November 6th - 18.02 - Fall 17

Worksheet: Green's theorem, Flux

Green's theorem. Let C be a closed simple curve oriented counterclockwise that bounds a region R of the plane. The work of a vector field $\vec{F} = (M, N)$ around C is given by

$$W := \oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \; dxdy,$$

where the curl is curl $\vec{F} = N_x - M_y$.

Green's theorem in normal form. We denote by $\vec{n}$ the outer unit normal vector to the curve C. The flux of a vector field $\vec{F} = (M, N)$ through C is given by

$$\phi := \oint_C \vec{F} \cdot \vec{n} \; ds = \iint_R \text{div } \vec{F} \; dxdy,$$

where the divergence is div $\vec{F} = M_x + N_y$.

Problem 1. The only thing you know about the path C is that it stays in the first quadrant and goes from $(a, 0)$ on the x-axis to $(0, b)$ on the y-axis. Compute the flux of the vector field $\vec{F} = (-xe^y, e^y)$ through C (we orient C so that its normal is generally pointing away from the origin).

Problem 2. What simple closed curve C, counterclockwise oriented, minimizes the line integral

$$\oint_C 4y\left(1 - \frac{y^2}{3}\right) dx + \frac{x^3}{3} dy?$$

Problem 3. We consider the fector field

$$\vec{F} = \frac{x}{x^2 + y^2}\vec{i} + \frac{y}{x^2 + y^2}\vec{j}.$$

(a) Let $\mathcal{C}_r$ be the counterclockwise circle of radius r centered at the origin. By direct computation of a line integral, find the flux of $\vec{F}$ out of $\mathcal{C}_r$.

(b) Use Green's theorem to compute the flux of $\vec{F}$ out of a curve C that does not surround the origin.

(c) Use Green's theorem together with (a) to show that the flux of $\vec{F}$ out of a curve C that surrounds the origin is equal to 2π.