OCTOBER 25TH - 18.02 - FALL 17

Worksheet: Line integrals

Line integrals. Given a vector field $\vec{F}(x, y)$ and a path C between two points P_0 and P_1 in the plane, we want to compute the line integral

$$I = \int_C \vec{F} \cdot d\vec{r}.$$

In order to carry out the computation, we need a parametrization $\vec{r}(t)$, $a \leq t \leq b$ of the path C. The line integral can then be computed as a usual integral in one variable:

$$I = \int_a^b \left(F(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \right) \, dt.$$

The line integral does not depend on the parametrization of C (see Bonus Problem 1).

Problem 1. Consider the vector field $\vec{F} = (x, 1)$. Let C be the piece of the curve $y = e^x$ running between $x = 0$ and $x = 2$.

Compute the integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

Problem 2. Let C be the quarter of the unit circle running from $(1, 0)$ to $(0, -1)$, and let $\vec{F} = y\vec{i}$.

(a) Without computation, guess whether the integral $I = \int_C \vec{F} \cdot d\vec{r}$ is positive or negative.
(b) Compute I to check your guess.

Problem 3. Let C be the straight line segment from $(0, 0)$ to $(1, 1)$. Let C' be a different path from $(0, 0)$ to $(1, 1)$, where we first move in a straight line from $(0, 0)$ to $(0, 1)$ and then move in a straight line from $(0, 1)$ to $(1, 1)$.

Consider the vector field $\vec{F} = -y\vec{i} + x\vec{j}$, and compute

$$I_C = \int_C \vec{F} \cdot d\vec{r}, \quad \text{and}$$
$$I_{C'} = \int_{C'} \vec{F} \cdot d\vec{r}.$$

Can you understand the value of I_C geometrically?

Problem 4. Consider the vector field $\vec{F} = (0, x - x^2)$, and the path C going from $(0,0)$ to $(1,1)$ via $y = x^2$.

Compute the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

Bonus 1. In this problem, we show that line integrals are independent of the path parametrizations used. Consider a vector field $\vec{F}$ as well as a path C, with two different parametrizations $\vec{r}(t)$, $a \leq t \leq b$ and $\vec{p}(s) = \vec{r}(t(s))$, $c \leq s \leq d$.

(a) Using parametrization $\vec{r}(t)$ and $\vec{p}(s)$ write down in two ways $I_{\vec{r}}$ and $I_{\vec{p}}$ the line integral $I = \int_C \vec{F} \cdot d\vec{r}$.

(b) Check that the two integrals of one variable obtained in (a) are equal. *Hint*: Use the chain rule to rewrite $I_{\vec{p}}$, then use the change of variable formula to compare it with $I_{\vec{r}}$.

Bonus 2. Consider a path C between two points P_0 and P_1. We are interested in the line integral $I = \int_C \vec{F} \cdot d\vec{r}$, for a vector field $\vec{F} = \vec{\nabla} f$ which is the gradient of a function $f(x, y)$. In this problem, we show that I does not depend on the whole path C but only on its endpoints P_0 and P_1.

Suppose we have a parametrization $\vec{r}(t)$, $a \leq t \leq b$ of the path C.

(a) Express the derivative of the function $g(t) = f(\vec{r}(t))$ in terms of the derivatives of f and $\vec{r}$.

(b) Using the parametrization $\vec{r}$, write down the line integral I as an integral of the one variable t. Use (a) to compute this integral in terms of the function g.

(c) Conclude.

Caution: For a general vector field $\vec{F}$, the line integral depends on the whole path, not only its endpoints.

Bonus 3. Consider the gravitational vector field in 3D, $\vec{F}(x, y, z) = (0, 0, -1)$, and let C be the path of a ball thrown from the origin, until it hits the ground: $\vec{r}(t) = (t, 2t, t - t^2)$ for $0 \leq t \leq 1$.

- What is the work done by the gravitational field on the ball from the highest point of the trajectory until it hits the ground?
- What is the work done during the whole trajectory?

Hint: The work is the line integral $\int_C \vec{F} \cdot d\vec{r}$.