Worksheet: Stokes' theorem, Surface integrals

The flux of a vector field $\vec{F}$ through a surface S given as the graph of a function $z = f(x, y)$ can be written as $\iint_S \vec{F} \cdot \vec{dS}$, where

$$\vec{dS} = \vec{n}\ dS = \pm(-f_x\vec{i} - f_y\vec{j} + \vec{k})\ dx\ dy.$$

The curl of a vector field $\vec{F} = (M, N, P)$ is the vector

$$\vec{\text{curl}}\vec{F} = (P_y - N_z)\vec{i} + (M_z - P_x)\vec{j} + (N_x - M_y)\vec{k}$$

A conservative vector field $\vec{F}$ is such that $\vec{\text{curl}}\vec{F} = \vec{0}$. From a conservative vector field, we can build a potential, i.e. a function f such that $\vec{\nabla} f = \vec{F}$.

Stokes' theorem. Consider a surface S in 3D, bounded by a curve C. We orient S and C such that, when looking at S from above, the curve C is positively (counterclockwise) oriented. Then, for any vector field $\vec{F}$,

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S \vec{\text{curl}}\vec{F} \cdot \vec{dS}.$$

Cylindrical and spherical surface integrals. The surface element on the sides of a cylinder of radius a is $dS = a\ d\theta dz$. The surface element on a sphere of radius a is $dS = a^2 \sin\phi\ d\theta d\phi$.

Problem 1. Consider the cap S of the sphere of radius 3, such that $\phi \leq \pi/6$. Compute the integral of the function $f = 1/r$ on S, where r is the polar radius (distance to the z-axis).

Problem 2. Consider the surface S which is the piece of the cylinder $r = 2$ contained between the plane $z = 0$ and the parabola $z = 2 + x^2 + (y - 1)^2$.

Compute the area of S.

Bonus 1. Application of Stokes' theorem. Your friend Maxwell tells you that the magnetic field $\vec{B}$ generated by a steady current $\vec{I}$ satisfies $\vec{\text{curl}}\vec{B} = \mu_0\vec{I}$ where μ_0 is some constant.

We consider an infinite wire following the z-axis, with a steady current $\vec{I} = I\vec{k}$ going upwards through it. The goal of this problem is to compute the magnetic field $\vec{B}$ outside of the wire, using Maxwell's law.

- Let $\mathcal{P}$ be a plane containing the wire. The plane $\mathcal{P}$ is a plane of symmetry for this setup. Using the fact that the magnetic force $\vec{F}$ exerted on a particle of charge q located on $\mathcal{P}$ and moving at velocity $\vec{v}$ is given by $\vec{F} = q\vec{v} \times \vec{B}$, show that $\vec{B}$ is orthogonal to $\mathcal{P}$, i.e. parallel to the cylindrical unit vector $\vec{\theta}$.

Hint: For any two vectors $\vec{v'}$ and $\vec{v}$ that are symmetric with respect to $\mathcal{P}$, then the two vectors $\vec{v} \times \vec{B}$ and $\vec{v'} \times \vec{B}$ also need to be symmetric with respect to $\mathcal{P}$. Check what happens when the velocity $\vec{v}$ is one of the cylindrical unit vectors $\vec{r}$, $\vec{\theta}$ and $\vec{k}$, and see that $\vec{B}$ needs to be normal to $\mathcal{P}$.

- From the previous question, we know that $\vec{B} = f(r, \theta, z)\vec{\theta}$. Using symmetries of the problem, show that $\vec{B} = f(r)\vec{\theta}$.
- Compute the line integral $\oint_C \vec{B} \cdot \vec{dr}$ on a circle going around the z-axis in two different ways (use Stokes' theorem) and deduce that the function $f(r) = \frac{\mu_0 I}{2\pi r}$.

  Note: As is common, we assume the wire to be infinitesimally thin. If that confuses you, you can instead choose the wire to have a small radius a, with the current inside the wire being $\vec{I} = \frac{I}{\pi a^2}\vec{k}$.