Notes on Sheffield's quantum zipper

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Abstract

These are reading notes of Sheffield's *Conformal welding of random surfaces: SLE and the quantum gravity zipper* [11]. In that paper, Sheffield constructs a stationary process $(h_t, \eta_t)_{t\in\mathbb{R}}$ whose marginal law is given by a Gaussian free field and an independent SLE curve. This process evolves with a deterministic cutting and unzipping dynamics: the curve is progressively unzipped, and the field varies according to a natural change of coordinates. The main result of [11] is to prove that the process in reverse time $(h_{-t}, \eta_{-t})_{t\in\mathbb{R}}$ also evolves deterministically.

In these notes, we go through Sheffield's proof of this result, with a minor variation in certain formulations using an idea of [2]. Let us also point to [3, Section 4] for other reading notes on the same topic.

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1 Introduction

In [11], Sheffield studies *quantum surfaces* (Riemann surfaces carrying a random metric) that are progressively cut by independent SLE curves. More precisely, on the upper-half plane, consider the Liouville metric : $e^{\gamma h}$: $\mathrm{d}z$ where h is a Gaussian free field, and draw on it an independent SLE_κ curve η, with $\kappa = \gamma^2$. One can then progressively cut the half-plane alongside η, and study the law of the domain equipped with a random metric that one obtains. The initial conditions we started with turn out to give a stationary measure for this cutting dynamics.

The result of [11] we focus on in these notes states that the reverse time process evolves deterministically, or in other words, that no information is lost in the (forward) cutting dynamics. Indeed, one expects that the field h as observed from the left and the right side of the curve η should be the same on the curve η itself, and it turns out that this is enough information to recover the original curve after unzipping. However, the free field is a very irregular object, and formalizing the intuition that its values match up on both sides of η is not straightforward. This is done by building a measure on η - that Sheffield calls a *quantum time* - somehow carrying the information of the values taken by h on η. A key step in the proof is then to understand the quantum time, which can be seen as a natural time scale for the cutting procedure.

We will recall the definitions and some basic properties of SLE and the free field in Section 2, before defining the zipper coupling in Section 3. We then procede to prove the main theorem (Theorem 3.2) in Section 4. Let us point out that a technically non-trivial step of this result is to control the constant part of the free field during certain operations. In these notes, we partially avoid this work (in the proof of Lemma 4.7, which is used to deduce Lemma 4.6 from Lemma 4.5) by assuming a non-trivial statement on SLE and its natural parametrization.

2 Background

2.1 Schramm-Loewner evolutions (SLE)

Chordal Schramm-Loewner evolutions (SLEs) are a one parameter family of conformally invariant random curves defined in simply-connected domains of the complex plane, with prescribed starting point and endpoint on the boundary.

Let us first give the definition of (forward) SLE_κ in the upper half-plane $(\mathbb{H}, 0, \infty)$. It is a random curve $\eta : \mathbb{R}^+ \to \overline{\mathbb{H}}$, growing from the boundary point 0 to ∞.

Suppose that such a curve η is given to us. Let H_s be the unbounded connected component of $\mathbb{H} \setminus \eta([0, s])$, and consider the uniformizing map $g_s : H_s \to \mathbb{H}$, normalized at ∞ such that

$$g_s(z) = z + 2a_s/z + o(1/z).$$

The quantity a_s is the so-called half-plane capacity of the compact hull $K_s = \mathbb{H} \setminus H_s$ generated by $\eta([0,s])$. Under additional assumptions[1], the half-plane capacity a_s is an increasing bijection of $\mathbb{R}^+$, and so we can reparametrize our curve by $t = a_s$.

With this parametrization, the family of functions g_t solves the Loewner differential equation:

$$\begin{cases} g_0(z) = z \\ \partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \end{cases}$$

where $W_t = g_t(\eta(t))$ is the (real-valued) driving function.

Conversely, starting from a continuous real-valued driving function, it is always possible to solve the Loewner equation, and hence to recover a family of compact sets K_t in $\overline{\mathbb{H}}$, growing from 0 to ∞, namely K_t is the set of initial conditions z that yield a solution $g_u(z)$ blowing up before time t. It may happen that the compact sets K_t coincides with the set of hulls generated by the trace of a curve γ, which can in this case be recovered as $\eta(t) = \lim_{\varepsilon \to 0} g_t^{-1}(W_t + i\varepsilon)$.

Definition 2.1. The process $\mathrm{SLE}_\kappa^{\mathbb{H}}(0 \to \infty)$ is the curve obtained from the solution of the Loewner equation with driving function $W_t = \sqrt{\kappa} B_t$, where B_t is a standard Brownian motion.

The law of $\mathrm{SLE}_\kappa^{\mathbb{H}}(0 \to \infty)$ is invariant by scaling. Hence, given a simply-connected domain (D, a, b) with two marked points on its boundary, we can define $\mathrm{SLE}_\kappa^{D}(a \to b)$ to be the image of an $\mathrm{SLE}_\kappa^{\mathbb{H}}(0 \to \infty)$ by any conformal bijection $(\mathbb{H}, 0, \infty) \to (D, a, b)$.

We now restrict to values of the parameter $\kappa \leq 4$. The SLE curves almost surely are simple curves of dimension $d = 1 + \frac{\kappa}{8}$ [1], and they carry a non-degenerate parametrization morally given by the Hausdorff measure of dimension d.

Definition 2.2 ([9],[8])**.** The Minkowski content μ of dimension $d = 1 + \frac{\kappa}{8}$ of the SLE is called its *natural parametrization.*

Remark 2.3. *Given a conformal isomorphism $\phi : D \to \phi(D)$, the Minkowski content of the* SLE *transforms as a d-dimensional measure:*

$$\mu_{\phi(D)} = |\phi'|^d \phi_\star \mu_D.$$

Let us finally note that SLE curves with their natural parametrizations have the following spatial Markov property:

Proposition 2.4. *The law of* $(\mathrm{SLE}_\kappa^{\mathbb{H}}(0 \to \infty), \mu_{\mathbb{H}})$ *after a stopping time τ conditioned on its past has the law of an* $(\mathrm{SLE}_\kappa^{H_\tau}(\eta_\tau \to \infty), \mu_{H_\tau})$.

2.2 The Neumann free field

We start by recalling general facts before defining the Neumann free field as the Gaussian on a certain function space.

[1] The curve η needs to be instantaneously reflected off its past and the boundary in the following sense: the set of times s larger than some time s_0 that η spends outside of the domain H_{s_0} should be of empty interior.

2.2.1 Gaussians

Gaussians are usually associated to vector spaces carrying a non-degenerate scalar product. However, it is natural to extend this definition in the degenerate case, by saying that a Gaussian of variance 0 is deterministically 0.

Definition 2.5. The Gaussian on a vector space V equipped with a symmetric positive semi-definite bilinear form $(\cdot, \cdot)$ is the joint data, for every vector $v \in V$, of a centered Gaussian random variable Γ_v, such that $v \mapsto \Gamma_v$ is linear, and such that for any couple $v, w \in V$, $\mathrm{Cov}(\Gamma_v, \Gamma_w) = (v, w)$.

Heuristically, one should think of Γ_v as being the scalar product (h, v) of v with a random vector h drawn according to the Gaussian law $e^{-\frac{1}{2}(h,h)}dh$. However, the linear form $v \to \Gamma_v$ is (in truly infinite-dimensional examples) a.s. not continuous, and so there does not exist a vector $h \in V$ such that $\Gamma_v = (h, v)$ for all $v \in V$. One can nonetheless try to find such a random object h in a superspace of V: this is the question of finding a continuous version of Brownian motion, or of seeing the Gaussian free field as a distribution.

Before moving on to defining the Neumann free field, let us recall a useful property.

Proposition 2.6 (Cameron-Martin formula)**.** *Let us fix a vector $m \in V$ and sample $(\widetilde{\Gamma}_v)_{v \in V}$ according to the Gaussian law Γ, biased by e^{Γ_m}.*

Then $\widetilde{\Gamma}$ has the law of $(\Gamma_v + (m, v))_{v \in V}$.

In other words, under the biased law, $\widetilde{h}$ has the law of $h + m$.

2.2.2 Definition of the Neumann free field

We now fix a smooth simply-connected Jordan domain D of the plane. Let us consider the (degenerate) Dirichlet scalar product

$$(f, g)_\nabla = \frac{1}{2\pi} \int_D \nabla f \nabla g$$

on the space $\mathcal{C}^\infty_\nabla\left(\overline{D}\right)$ of continuous functions f on $\overline{D}$ that are smooth on D and such that the Dirichlet norm $||f||_\nabla := (f, f)_\nabla^{1/2}$ is finite. We denote by $H(D)$ the completion of this space with respect to the (non-degenerate) metric $||f||_\nabla + |f(x_0)|$, where $x_0 \in D$ is an arbitrary point.

Definition 2.7. The Gaussian free field with Neumann boundary conditions on D (or Neumann free field) is the Gaussian on the space $H(D)$ equipped with the Dirichlet scalar product $(\cdot, \cdot)_\nabla$. It is the joint data, for any function $f \in H(D)$, of a random variable Γ_f.

Remark 2.8. *The Dirichlet product being conformally invariant, so is the Neumann free field.*

2.2.3 The Neumann free field as a random distribution

In order to see the Neumann free field Γ as a random distribution h, i.e. to be able to write $\Gamma_f = (h, f)_\nabla$, we are looking for a consistent way to define the quantities

$$(h, g) := \int_D h(z) g(z) \mathrm{d}z$$

for every test function (i.e. smooth and compactly supported function) g. Let $\Delta = \partial_x^2 + \partial_y^2$ be the Laplacian, and ∂_n denotes the outward normal derivative on the boundary. If H is a regular enough distribution, and f is a smooth enough function, then Green's formula holds:

$$2\pi(H, f)_\nabla = -(H, \Delta f) + \int_{\partial D} H \partial_n f. \tag{1}$$

If we can solve the Poisson problem with Neumann boundary conditions for a fixed smooth function g, i.e. if we can find a function f so that

$$\begin{cases} \partial_n f = 0 \text{ on } \partial D \\ \Delta f = g \text{ on } D, \end{cases}$$

we can then tentatively define (h, g) in formal agreement with Green's formula (1) by

$$(h, g) := -\Gamma_f.$$

Proposition 2.9. *The Poisson problem with Neumann boundary conditions admits solutions if and only if the function g satisfies the integral condition $\int_D g = 0$. The solution is then unique up to an additive constant.*

Proof. Assuming a solution f for the Poisson problem exists, applying Green's formula (1) with H being the constant function 1 yields the integral condition $\int_D g = 0$.

We now show that solutions exist provided the integral condition holds. By conformal covariance, it is enough to solve the Poisson problem in the upper half-plane $\mathbb{H}$. This can be achieved by using the fundamental solution

$$G(z, w) := -\log|z - w| - \log|z - \overline{w}|. \tag{2}$$

Indeed, the function

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{H}} G(z, w) g(w) \mathrm{d}y$$

is a solution of the Poisson problem, as, in particular the normal derivative $\partial_n f$ on the boundary is a Dirac mass at ∞, of total mass $\int_{\mathbb{H}} g$.

Let us now consider f_1 and f_2 two solutions for the same Poisson problem. Then, for any function $H \in H(D)$, Green's formula yields $(H, f_1 - f_2)_\nabla = 0$. In particular $f_1 - f_2$ is of zero Dirichlet norm, so is a constant. $\square$

In terms of finding a distributional representation h of the Neumann free field Γ, Proposition 2.9 implies that we only have a natural way to define the pairing (h, g) for test functions g such that $\int_D g = 0$. In other words, the distribution h is canonically defined only in the space of distributions modulo constants. On the other hand, with h is defined in this way, we can a posteriori check consistency with Green's formula (1).

Remark 2.10. *The (up to constant) distribution h is almost surely regular enough so that for a smooth function f, the integral $\int_{\partial D} h \partial_n f$ is well-defined, and vanishes when $\partial_n f$ is identically zero on the boundary ∂D. This can be seen by considering the trace of the field on the boundary, as in [5, Section 4.3].*

2.2.4 Choice of constant for the field

Even though there is no canonical choice of a distribution h representing the Neumann free field, we can still make some choice for h and thus fix the constant. This is done by picking a function g_0 of non-zero mean, and by deciding that (h, g_0) should have a certain joint distribution with the set of random variables (h, g) where g runs over all test functions. However, none of these choices will preserve the conformal invariance of the field.

In the following, unless otherwise noted, we will always assume that some choice of constant for the Neumann free field has been made.

2.2.5 Covariance

Definition 2.11. The pointwise covariance of the field is a choice of generalized function $K(x, y)$ that represents the bilinear form $(g, \tilde{g}) \mapsto \mathbb{E}[(h, g)(h, \tilde{g})]$, where g and $\tilde{g}$ are mean-zero test functions:

$$\int_{z,w\in D} g(z)K(z,w)\tilde{g}(w)\mathrm{d}z\mathrm{d}w := \mathbb{E}[(h,g)(h,\tilde{g})].$$

In the upper-half plane $\mathbb{H}$, the covariance is given by Green's function (2).

2.3 Liouville quantum gravity (LQG)

2.3.1 Quantum surfaces

The goal of Liouville quantum gravity (LQG) is to study *quantum surfaces*, i.e. complex domains carrying a natural random metric - the Liouville metric. This Liouville metric is of the form $e^{\gamma h}g$ where g is some metric compatible with the complex structure, and h is a field related to the Neumann free field. However, the exponential $e^{\gamma h}$ of the free field is ill-defined and building the Liouville metric is not yet understood in general. We can however build as chaos measures (see Section 2.5) certain Hausdorff volume measures associated with the Liouville metric.

The field h only appears as a tool to construct the Liouville metric and objects related to it in fixed coordinates, and hence should change appropriately when we change coordinates, so that the geometric objects are left unchanged.

Definition 2.12. Given a conformal isomorphism $\phi : D \to \phi(D)$ between complex domains, the Liouville coordinate change formula is given by:

$$h_{\phi(D)} = h_D \circ \phi^{-1} + Q \log |\phi^{-1\prime}|,$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

Natural volume measures are then invariant under this change of coordinates (Proposition 3.3). We call *quantum surface* a class of field-carrying complex domains (D, h) modulo Liouville changes of coordinates. A particular representative (D, h) of a given quantum surface is called a *parametrization*.

2.3.2 The circle-average coordinates

Let us consider a quantum surface $(\mathbb{H}, h, 0, \infty)$ with two marked points. It will be sometimes convenient to work in fixed coordinates, independent of a normalization at ∞. In order to do so, let us note that the Dirichlet space $H(\mathbb{H})$ admits an orthogonal decomposition $H^r \oplus H^m$ for the Dirichlet product, where H^r is the closure of radially symmetric functions $f(|\cdot|)$, and H^m is the closure of functions that are of mean zero on every half-circle $C_R = \{z \in \mathbb{H}, |z| = R\}$. Any field h correspondingly split as a sum $h^r + h^m$.

Note that the Liouville change of coordinates corresponding to a rescaling of the half-plane by a factor e^C is given by

$$h_C(\cdot) = h(e^{-C}\cdot) - QC.$$

In particular, the average of the rescaled field h_C on the unit half-circle is given by

$$h^r_C(0) = h^r(e^{-C}) - QC.$$

Definition 2.13. The *circle-average coordinates* of a quantum surface $(\mathbb{H}, h, 0, \infty)$ is its unique parametrization $(\mathbb{H}, \hat{h}, 0, \infty)$ such that

$$\inf\{C \in \mathbb{R} | \hat{h}^r(e^{-C}) - QC \leq 0\} = 0.$$

Note that this is well-defined as soon as the function $C \mapsto h^r(e^{-C}) - QC$ diverges to $+\infty$ (resp. $-\infty$) when C goes to $-\infty$ (resp. $+\infty$).

2.4 Wedge fields

2.4.1 The radial part of the free field

Before defining the wedge field, we make some observations about the Neumann free field. Recall that a Neumann free field h splits as the sum $h^r + h^m$ of its radial component with a mean-zero component on each half-circle. If one fixes the constant of the field by requiring that $h^r(1) = 0$, the components h^r and h^m are independent. Moreover, the law of the radial component $h^r(e^{-\frac{t}{2}})$ can be explicitly computed (see the related [6, Proposition 3.3]): it is a double-sided Brownian motion B_t, i.e. the functions $\left(h^r(e^{-\frac{t}{2}})\right)_{t \geq 0}$ and $\left(h^r(e^{\frac{t}{2}})\right)_{t \geq 0}$ are independent standard Brownian motions. Indeed, with m_R being the uniform measure on the upper half-circle of radius R around 0, and radii $0 < R_1 < R_2$, we see that

$$\begin{aligned}
\mathbb{E}\left[h^r(R_1)h^r(R_2)\right] &= \mathbb{E}\left[(h, m_{R_1})(h, m_{R_2})\right] \\
&= \int_{\mathbb{H}^2} (m_{R_1} - m_1)(\mathrm{d}z) G(z, w)(m_{R_2} - m_1)(\mathrm{d}w) \\
&= 2 \log |R_2/R_1| .
\end{aligned}$$

Moreover, note that adding a drift at to $h^r(e^{-t})$ corresponds to adding the function $-a \log |\cdot|$ to the field.

2.4.2 Definition of the wedge field

We now define, in the upper half plane $\mathbb{H}$, an object closely related to the Neumann free field: the *wedge field*. Let us fix some real number $\alpha < Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

Definition 2.14. The α-wedge field is a random distribution that splits in $H^r \oplus H^m$ as an independent sum $h_W = h^r_W + h^m$, where h^m is as for the Neumann free field, and $h^r_W(e^{-t})$ has the law of A_t, as defined below.

For $t > 0$, $A_t = B_{2t} + \alpha t$, where B is a standard Brownian motion started from 0. For $t < 0$, $A_t = \widehat{B}_{-2t} + \alpha t$, where $\widehat{B}$ is a standard Brownian motion started from 0 independent of B and conditioned on the singular event

$$A_t - Qt = \widehat{B}_{-2t} + (\alpha - Q)t > 0$$

for all negative times t.

Proposition 2.15. *Let $(B_t)_{t\in\mathbb{R}}$ be a double-sided standard Brownian motion, and for all real number M, let*

$$T_M = \frac{1}{2}\inf\{t \in \mathbb{R} | B_{2t} + (\alpha - Q)t + M \leq 0\}.$$

Then the process

$$(B_{2T_M+2t} + \alpha t + M)_{t\in\mathbb{R}}$$

converges in law towards $(A_t)_{t\in\mathbb{R}}$ as M tends to $+\infty$

Proof. Note that, for large M, the time T_M equals with high probability the first positive time T'_M such that drifted Brownian motion hits the value $-M$:

$$T'_M = \frac{1}{2}\inf\{t \geq 0 | B_{2t} + (\alpha - Q)t = -M\}.$$

The claim follows. □

Proposition 2.16. *Let $h = \tilde{h} + a\log|\cdot|$ where $\tilde{h}$ is a Neumann free field with an arbitrary choice of constant, and let h_W be an α-wedge field in $\mathbb{H}$. Let K be a compact subset of the punctured half-disk $\{z \in \overline{\mathbb{H}}, 0 < |z| \leq 1\}$. There exists a random constant c such that h and $h_W + c$ are absolutely continuous on K.*

Proof. Without loss of generality, we can fix the constant of the Neumann free field $\tilde{h}$ as we see fit: we ask that its radial component satisfies $h^r(1) = \tilde{h}^r(1) = 0$. Let us also define a field $\hat{h} := h_W + c$, where h_W is a wedge field, and where the random constant c is chosen such that $\hat{h}^r(2b) = 0$.

The two fields $\hat{h}$ and h are absolutely continuous on K: their H^m components have same law, and their independent H^r components $\hat{h}^r(e^{-\frac{t}{2}})$ and $h^r(e^{-\frac{t}{2}})$ for positive times have the law of drifted Brownian motions started from 0, with possibly different drifts. □

2.4.3 The wedge field as a scaling limit

Lemma 2.17. *Consider the field $h = \tilde{h} - \alpha\log|\cdot|$, where $\tilde{h}$ is a Neumann free field with an arbitrary choice of constant and $\alpha < Q$. Then $h + M$ converges in law towards the α-wedge field h_W as M goes to ∞, in the sense of convergence on compact sets in circle-average coordinates.*

Proof. We first assume that the constant of the Neumann free field is fixed such that $h^r(0) = 0$. Going to circle-average coordinates after adding a constant M to the field h amounts to zooming in towards 0, with a random scaling factor depending on h^r alone, that almost surely goes to ∞ as M goes to ∞. The space H^m equipped with the Dirichlet scalar product is invariant by scaling, and so the law of the H^m component is left unchanged by this operation and is independent of the H^r component. The rescaling operation acts explicitly on the space H^r, and under this operation, double-sided Brownian motion converges towards the process A (see Proposition 2.15).

We go back to the general case. A general choice of constant for $\tilde{h}$ can be written as the previous choice plus some random variable c depending on h^r and h^m. However, the σ-algebra generated by the information contained in an arbitrary neighborhood of the origin is trivial, and so c tends to be independent of the picture we look at in the scaling limit $M \to \infty$. □

Corollary 2.18. *The law of an α-wedge field is invariant as a quantum surface (i.e. up to a Liouville change of coordinates) when one adds a deterministic constant to the wedge field.*

2.5 Chaos measures

We now explain how to associate natural measures to a field h.

2.5.1 Definition

Given a field h and a reference measure σ, one can build interesting random measures $: e^{\tilde{\gamma} h} : \sigma$ called *chaos measures*, that were first studied for their multiplicative structure. The field h is usually too irregular for its exponential to make sense, and so defining chaos requires some renormalization.

Let σ be a Radon measure whose support has Hausdorff dimension at least d, and let h be a Gaussian field with covariance K blowing up like $-\log$, meaning that we have, for any point z in the support of σ,

$$K(z, z+\delta) = -\log|\delta| + O_{|\delta|\to 0}(1).$$

We can then build non-trivial *chaos measures* $\sigma[\tilde{\gamma}, h]$ for values of the parameter $0 < \tilde{\gamma} < \sqrt{2d}$ (see [4] and references therein).

Let us first define these chaos measures in the upper half-plane $\mathbb{H}$, when the field h is of the form $\tilde{h} + m$, where $\tilde{h}$ is a Neumann free field, and m is a smooth function.

Definition 2.19. The $\tilde{\gamma}$-chaos of a Radon measure σ with respect to the field h is defined in the following way. For a measure σ supported in the bulk $\mathbb{H}$, and $\tilde{\gamma} < \sqrt{2d}$:

$$\sigma[\tilde{\gamma}, h](\mathrm{d}z) =: e^{\tilde{\gamma} h}\sigma : (\mathrm{d}z) := \lim_{\varepsilon \to 0} e^{\tilde{\gamma}(\theta_z^\varepsilon, h)} \varepsilon^{\frac{\tilde{\gamma}^2}{2}} \sigma(\mathrm{d}z),$$

and for a measure σ supported on the boundary $\mathbb{R}$ and $\tilde{\gamma} < \sqrt{d}$ (the covariance of h blows up like $-2\log$ on the boundary):

$$\sigma[\tilde{\gamma}, h](\mathrm{d}z) =: e^{\tilde{\gamma} h} : \sigma(\mathrm{d}z) := \lim_{\varepsilon \to 0} e^{\tilde{\gamma}(\theta_z^\varepsilon, h)} \varepsilon^{\tilde{\gamma}^2} \sigma(\mathrm{d}z),$$

where $(\theta_z^\varepsilon, h)$ is some ε-regularization of the field h by a smooth test function $\theta_z^\varepsilon(\cdot)$ of total mass one and of fixed shape, and supported on the ball (in the bulk case) or on the half-ball (in the boundary case) of radius ε around z.

For example, one can fix a radially-symmetric smooth function θ_i^1 of total mass 1 supported in the unit ball around $i \in \mathbb{H}$, and let the bulk regularizing function be given by

$$\theta_z^\varepsilon(w) = \varepsilon^{-2}\theta_i^1\left(\frac{w-z}{\varepsilon} + i\right).$$

Remark 2.20. *Note that, if m is a smooth function, then*

$$\sigma[\tilde{\gamma}, h+m](\mathrm{d}z) = e^{\tilde{\gamma}m}\sigma[\tilde{\gamma}, h](\mathrm{d}z).$$

As the chaos measures defined above are non-atomic, we can extend the previous definition when the mean m of the field is singular, for example when $m = a\log|\cdot|$.

Remark 2.21. *The theory of chaos measure was developed in the context of Gaussian fields. However, chaos are also well-defined for the Neumann free field with a non-Gaussian choice of constant: indeed such a field can be written as the sum of a Gaussian field and a random constant. Moreover, one can build chaos measures for wedge fields, by local absolute continuity (Proposition 2.16).*

Note that we can recover a measure σ from one of its chaos.

Lemma 2.22. *Let h be a Neumann free field with a Gaussian choice of constant, and let σ be a Radon measure supported in the bulk. We can recover the measure σ from its chaos $\sigma[\tilde{\gamma}, h]$:*

$$\sigma(dx) = e^{-\frac{\tilde{\gamma}^2}{2}\widehat{K}(x)}\mathbb{E}[\sigma[\tilde{\gamma}, h](dx)],$$

where $\widehat{K}(x) = \lim_{\varepsilon\to 0}\mathrm{Var}(\theta_x^\varepsilon, h) + \log\varepsilon$.

The same statement holds in the boundary case.

Proof. We do the computation in the bulk case. The first equality holds by uniform integrability (see [4, Section 3]).

$$\begin{aligned}
\mathbb{E}[\sigma[\tilde{\gamma}, h](dx)] &= \lim_{\varepsilon\to 0}\mathbb{E}[e^{\tilde{\gamma}(\theta_x^\varepsilon, h)}\varepsilon^{\frac{\tilde{\gamma}^2}{2}}\sigma(dx)] \\
&= \lim_{\varepsilon\to 0}\mathbb{E}[e^{\tilde{\gamma}(\theta_x^\varepsilon, h)}]\varepsilon^{\frac{\tilde{\gamma}^2}{2}}\sigma(dx) \\
&= \lim_{\varepsilon\to 0} e^{\frac{\tilde{\gamma}^2}{2}\mathrm{Var}(\theta_x^\varepsilon, h)}\varepsilon^{\frac{\tilde{\gamma}^2}{2}}\sigma(dx) \\
&= \lim_{\varepsilon\to 0} e^{\frac{\tilde{\gamma}^2}{2}(-\log\varepsilon + \widehat{K}(x) + o_\varepsilon(1))}\varepsilon^{\frac{\tilde{\gamma}^2}{2}}\sigma(dx) \\
&= e^{\frac{\tilde{\gamma}^2}{2}\widehat{K}(x)}\sigma(dx).
\end{aligned}$$

□

Together with Proposition 4.1, this provides a way to construct the natural parametrization of SLE from the free field.

2.5.2 The Cameron-Martin formula for chaos measures

In this section, we want to describe the law of a Neumann free field biased by the total mass $\sigma[\tilde{\gamma}, h](\mathbb{H})$ of a certain chaos measure. In order for the choice of constant for the field to not matter, we introduce a compensation that takes the form of an additional bias of the form $e^{-\tilde{\gamma}(\rho,h)}$ for a non-negative test function ρ of total mass 1, i.e. such that $\int_{\mathbb{H}} \rho = 1$.

Let σ be a bulk-supported finite measure on $\mathbb{H}$ and $\tilde{\gamma} > 0$ be, such that the chaos $\sigma[\tilde{\gamma}, h]$ is non-trivial for h a Neumann free field. Let dh denote the law of the Neumann free field on $\mathbb{H}$ with some arbitrary choice of constant. Let us also consider the probability measure

$$M(\mathrm{d}z) := Z^{-1}\mathbb{E}[e^{-\tilde{\gamma}(\rho,h)}\sigma[\tilde{\gamma}, h](\mathrm{d}z)],$$

where Z is a normalization constant.

Lemma 2.23. *A couple (h, z) sampled according to*

$$Z^{-1}dh(h)e^{-\tilde{\gamma}(\rho,h)}\sigma[\tilde{\gamma}, h](\mathrm{d}z)$$

can also be described in the following way. First, one samples z according to the measure $M(\mathrm{d}z)$. Then, one samples h with the law of

$$\tilde{h} + \tilde{\gamma}G(z, \cdot) - \int_{\mathbb{H}} \tilde{\gamma}\rho(w)G(w, \cdot)\mathrm{d}w,$$

where $\tilde{h}$ is a Neumann free field (with a complicated choice of constant that can depend on z), and G is the Green's function (2).

Proof. Let us consider the ε-approximation of the chaos measure:

$$Z^{-1}e^{-\tilde{\gamma}(\rho,h)}\sigma[\tilde{\gamma}, h](\mathrm{d}z) = Z^{-1} \lim_{\varepsilon\to 0} e^{\tilde{\gamma}(\theta_z^\varepsilon-\rho,h)}\varepsilon^{\frac{\tilde{\gamma}^2}{2}}\sigma(\mathrm{d}z).$$

The measure on (h, z) on the right hand side can be read as first picking a point z according to

$$Z^{-1}\varepsilon^{\frac{\tilde{\gamma}^2}{2}}\mathbb{E}[e^{\tilde{\gamma}(\theta_z^\varepsilon-\rho,h)}]\sigma(\mathrm{d}z)$$

(which tends to $M(\mathrm{d}z)$ as ε goes to 0), and conditionally on z, sampling a field according to $e^{\tilde{\gamma}(\theta_z^\varepsilon-\rho,h)}dh$. This last step can be understood by Cameron-Martin as sampling a field according to dh and then adding to it the function

$$\int_{\mathbb{H}} \tilde{\gamma}(\theta_z^\varepsilon(w) - \rho(w))G(w, \cdot)\mathrm{d}w,$$

which converges as $\varepsilon \to 0$ towards

$$\tilde{\gamma}G(z, \cdot) - \int_{\mathbb{H}} \tilde{\gamma}\rho(w)G(w, \cdot)\mathrm{d}w.$$

□

3 The zipper coupling of LQG and SLE

From now on, we work in the upper half plane $(\mathbb{H}, 0, \infty)$, and we tune LQG and SLE parameters so that $\gamma^2 = \kappa < 4$.

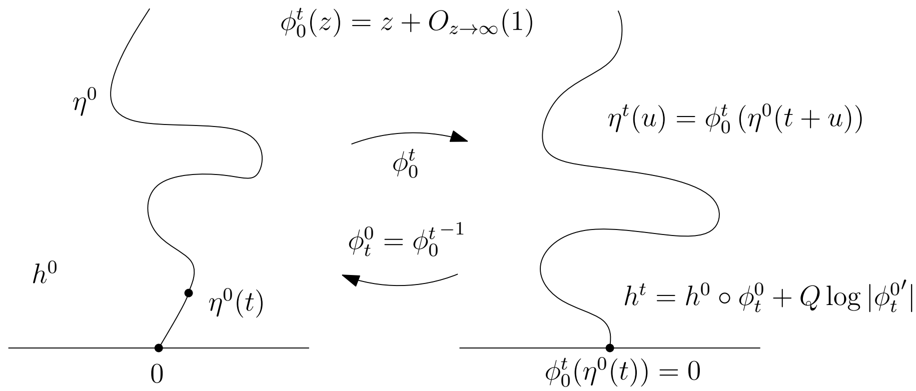

Figure 1: The capacity zipper

3.1 The capacity zipper

The *capacity zipper* $(h^t, \eta^t)_{t\geq 0}$ is a process of pairs consisting of a distribution and a curve. The curve η^0 is an SLE_κ (parametrized by half-plane capacity) sampled independently of a field h^0 which has the law of $\tilde{h} + \frac{2}{\gamma}\log|\cdot|$, where $\tilde{h}$ is a Neumann free field (with some arbitrary choice of constant). The process evolves by deterministically unzipping the SLE (see Figure 1): for a time $t > 0$, call $\phi_0^t : \mathbb{H}\backslash\eta^0([0,t]) \to \mathbb{H}$ the uniformizing map normalized at ∞ so that $\phi_0^t(z) = z + O_{z\to\infty}(1)$ and $\phi_0^t(\eta^0(t)) = 0$, and let ϕ_t^0 be its inverse. The curve η^t is then the simple curve parametrized by capacity given by

$$\left(\eta^t(u)\right)_{u\geq 0} = \left(\phi_0^t(\eta^0(t+u))\right)_{u\geq 0}$$

and the field h^t is given by the Liouville change of coordinates

$$h^t := h^0 \circ \phi_t^0 + Q\log|\phi_t^{0\prime}|.$$

We also define zipping/unzipping maps for all couples of non-negative times s, t by

$$\phi_s^t = \phi_0^t \circ \phi_s^0.$$

Proposition 3.1 ([11, Theorem 1.2])**.** *The capacity zipper $(h^t, \eta^t)_{t\geq 0}$ has stationary law, when the fields h^t are seen as distributions up to a constant.*

In particular, we can naturally extend the capacity zipper by stationarity to a (stationary up to constant) process $(h^t, \eta^t)_{t\in\mathbb{R}}$, as the choice of constant for h^0 can be propagated to negative times.

Proof. We only sketch the proof here, and refer to [11] for details.

Let us fix a time $t > 0$. The curve η^t is an SLE independent of h^0 and $\eta^0([0,t])$, so in particular of $h^t = h^0 \circ \phi_t^0 + Q\log|\phi_t^{0\prime}|$. We thus only need to show that the field h^t is a Neumann free field.

For any point $z \in \mathbb{H}$, SLE computations via Itô calculus gives us a martingale

$$M_s(z) = \frac{2}{\gamma}\log|\phi_t^{t-s}(z)| + Q\log|(\phi_t^{t-s})'(z)|,$$

whose quadratic covariations are explicit:

$$d\left\langle M_s(z), M_s(w)\right\rangle = -dG_s(z,w), \tag{3}$$

where G_s is Green's function in the domain $\mathbb{H} \setminus \eta^s([0, t-s]) = \phi_t^{t-s}(\mathbb{H})$, i.e.

$$G_s(z, w) := G(\phi_t^{t-s}(z), \phi_t^{t-s}(w)).$$

Now, note that we can rewrite the field h^t as

$$h^t = h^0 \circ \phi_t^0 + Q \log |\phi_t^{0\prime}| = M_t + (h^0 - M_0) \circ \phi_t^0,$$

which is the sum of the log singularity M_0 plus an η^0-measurable centered random variable $\mathcal{B} = M_t - M_0$ plus a random variable $\mathcal{N} = (h^0 - M_0) \circ \phi_t^0$.

We now prove that (the up to constant part of) the field $h^t - M_0$ is a centered Gaussian field of covariance given by Green's function G. In other words, we now show that, for any test function ρ of mean zero, the random variable $(h^t - M_0, \rho)$ is a centered Gaussian of variance

$$\int_{\mathbb{H}^2} \rho(z) G(z, w) \rho(w) \mathrm{d}z \mathrm{d}w. \tag{4}$$

Let us consider the martingale

$$M_s^\rho = \int_{\mathbb{H}} \rho(z) M_s(z) \mathrm{d}z.$$

From the covariation computation (3), we see that $(\mathcal{B}, \rho) = M_t^\rho - M_0^\rho$ has the law of an η^0-measurable standard Brownian motion evaluated at the (non-negative) stopping time

$$\int_{\mathbb{H}^2} \rho(z) \left(G(z, w) - G_t(z, w)\right) \rho(w) \mathrm{d}z \mathrm{d}w. \tag{5}$$

Moreover, conditionally on the curve η^0, the up to constant part of the field $\mathcal{N} = (h^0 - M_0) \circ \phi_t^0$ is a centered Gaussian of covariance G_t. In particular, the random variable $(\mathcal{N}, \rho)$ is a Gaussian of variance

$$\int_{\mathbb{H}^2} \rho(z) G_t(z, w) \rho(w) \mathrm{d}z \mathrm{d}w. \tag{6}$$

Now, note that if B_T is the value taken by Brownian motion at a stopping time $T \leq 1$, and N is conditionally on B a centered Gaussian of variance $1-T$, then $(B_T, N, B_T + N)$ has the law of $(B_T, B_1 - B_T, B_1)$. In particular, the random variable $(h^t - M_0, \rho) = (\mathcal{B}, \rho) + (\mathcal{N}, \rho)$ is a centered Gaussian of variance given by (4)=(5)+(6). □

The main result of [11] is the following. We postpone its proof to Section 4.

Theorem 3.2 ([11, Theorem 1.4]). *The zipping up dynamics $(h^{-t}, \eta^{-t})_{t \in \mathbb{R}}$ on the capacity zipper almost surely evolves deterministically.*

3.2 Some volume measures for the zipper

We can build several natural chaos measures (Figure 2) from a fixed time frame (h^t, η^t) of the capacity zipper.

- The boundary Liouville measure $\lambda\left[\frac{\gamma}{2}, h^t\right]$, a chaos on the Lebesgue measure λ on $\mathbb{R}$.
- The chaos $\mu^t\left[\frac{\gamma}{2}, h^t\right]$ on the natural parametrization μ^t of the SLE η^t.

Note that these measures depend on the normalization of the free field: adding a random constant c to the field will introduce a scaling factor $e^{\frac{\gamma}{2}c}$.

Proposition 3.3. *The boundary Liouville measure and the $\gamma/2$-chaos on the natural parametrization are invariant by Liouville change of coordinates This includes invariance under rescaling, but also the fact that for any couple of times $s<t$,*

$$(i) \qquad \lambda\left[\frac{\gamma}{2},h^t\right]_{|\phi_s^t(\mathbb{R})}\circ\phi_s^t=\lambda\left[\frac{\gamma}{2},h^s\right] \quad \text{and}$$
$$(ii) \qquad \mu^t\left[\frac{\gamma}{2},h^t\right]=\mu^s\left[\frac{\gamma}{2},h^s\right]_{|\phi_t^s(\eta^t)}\circ\phi_t^s.$$

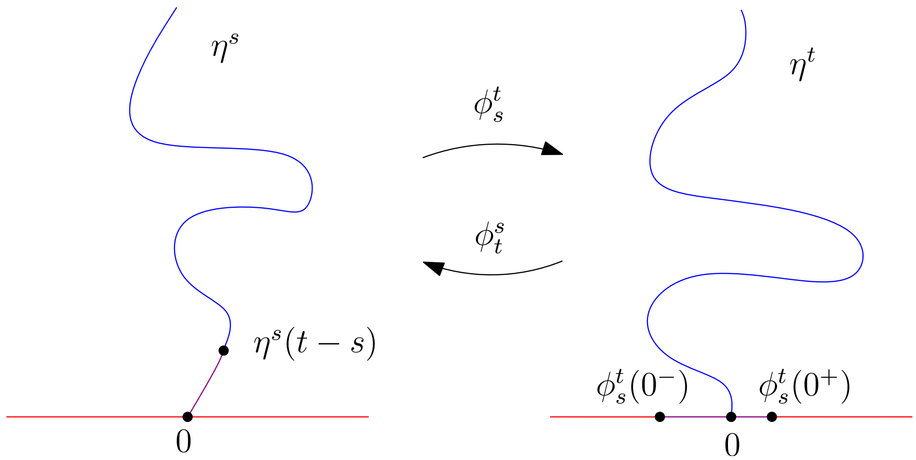

Figure 2: Under the unzipping operation, the natural volume measures on the curve and on the boundary are preserved: Proposition 3.3 (i) and (ii) respectively claim that the red (resp. blue) regions carry the same measure. On the purple regions, the natural measures on the curve and on the boundary coincide (Proposition 4.1) .

Proof. We only prove (ii). The other claims are proved similarly. Recall that the dimension of SLE is given by $d=1+\frac{\gamma^2}{8}$, and that the Liouville change of coordinates for the capacity zipper reads $h^t=h^s\circ\phi_t^s+Q\log|\phi_t^{s\prime}|$. We have that:

$$\begin{aligned}
\mu^s\left[\frac{\gamma}{2},h^s\right]_{|\phi_t^s(\eta^t)}\circ\phi_t^s(\mathrm{d}z) &= \varepsilon^{\frac{\gamma^2}{8}}\lim_{\varepsilon\to0}e^{\frac{\gamma}{2}(\theta^\varepsilon_{\phi_t^s(z)},h^s)}d\mu^s\circ\phi_t^s(\mathrm{d}z)\\
&= \lim_{\varepsilon\to0}\varepsilon^{\frac{\gamma^2}{8}}e^{\frac{\gamma}{2}(\theta_z^{\varepsilon|\phi_s^{t\prime}\circ\phi_t^s(z)|},h^s\circ\phi_t^s)}|\phi_t^{s\prime}|^d d\mu^t(\mathrm{d}z)\\
&= \lim_{\delta\to0}(|\phi_t^{s\prime}|\delta)^{\frac{\gamma^2}{8}}e^{\frac{\gamma}{2}(\theta_z^\delta,h^t)-\frac{\gamma}{2}Q\log|\phi_t^{s\prime}|}|\phi_t^{s\prime}|^d d\mu^t(\mathrm{d}z)\\
&= |\phi_t^{s\prime}|^{\frac{\gamma^2}{8}-\frac{\gamma}{2}Q+d}\lim_{\delta\to0}\delta^{\frac{\gamma^2}{8}}e^{\frac{\gamma}{2}(\theta_z^\delta,h^t)}d\mu^t(\mathrm{d}z)\\
&= |\phi_t^{s\prime}|^{\frac{\gamma^2}{8}-\frac{\gamma}{2}(\frac{\gamma}{2}+\frac{2}{\gamma})+1+\frac{\gamma^2}{8}}\mu^t\left[\frac{\gamma}{2},h^t\right](\mathrm{d}z)\\
&= \mu^t\left[\frac{\gamma}{2},h^t\right](\mathrm{d}z).
\end{aligned}$$

Note that to go from the first to the second line, we used that

$$(\theta^\varepsilon_{\phi_t^s(z)},h^s)-(\theta_z^{\varepsilon|\phi_s^{t\prime}\circ\phi_t^s(z)|},h^s\circ\phi_t^s)$$

is a Gaussian of variance $o(1)$ as ε goes to 0. This is seen in the following way: with

$$F_\varepsilon(\cdot) = \theta^\varepsilon_{\phi^s_t(z)}(\cdot) - |\phi^{t\prime}_s(\cdot)|^2 \theta_z^{\varepsilon|\phi^{t\prime}_s \circ \phi^s_t(z)|}(\phi^t_s(\cdot)),$$

and G being Green's function, one has that

$$\int_{\mathbb{H}^2} F_\varepsilon(w) G(w,y) F_\varepsilon(w) \mathrm{d}w \mathrm{d}y = o_\varepsilon(1).$$

□

4 The zipping operation is deterministic

4.1 Proof of the main theorem (Theorem 3.2)

The main theorem will be a consequence of the following result, which is in the line of ideas from [2] (see Figure 2).

Proposition 4.1. *Let $(h^t, \eta^t)_{t\in\mathbb{R}}$ be a capacity zipper. The following identity holds for $s < t$:*

$$\lambda\left[\frac{\gamma}{2}, h^t\right]_{|[0,\phi^t_s(0^+)]} \circ \phi^t_s = C\mu^s\left[\frac{\gamma}{2}, h^s\right]_{|\eta^s([0,t-s])},$$

where C is a universal constant[2]*.*

By symmetry, we deduce the following.

Corollary 4.2 ([11, Theorem 1.3])**.** *The push-forwards on the SLE curve of the boundary Liouville measure from the positive and from the negative half-lines agree:*

$$\lambda\left[\frac{\gamma}{2}, h^t\right]_{|[0,\phi^t_s(0^+)]} \circ \phi^t_s = \lambda\left[\frac{\gamma}{2}, h^t\right]_{|[\phi^t_s(0^-),0]} \circ \phi^t_s.$$

We can now prove that the zipping up process $(h^{-t}, \eta^{-t})_{t\in\mathbb{R}}$ evolves deterministically (Figure 3).

Proof of Theorem 3.2. Let us fix a time $t > 0$. Let $\widetilde{\eta}$ denote the range of a simple curve in $\mathbb{H}$ starting from 0, of half-plane capacity t. Consider the conformal map $\widetilde{\phi}$ from the upper half-plane $\mathbb{H}$ to $\mathbb{H} \setminus \widetilde{\eta}$ normalized so that $\widetilde{\phi}(z) = z + O_{z\to\infty}(1)$ and $\widetilde{\phi}(0)$ is the endpoint of $\widetilde{\eta}$. We assume that the preimages x_- and x_+ by $\widetilde{\phi}$ of any point in $\widetilde{\eta}$ are such that

$$\lambda\left[\frac{\gamma}{2}, h^t\right]([x_-, 0]) = \lambda\left[\frac{\gamma}{2}, h^t\right]([0, x_+]). \tag{7}$$

By Corollary 4.2, the map ϕ^0_t satisfies this property, and we only need to show that any map $\widetilde{\phi}$ satisfying the above property has to be equal to ϕ^0_t. Indeed, this would imply that the map ϕ^0_t is determined by h^t. The couple (h^0, η^0) can then be easily recovered from the data of (h^t, η^t, ϕ^0_t).

Let us then consider the map $f = \widetilde{\phi} \circ \phi^t_s$ where the time $s \in (-\infty, t)$ is such that $f(0^-) = 0^-$. We see by (7) that the map f is a continuous bijection of $\mathbb{H}$. The map f is moreover conformal off the SLE curve $\eta^s([0, t-s])$. However, SLE is removable [7, 10], which implies that the map f is conformal on the whole of $\mathbb{H}$, hence a Moebius transformation. From the behavior of f at ∞ and 0, we see that f is the identity. In particular, $\eta^s([0, t-s]) = \widetilde{\eta}$. By comparing the half-plane capacities of the two curves, we see that $t - s = t$, i.e. $s = 0$. □

[2] The constant C depends on the somewhat arbitrary renormalization procedure used to build chaos measures.

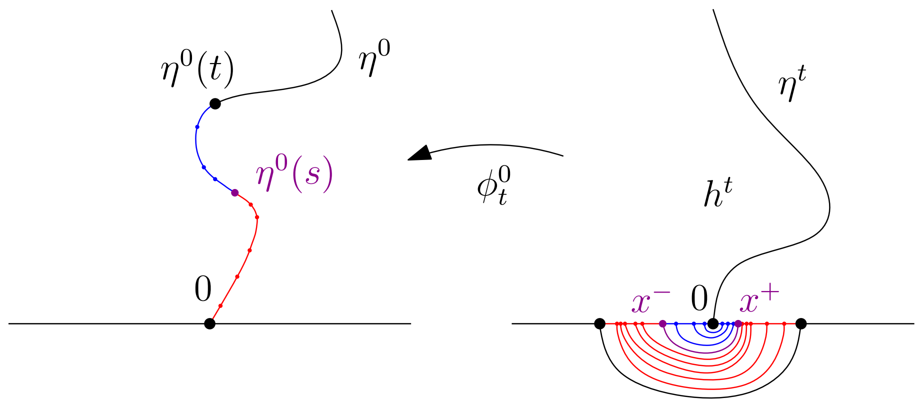

Figure 3: From the field h^t, we can construct a bijection between the negative and positive half-lines, by associating to each boundary point $x^- \in \mathbb{R}^-$ the point $x^+ \in \mathbb{R}^+$ such that $\lambda[\frac{\gamma}{2}, h^t]([x^-, 0]) = \lambda[\frac{\gamma}{2}, h^t]([0, x^+])$. Each of these couples (x^-, x^+) should be mapped to a same point $\eta^0(s)$ by the zipping map ϕ_t^0. It turns out that the data of this bijection is enough to completely recover the map ϕ_t^0 and the curve $\eta^0([0, t])$.

We still need to prove Proposition 4.1. In order to be able to make use of the stationarity of the unzipping dynamics, one has to modify the zipper coupling so that the volume measures $\lambda[\frac{\gamma}{2}, h]$ and $\mu[\frac{\gamma}{2}, h]$ are stationary, and not only up to a scaling constant. One can achieve this by considering a quantum zipper on a $(\gamma - \frac{2}{\gamma})$-wedge field h_W.

4.2 Unzipping wedges: the quantum zipper

The *quantum zipper* $(h_W^t, \eta^t)_{t\geq 0}$ is a process of pairs consisting of a distribution and a curve. The initial conditions are given by a $(\gamma - \frac{2}{\gamma})$-wedge field h_W^0, and an independent SLE_κ curve η^0, that we choose to parametrize by the chaos $\mu^0[\frac{\gamma}{2}, h_W^0]$, i.e. such that

$$\mu^0\left[\frac{\gamma}{2}, h_W^0\right]\left(\eta^0([0, t])\right) = t.$$

The quantum zipper evolves by deterministically unzipping the curve η^0 according to the time clock $\mu^0[\frac{\gamma}{2}, h_W^0]$, and applying the Liouville change of coordinates to transform the field, so that

$$\left(\eta^t(u)\right)_{u\geq 0} = \left(\phi_0^t(\eta^0(t+u))\right)_{u\geq 0}$$

and

$$h_W^t := h_W^0 \circ \phi_t^0 + Q \log|\phi_t^{0\prime}|.$$

Proposition 4.3 ([11, Theorem 1.8]). *The quantum zipper $(h_W^t, \eta^t)_{t\geq 0}$ has stationary law, when the fields are considered up to Liouville changes of coordinates.*

Remark 4.4. *We built two stationary processes. In the capacity setting (Proposition 3.1), we fix coordinates on $\mathbb{H}$ through a normalization at ∞, and the field up to a constant is invariant in law. In the quantum setting (Proposition 4.3), we keep track of the constant of the field. The time-parametrization of the zipper is coordinate-invariant, in particular invariant by scaling (Proposition 3.3). The law of the field as a quantum surface (i.e. up to change of coordinates) is invariant in law .*

We will deduce Proposition 4.3 from its analog for the capacity zipper, Proposition 3.1. This is done by introducing a bias

$$\mu^0\left[\frac{\gamma}{2}, h^0\right]\left(\eta^0([0,1])\right)$$

on the law of the capacity zipper, and sampling a point z on the curve η^0 according to the chaos measure $\mu^0[\frac{\gamma}{2}, h^0]_{|\eta^0([0,1])}(\mathrm{d}z)$. The quantum zipper can then be recovered by unzipping the curve η^0 all the way to the marked point z, and subsequently zooming in at 0. We need to introduce a second bias of the form $e^{-\frac{\gamma}{2}(\rho,h)}$ for technical reasons, namely so that the choice of constant of the field h^0 has no effect. We hence choose a non-negative test function ρ of total mass 1, that is supported on a compact set at distance say at least 10 from the real line, so that it is almost surely disjoint from $\eta^0([0,1])$. For any real $\beta \geq 1$, we also define the rescaled test function $\rho_\beta(w) = \beta^{-2}\rho(\frac{w}{\beta})$.

Let us consider the capacity process $(h^t, \eta^t)_{t\in\mathbb{R}}$ biased by

$$Z^{-1}e^{-\frac{\gamma}{2}(\rho_\beta,h^0)}\mu^0\left[\frac{\gamma}{2}, h^0\right]\left(\eta^0([0,1])\right),$$

where Z is a normalization constant. We also sample a point z on η^0 proportionally to $\mu^0[\frac{\gamma}{2}, h^0]_{|\eta^0([0,1])}(\mathrm{d}z)$, and let T_z be the time such that $\phi_0^{T_z}(z) = 0$.

Lemma 4.5. *As the parameter β goes to ∞ alongside a well-chosen subsequence, the couple (h^{T_z}, η^{T_z}) converges in law towards an* SLE_κ *curve and a field $\tilde{h} + \left(\frac{2}{\gamma} - \gamma\right)\log|\cdot|$, where the up to constant part of $\tilde{h}$ is a Neumann free field independent from the curve.*

In the rest if this section, $\hat{h}$ will exclusively stand for a Neumann free field defined up to a constant. We first prove the following:

Lemma 4.6. *The couple (h^{T_z}, η^{T_z}) (with h^{T_z} seen as a distribution up to constant) converges in law when β goes to ∞ towards an* SLE_κ *curve together with an independent (up to constant) field $\hat{h} + \left(\frac{2}{\gamma} - \gamma\right)\log|\cdot|$.*

Proof. Let us first note that the Markov property of SLE implies that the curve η^{T_z} is an SLE_κ independent from the field h^{T_z}. Indeed, note that under the biased law, conditionally on h^0, the time T_z is a stopping time for the filtration generated by η^0. Hence, the claim will follow if we understand the law of the field h^{T_z} as β goes to ∞.

Let us now fix an integer n, and let us cut the interval $[0,1)$ into n consecutive intervals $I_{n,l} := [l/n, (l+1)/n)$. We call k the integer in $\{0, 1, \cdots, n-1\}$ such that the point z belongs to $\eta^0(I_{n,k})$, and let $T_n := k/n$. The (biased) law of (h^{T_n}, η^{T_n}, z) can be sampled as follows: let k be uniformly chosen integer between 0 and $n-1$, and unzip an (unbiased) couple (h^0, η^0) until time T_n. Then, bias the resulting couple (h^{T_n}, η^{T_n}) by (we do not keep track of normalizing constants)

$$e^{-\frac{\gamma}{2}(\rho_\beta,h^0)}\mu^{T_n}\left[\frac{\gamma}{2}, h^{T_n}\right]\left(\eta^{T_n}([0,1/n])\right).$$

The point $y = \phi_0^{T_n}(z)$ can finally be picked according to $\mu^{T_n}[\frac{\gamma}{2}, h^{T_n}]_{|\eta^{T_n}([0,1/n])}(\mathrm{d}y)$.

In particular, h^{T_n} has the law of a field $\hat{h} + \frac{2}{\gamma}\log|\cdot|$ biased by $\alpha_n\widetilde{\alpha}_n$, where

$$\begin{aligned}\alpha_n &= e^{-\frac{\gamma}{2}(\rho_\beta,h^{T_n})}\mu^{T_n}\left[\frac{\gamma}{2}, h^{T_n}\right]\left(\eta^{T_n}([0,1/n])\right) \text{ and} \\ \widetilde{\alpha}_n &= e^{\frac{\gamma}{2}(\rho_\beta,h^{T_n}-h^0)}.\end{aligned}$$

Let us first bias by α_n only. By Lemma 2.23, the (up to constant) field h^{T_n} biased by α_n has the law of

$$\hat{h} + \frac{2}{\gamma}\log|\cdot| + \frac{\gamma}{2}G(y,\cdot) - \frac{\gamma}{2}\int_{\mathbb{H}} \rho_\beta(w)G(w,\cdot)\mathrm{d}w,$$

where the point y belongs to $\eta^{T_n}([0,1/n])$. We now let n go to ∞ so that the time T_n goes to T_z. Note that, the diameter of $\eta^{T_n}([0,1/n])$ almost surely goes to 0 as n goes to ∞. Indeed, for the (unbiased) SLE curve η^0, we almost surely have that

$$\lim_{n\to\infty} \sup_{t\in[0,1]} \operatorname{diam}\left(\eta^t([0,1/n])\right) = 0.$$

As for any almost sure property, it also holds under any biased law. As a consequence, the point y almost surely goes to 0. The (up to constant) law of h^{T_n} biased by α_n hence converges as n goes to ∞ towards the law of a field

$$\hat{h} + \left(\frac{2}{\gamma} - \gamma\right)\log|\cdot| - \frac{\gamma}{2}\int_{\mathbb{H}} \rho_\beta(z)G(z,\cdot)\mathrm{d}z.$$

Note that, on any compact set K,

$$\sup_{x\in K}\left|\int_{\mathbb{H}} \rho_\beta(w)\left(G(w,x) - G(w,0)\right)\mathrm{d}w\right| = o_{\beta\to\infty}(1).$$

In other words, the term $-\frac{\gamma}{2}\int_{\mathbb{H}} \rho_\beta(w)G(w,\cdot)\mathrm{d}w$ is negligible (up to a constant) in the limit $\beta\to\infty$.

We prove in Lemma 4.7 below that, under the law of (h^{T_n}, η^{T_n}) biased by α_n, the additional bias $\widetilde{\alpha}_n$ is uniformly integrable and converges almost surely towards 1 as first n then β go to ∞, and this concludes the proof. □

Lemma 4.7. *With the notations of the proof of Lemma 4.6, under the law of the quantum zipper biased by α_n, the random variable $\widetilde{\alpha}_n$ is uniformly integrable in n,β and almost surely converges to 1, uniformly in n as β goes to ∞.*

Proof. We only show uniform integrability, as the convergence to 1 is slightly easier and follows from similar estimates. This proof assumes that a non-trivial statement about SLE and natural parametrization holds.

We want to control the quantity

$$\widetilde{\alpha}_n = e^{\frac{\gamma}{2}(\rho_\beta, h^{T_n} - h^0)} = e^{\frac{\gamma}{2}(|\phi_0^{T_n}{}'|^2\rho_\beta\circ\phi_0^{T_n} - \rho_\beta, h^0)} e^{\frac{\gamma}{2}(\rho_\beta, Q\log|\phi_{T_n}^0{}'|)}.$$

The second term of the product in the right-hand side is a geometric term which can be deterministically bounded.

The goal is then to show uniform integrability of the quantity

$$\mathcal{N}(n,\beta) = (|\phi_0^{T_n}{}'|^2\rho_\beta\circ\phi_0^{T_n} - \rho_\beta, h^0)$$

under the law of the capacity zipper biased by an ε-approximation of the chaos:

$$\alpha(n,\beta,\varepsilon) = e^{-\frac{\gamma}{2}(\rho_\beta, h^{T_n})}\int_{\eta^0([T_n, T_n+1/n])} e^{\frac{\gamma}{2}(\theta_z^\varepsilon, h^0)}\varepsilon^{\frac{\gamma^2}{8}}\mu^0(\mathrm{d}z).$$

By conditionning on z, this follows from uniform integrability in (n,β,ε,z) of $e^{\frac{\gamma}{2}\mathcal{N}(n,\beta)}$ under the biased law by

$$\alpha(n,\beta,\varepsilon,z) := e^{-\frac{\gamma}{2}(\rho_\beta,h^{T_n})}e^{\frac{\gamma}{2}(\theta_z^\varepsilon,h^0)}\varepsilon^{\frac{\gamma^2}{8}} = e^{\frac{\gamma}{2}(\theta_z^\varepsilon-|\phi_0^{T_n}{}'|^2\rho_\beta\circ\phi_0^{T_n},h^0)}e^{-\frac{\gamma}{2}(\rho_\beta,Q\log|\phi_{T_n}^0{}'|)}\varepsilon^{\frac{\gamma^2}{8}}.$$

The constant $\varepsilon^{\frac{\gamma^2}{8}}$ cancels out with the normalization factor, and the quantity $(\rho_\beta, Q\log|\phi_{T_n}^0{}'|)$ can be deterministically bounded.

We hence reduced the problem to understanding the law of $\mathcal{N}(n,\beta)$ under the bias $e^{\frac{\gamma}{2}\mathcal{N}'(n,\beta,\varepsilon,z)}$, with

$$\mathcal{N}'(n,\beta,\varepsilon,z) = (\theta_z^\varepsilon - |\phi_0^{T_n}{}'|^2\rho_\beta\circ\phi_0^{T_n},h^0).$$

We now condition on η_0. The random variables $\mathcal{N}(n,\beta)$ and $\mathcal{N}'(n,\beta,\varepsilon,z)$ are jointly Gaussians. By Cameron-Martin, $\mathcal{N}(n,\beta)$ has the law of

$$\begin{aligned}&\widetilde{\mathcal{N}}(n,\beta,\varepsilon,z) = (|\phi_0^{T_n}{}'|^2\rho_\beta\circ\phi_0^{T_n}-\rho_\beta,h^0) \\ +\ &\int_{\mathbb{H}^2}\left(\theta_z^\varepsilon(w)-|\phi_0^{T_n}{}'(w)|^2\rho_\beta\circ\phi_0^{T_n}(w)\right)G(w,y)\left(|\phi_0^{T_n}{}'(y)|^2\rho_\beta\circ\phi_0^{T_n}(y)-\rho_\beta(y)\right)\mathrm{d}w\mathrm{d}y,\end{aligned}\tag{8}$$

where h^0 is a Neumann free field.

Note that in the second term of (8), the two functions we integrate against G are of mean zero. Moreover, $G(\beta w,\beta y) = -2\log\beta + G(w,y)$. Hence, the second term of (8) can be rewritten by scaling the integration variables as:

$$\int_{\mathbb{H}^2}\left(\theta_{z/\beta}^{\varepsilon/\beta}(w)-\rho_\beta^n(w)\right)G(w,y)\left(\rho_\beta^n(y)-\rho(y)\right)\mathrm{d}w\mathrm{d}y,\tag{9}$$

where

$$\rho_\beta^n(w) = |\phi_0^{T_n}{}'(\beta w)|^2\rho\left(\frac{\phi_0^{T_n}(\beta w)}{\beta}\right).$$

Let d be the diameter of $\eta^0([0,1])$. We will bound (9) by a quantity depending on d only. On the complementary of this event, and we have that $|x|$ and $|z|$ are of order at most d, where x is the real number (depending on n and on the SLE) such that $\phi_0^{T_n}(w) = w + x + o_\infty(1)$.

Let D be twice the maximum distance to 0 of points in the support of ρ. We can find a deterministic compact subset $K\subset\mathbb{H}$ such that, on the event $d\le D$, the supports of the functions $\rho_\beta^n(\cdot)$ and $\theta_{z/\beta}^{\varepsilon/\beta}$ for $\varepsilon\le 1$ are included in K.

Moreover, note that by choosing ρ with support at distance 10 from the real line, we ensured that points in the support of one of the ρ_β^n are at a distance no less than 5 than points in the support of one of the $\theta_{z/\beta}^{\varepsilon/\beta}$, as long as $\varepsilon\le 1$. This allows to look only at the off-diagonal (i.e. negative) part of G when trying to bound terms of the form $\int\theta G\rho$.

We hence see that when $d\le D$, the quantity (9) can be bounded by:

$$M := 2\sup_{K\times K}(-G\vee 0) + \sup_K\left|\rho_\beta^n-\rho\right|\sup_K\left|\rho_\beta^n\right|\left|\int_{K\times K}|G(w,y)|\mathrm{d}w\mathrm{d}y\right|$$

Note that $\sup_K\left|\rho_\beta^n\right|$ can be bounded by a constant depending on $\sup|\rho|$ only. When $d\ge D'$, we can bound (9) by $M+10\log d$.

We assume that the probability that the diameter d of $\eta^0([0,1])$ is larger than some positive number D, under the law of SLE biased by

$$\mathbb{E}^{h^0}\left[\mu^0\left[\frac{\gamma}{2},h^0\right]\left(\eta^0([0,1])\right)\right]=\int_{\eta^0([0,1])}|2\Im(w)|^{-\frac{\gamma^2}{8}}e^{-\frac{\gamma^2}{4}\int_{\mathbb{H}}\rho(y)G(y,w)\mathrm{d}y}d\mu^0(w),$$

decreases faster than any polynomial in D.

Together with the estimate on the tail of the law d, we get the wanted uniform integrability of the exponential of the second term of (8).

The first term of (8), $(|\phi_0^{T_n}{}'|^2\rho_\beta\circ\phi_0^{T_n}-\rho_\beta,h^0)$, is a centered Gaussian of variance

$$\int\left(\rho_\beta^n(w)-\rho(w)\right)G(w,y)\left(\rho_\beta^n(y)-\rho(y)\right)\mathrm{d}w\mathrm{d}y.$$

This variance can be controlled as above in a way such that $(|\phi_0^{T_n}{}'|^2\rho_\beta\circ\phi_0^{T_n}-\rho_\beta,h^0)$ is seen to be uniformly integrable. This concludes the proof. □

Proof of Lemma 4.5. The claim follows from tightness in β of h^{T_z} under the law of the biased capacity zipper. By Lemma 4.6, the law of h^{T_z} up to a constant is tight. However, the law of h^{T_z} can be seen as a coupling of h^{T_z} up to a constant with (h^{T_z},ρ_1). As the set of all couplings of tight random variables is itself tight, it is enough to show that (h^{T_n},ρ_1) is tight in (n,β). This follows from the proof of Lemma 4.7, which shows tightness of the random variable $(h^{T_n},\rho_1)-(h^0,\rho_1)$ under the law of the capacity zipper biased by α_n, together with uniform integrability of the additional bias $\widetilde{\alpha}_n$. □

Proof of Proposition 4.3. Let us consider a $(\gamma-\frac{2}{\gamma})$-wedge field and an independent SLE $(h_W^0,\hat{\eta}^0)$. We want to show that for any fixed $a>0$, the couple $(h_W^a,\hat{\eta}^a)$ obtained by unzipping the picture by a units of $\mu^0[\frac{\gamma}{2},h_W^0]$ mass has the law of $(h_W^0,\hat{\eta}^0)$.

In the setup and with the notations of Lemma 4.5, consider $\varepsilon=e^{-\frac{\gamma}{2}M}a$ where M is a large real number, and let z_ε be the point such that

$$\mu^0\left[\frac{\gamma}{2},h^0\right]\left(\eta^0([T_z,T_{z_\varepsilon}])\right)=\varepsilon.$$

By Lemmas 2.17 and 4.5, $(h^{T_z}+M,\eta^{T_z})$ converges in law to $(h_W^0,\hat{\eta}^0)$ when β first goes to ∞ along a well-chosen subsequence, and we then let M go to ∞. As a consequence, the couple $(h^{T_z}+M,\eta^{T_z},h^{T_{z_\varepsilon}}+M,\eta^{T_{z_\varepsilon}})$ converges in law to $(h_W^0,\hat{\eta}^0,h_W^a,\hat{\eta}^a)$.

On the other hand, the laws of (h^0,η^0,z) and $(h^0,\eta^0,z_\varepsilon)$ are at a total variation distance of order ε, that goes to 0 when M goes to ∞. Hence, $(h^{T_z}+M,\eta^{T_z})$ and $(h^{T_{z_\varepsilon}}+M,\eta^{T_{z_\varepsilon}})$ have the same limit in law as M goes to ∞. In other words, $(h_W^a,\hat{\eta}^a)$ has the law of $(h_W^0,\hat{\eta}^0)$. □

4.3 The push-forward of the Liouville boundary measure is a chaos on the natural parametrization (Proof of Proposition 4.1)

We first prove the analog of Proposition 4.1 for the quantum zipper.

Proposition 4.8. *Let us consider the quantum zipper $(h_W^t,\eta^t)_{t\in\mathbb{R}}$. Then, for any times $s<t$:*

$$\lambda\left[\frac{\gamma}{2},h_W^t\right]_{|[0,\phi_s^t(0^+)]}\circ\phi_s^t=C\mu^s\left[\frac{\gamma}{2},h_W^s\right]_{|\eta^s([0,t-s])},$$

where C is a universal constant.

Proof. Let $m(t) = \lambda[\frac{\gamma}{2}, h_W^t]([0, \phi_s^t(0^+)])$ be the right-hand side Liouville boundary mass of the part of the SLE path that has been unzipped between times 0 and t. Note that for any $0 < s < t$,

$$m(t) = \lambda\left[\frac{\gamma}{2}, h_W^s\right]([0, \phi_0^s(0^+)]) + \lambda\left[\frac{\gamma}{2}, h_W^t\right]([0, \phi_s^t(0^+)]).$$

On the other hand, by stationarity of the quantum zipper up to Liouville change of coordinates (Proposition 4.3), and by invariance of volume measures under such changes (Proposition 3.3), the quantity $\lambda[\frac{\gamma}{2}, h_W^t]([0, \phi_{t-1}^t(0^+)])$ has stationary law. By the Birkhoff ergodic theorem, the quantity $\frac{m(n)}{n}$ almost surely converges towards a random variable $C(\omega)$, for integer times n going to ∞. The function $m(t)$ being monotone, this implies that $\frac{m(t)}{t}$ converges towards $C(\omega)$ as the time t goes to ∞.

Let us spell out the previous statement: there is a random variable $C(\omega)$ such that for any $\varepsilon > 0$, we can find a deterministic time T, such that with probability at least $1 - \varepsilon$, we have that

$$\sup_{t \geq T} \left| \frac{m(t)}{t} - C(\omega) \right| < \varepsilon.$$

Let us now add a constant $\frac{2}{\gamma} \log \frac{\tau}{T}$ to the field h_W^0, where $\tau > 0$ is an arbitrary small time. The law of the quantum zipper is preserved (Corollary 2.18), but the time scale t and the quantity $m(t)$ are both scaled by $\frac{\tau}{T}$. Hence, for any $\varepsilon > 0$, for any time $\tau > 0$, with probability at least $1 - \varepsilon$:

$$\sup_{t \geq \tau} \left| \frac{m(t)}{t} - C(\omega) \right| < \varepsilon.$$

In other words, $m(t) = C(\omega)t$ for all positive times. The random constant $C(\omega)$ is then measurable with respect to the curve and the field in any neighborhood of 0. However, the corresponding σ-algebra is trivial, and the constant $C(\omega)$ is hence deterministic. □

Remark 4.9. *Chaos measures are only defined almost surely. However, the fields h_W^t at different times are related through the Liouville change of coordinates formula, and so it is almost surely possible to define simultaneously all chaos $\lambda[\frac{\gamma}{2}, h_W^t]$ and $\mu^t[\frac{\gamma}{2}, h_W^t]$ in such a way that the statements of Proposition 3.3 (i) and (ii) hold for all couples of times $s < t$. As a consequence, we can assume that Proposition 4.8 almost surely holds simultaneously for any couple of times $s < t$.*

Proof of Proposition 4.1. We now consider the capacity zipper, and show that the chaos on the natural parametrization and the push-forward of the right-sided Liouville boundary measure agree on $\eta^0([\varepsilon, T])$:

$$\lambda\left[\frac{\gamma}{2}, h^T\right]_{|[0, \phi_0^T(\eta^0(\varepsilon)^+)]} \circ \phi_0^T = C\mu^0\left[\frac{\gamma}{2}, h^0\right]_{|\eta^0([\varepsilon, T])}, \tag{10}$$

where ε is a small parameter, and T is the first time such that η^0 exits the ball of radius 1 around the origin:

This is enough, as we can rescale the initial conditions of the capacity zipper to make any fixed interval of η^0 appears before the exit time of the ball of radius 1 (at the cost of changing the law of the constant of the field h^0).

Let us consider a quantum zipper $(h_W^t, \hat{\eta}^t)_{t \geq 0}$ such that h_W^0 is in circle-average coordinates, and $\hat{\eta}^0 = \eta^0$, i.e. such that $\hat{\eta}^t$ is a time-reparametrization of η^t. Let τ be the random time such that $\hat{\eta}^\tau = \eta^T$. We unzip η^0 by $\hat{\phi}_0^\tau = \phi_0^T$. By Remark 4.9, even though τ is random, we almost surely have that

$$\lambda\left[\frac{\gamma}{2}, h_W^\tau\right]_{|[0,\hat{\phi}_0^\tau(0+)]} \circ \hat{\phi}_0^\tau = C\mu^0\left[\frac{\gamma}{2}, h_W^0\right]_{|\hat{\eta}^0([0,\tau])}. \tag{11}$$

We now condition on $\eta^0([0,T])$. The unzipping operation is deterministic given the curve η^0, and so whether (10) holds is determined by the sample of the field at time 0 in any neighborhood of $\eta^0([\varepsilon,T])$. By Proposition 2.16, there exists a random constant c such that h^0 and $\widetilde{h} := h_W^0 + c$ are absolutely continuous on a neighborhood of $\eta^0([\varepsilon,T])$. The identity (11) also holds for $\widetilde{h}$, as adding a constant c amounts to multiplying both sides of the equality by $e^{\frac{\gamma}{2}c}$. By absolute continuity of h^0 and $\widetilde{h}$, we thus see that (11) implies (10), and this concludes the proof.

□

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