Worksheet: Cross product, determinants

Cross product. The cross product $\vec{V} \times \vec{W}$ of two 3D vectors is perpendicular to both $\vec{V}$ and $\vec{W}$. It is $\vec{0}$ if and only if $\vec{V}$ and $\vec{W}$ are parallel to each other. Computing a cross product is similar to computing the determinant of a 3×3 matrix.

Computing determinants. The determinant of a matrix can be interpreted as a volume. For example, the determinant of a 2×2 matrix with rows $\vec{V}$ and $\vec{W}$ is (up to a sign) the area of a parallelogram with sides $\vec{V}$ and $\vec{W}$.

- The determinant of the 2×2 matrix
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
is $|A| = ad - bc$.
- The i, j-cofactor c_{ij} of a 3×3 matrix A is the determinant of the 2×2 matrix obtained from A by erasing its ith row and its jth column.
- The determinant of an 3×3 matrix A can be computed by expansion along any row or column: Say we want to compute it by expanding along the ith row, then:
$$|A| = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} c_{ij},$$
where a_{ij} is the number located in the ith row and jth column of the matrix A, and where c_{ij} is the i, j-cofactor. Note that the sum (in j) can be seen as a sum over the columns of the matrix, and that $(-1)^{i+j}$ is an alternating sign.
- Determinants can sometimes be computed easily by using a few tricks:
  - Expand the determinant along a row or column that has a lot of zero coefficients.
  - Exchanging two rows flips the sign of the determinant (same for columns).
  - Adding (or substracting) a row to another does not change the determinant (same for columns).

Problem 1. Compute the determinant of the matrix
$$A = \begin{pmatrix} 1 & -1 & 3 \\ 1 & -2 & 0 \\ 1 & 4 & 2 \end{pmatrix}$$
in two different ways: by expanding along the third column and by expanding along the second line.

Note: Both are a good choice to expand because of the zero.

Problem 2. Find a vector perpendicular to both $\vec{V} = (1,3,3)$ and $\vec{W} = (2,2,0)$.
Hint: Compute a cross product.

Problem 3. Find the area of the parallelogram with vertices (in counterclockwise order) at $(-2,1), (1,2), (3,3)$ and $(0,2)$ by computing a determinant.

Problem 4. Show that the parallelepiped with sides given by the vectors $(1,1,1)$ $(0,2,3)$ and $(2,4,5)$ is flat.
Hint: You can compute its volume as a determinant.

Bonus 1. Find an efficient way to compute the determinants of the following matrices:

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 4 & 2 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 2 \\ 1 & 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Bonus 2. Let $\vec{V}$ and $\vec{W}$ be two vectors in the plane. We claimed that adding a matrix row to another does not change the determinant. Hence we should have the following inequality

$$\begin{vmatrix} \vec{V} \\ \vec{W} \end{vmatrix} = \begin{vmatrix} \vec{V} \\ \vec{W} + \vec{V} \end{vmatrix}.$$

Show this equality by using the interpretation of the determinant as the area of a certain parallelogram.
Hint: You can show by cutting and pasting that the parallelograms $\mathcal{P}$ with sides $\vec{V}$ and $\vec{W}$ and $\mathcal{P}'$ with sides $\vec{V}$ and $\vec{W} + \vec{V}$ have same area.

Bonus 3. Show that the determinant of the following matrix is zero:

$$A = \begin{pmatrix} 2 & 1 & 6 & 3 \\ -2 & -1 & 1 & 3 \\ 4 & 2 & 2 & 2 \\ 2 & 1 & 2 & 5 \end{pmatrix}.$$

Hint: The determinant is unchanged when substracting a column from another. What happens if you substract the second colum from the first column twice?