : [0, 1] \to M, \ x(0) = \varphi^p(x(1)) \} \]

\[ \equiv \{ x_0 : [0, \frac{1}{p}] \to M, \ x_0(0) = \varphi(x_1(\frac{1}{p})) \]

\[ x_1 : [\frac{1}{p}, \frac{2}{p}] \to M, \ x_1(0) = \varphi(x_2(\frac{1}{p})) \]

\[ \ldots \]

\[ x_{p-1} : [\frac{p-1}{p}, 1] \to M, \]

\[ x_{p-1}(0) = \varphi(x_{0}(\frac{1}{p})) \}

It carries a natural \( \mathbb{Z}_p \)-action, which is compatible with the \( p \)-fold pair-of-pants product,

\[ \text{HF}^*(\varphi)^\otimes_p \to \text{HF}^*(\varphi^p) \]

By exploiting that symmetry, we get an equivariant \( p \)-th power map on Floer cohomology,

\[ \text{HF}^*(\varphi) \xrightarrow{p \text{-th power}} \text{HF}^{p*}(\varphi^p) \]

\[ \text{H}^*(M; \mathbb{K}) \xrightarrow{\varphi^p} (\text{H}(M; \mathbb{K}) \otimes \text{H}_{/p}(\text{point}))^{p*} \]

unlike the ordinary pair-of-pants product, the equivariant version “knows that \( \varphi(x) = x \Rightarrow \varphi^p(x) = x \)”
Quasi-rotations This is an extreme case of the Conley conjecture. Suppose $\text{Hom}(M; \mathbb{Z})$ is torsion-free. $\varphi: M \rightarrow M$ is called a pseudo-rotation if

$$\# \text{Fix}(\varphi) = \text{rank } H^*(M)$$

(the least number of fixed points allowed by the Arnold's conjecture). As usual, we also impose a non-degeneracy assumption. The consequence is that

$$\text{CF}^*(\varphi) = \text{CF}^*(\varphi^k)$$

for all $k$ (ignoring grading issues), with zero differentials.

**Theorem** (Shelukhin; May 2019)

Let $M$ satisfy

1. $c_1(M) = [\omega_M]$
2. For $p = 2$, $Q\text{Sq}(\text{point})$ is equal to $S\text{q}(\text{point})$
3. $c_i(M) \in H^2(M; \mathbb{Z})$ is divisible by some integer $n = \frac{1}{2} \dim(M)$

$\uparrow$ true for $M = \mathbb{C}P^n$, but otherwise rare

Then $M$ has no pseudo-rotations.

**Theorem** (Cineli-Gintburg-Gürel; Shelukhin; September 2019)

(3) can be dropped $\leftarrow$ important
Recall that

\[ H^*_\mathbb{Z}_p(\text{point}; K) = k[[t]] \oplus \mathbb{K}[[t]] \]

For a given prime \( p \), let \( \ell_p(M) \) be the largest integer such that

\[ H^*(M; K) \xrightarrow{Q_S^t} H^*(M; K) \otimes_{t^{\ell_p(M)}} H^*_\mathbb{Z}_p(\text{point}) \]

is not injective. For classical Steenrod operations,

\[ S^t([\text{point}]) = t^{n(p-1)} [\text{point}] \]

so the counterpart of our quantum notion would be \( n(p-1) \).

Unpublished (S-Shekuhchii)

If \( M \) admits a pseudorotation,

\[ \limsup_p \frac{\ell_p(M)}{p} < \infty. \]

The end (for now...).