Products of Cycles

Richard P. Stanley

M.I.T.

Had

Elegant

Research

Breakthroughs

Which

Include

Lovely

Formulas

Separation of elements

$\mathfrak{S}_n$: permutations of $1, 2, \ldots, n$

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$\mathfrak{S}_n$: permutations of $1, 2, \ldots, n$

Let $n \geq 2$. Choose $w \in \mathfrak{S}_n$ (uniform distribution). What is the probability $\rho_2(n)$ that $1, 2$ are in the same cycle of w?

The “fundamental bijection”

Write w as a product of disjoint cycles, least element of each cycle first, decreasing order of least elements:

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The map $\boldsymbol{f} : \mathfrak{S}_n \to \mathfrak{S}_n$, $f(w) = \widehat{w}$, is a bijection (**Foata**).

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$\Rightarrow$ **Theorem.** $\rho_2(n) = 1/2$

α-separation

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a **composition** of m, i.e., $\alpha_i \geq 1$, $\sum \alpha_i = m$.

Let $n \geq m$. Define $w \in \mathfrak{S}_n$ to be **$\boldsymbol{\alpha}$-separated** if $1, 2, \ldots, \alpha_1$ are in the same cycle C_1 of w, $\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_1 + \alpha_2$ are in the same cycle $C_2 \neq C_1$ of w, etc.

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Example. $w = (\mathbf{1}, \mathbf{2}, 10)(\mathbf{3}, 12, 7)(\mathbf{4}, 6, \mathbf{5}, 9)(8, 11)$ is $(2, 1, 2)$-separated.

Generalization of $\rho_2(n) = 1/2$

Let $\rho_\alpha(n)$ be the probability that a random permutation $w \in \mathfrak{S}_n$ is α-separated, $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\sum \alpha_i = m$.

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Let $\rho_\alpha(n)$ be the probability that a random permutation $w \in \mathfrak{S}_n$ is α-separated, $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\sum \alpha_i = m$.

Similar argument gives:

Theorem.

$$\rho_\alpha(n) = \frac{(\alpha_1 - 1)! \cdots (\alpha_k - 1)!}{m!}.$$

Conjecture of M. Bóna

Conjecture (**Bóna**). Let u, v be random n-cycles in $\mathfrak{S}_n$, n **odd**. The probability $\boldsymbol{\pi_2(n)}$ that uv is (2)-separated (i.e., 1 and 2 appear in the same cycle of uv) is 1/2.

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Corollary. Probability that uv is $(1,1)$-separated:

$$\pi_{(1,1)}(n) = 1 - \frac{1}{2} = \frac{1}{2}.$$

$n = 3$ and even n

Example ($n = 3$).

$$(1,2,3)(1,3,2) = (1)(2)(3): \quad (1,1) \text{ – separated}$$
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What about n even?

Probability $\pi_2(n)$ that uv is (2)-separated:

n	2	4	6	8	10
$\pi_2(n)$	0	7/18	9/20	33/70	13/27

Theorem on (2)-separation

Theorem. *We have*

$$\pi_2(n) = \begin{cases} \frac{1}{2}, & n \text{ odd} \\ \frac{1}{2} - \frac{2}{(n-1)(n+2)}, & n \text{ even.} \end{cases}$$

Sketch of proof

Let $w \in \mathfrak{S}_n$ have cycle type $\boldsymbol{\lambda} \vdash n$, i.e.,

$$\lambda = (\lambda_1, \lambda_2, \ldots), \ \lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \ \sum \lambda_i = n,$$

cycle lengths $\lambda_i > 0$.

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cycle lengths $\lambda_i > 0$.

$$\text{type}((1,3)(2,9,5,4)(7)(6,8)) = (4,2,2,1)$$

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E.g., $q_{(1,1,\ldots,1)} = 0$.

a_λ

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Easy: $\pi_2(n) = \frac{1}{(n-1)!^2} \sum_{\lambda \vdash n} a_\lambda q_\lambda.$

The key lemma

Let $n!/z_\lambda = \#\{w \in \mathfrak{S}_n \ : \ \mathrm{type}(w) = \lambda\}$. E.g.,

$$\frac{n!}{z_{(1,1,\ldots,1)}} = 1, \quad \frac{n!}{z_{(n)}} = (n-1)!.$$

Lemma (**Boccara**, 1980).

$$a_\lambda = \frac{n!(n-1)!}{z_\lambda} \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx.$$

A “formula” for $\pi_2(n)$

$$
\begin{aligned}
\pi_2(n) &= \frac{1}{(n-1)!^2} \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} \left(\sum_i \frac{\lambda_i(\lambda_i - 1)}{n(n-1)} \right) \\
&\qquad \cdot (n-1)! \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i} \right) dx \\
&= \frac{1}{n-1} \sum_{\lambda \vdash n} z_\lambda^{-1} \left(\sum_i \lambda_i(\lambda_i - 1) \right) \\
&\qquad \cdot \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i} \right) dx.
\end{aligned}
$$

The exponential formula

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Let $\boldsymbol{p_r(x)} = x_1^r + x_2^r + \cdots$,

$$\boldsymbol{p_\lambda(x)} = p_{\lambda_1}(x)p_{\lambda_2}(x)\cdots .$$

"Exponential formula, permutation version"

$$\exp \sum_{r \geq 1} \frac{1}{r} p_r(x) = \sum_{\lambda} z_\lambda^{-1} p_\lambda(x).$$

The "bad" factor

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Compare

$$\pi_2(n) = \frac{1}{n-1} \sum_{\lambda \vdash n} z_\lambda^{-1} \left(\sum_i \lambda_i(\lambda_i - 1) \right)$$

$$\cdot \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i} \right) dx.$$

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Bad: $\sum \lambda_i(\lambda_i - 1)$

A trick

Straightforward: Let $\boldsymbol{\ell(\lambda)}$ = number of parts.

$$2^{-\ell(\lambda)+1}\left(\frac{\partial^2}{\partial a^2}-\frac{\partial^2}{\partial a\partial b}\right)p_\lambda(a,b)\big|_{a=b=1}=\sum\lambda_i(\lambda_i-1).$$

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Exponential formula gives:

$$\sum(n-1)\pi_2(n)t^n = 2\int_0^1\left(\frac{\partial^2}{\partial a^2}-\frac{\partial^2}{\partial a\partial b}\right)$$

$$\exp\left[\sum_{k\geq 1}\frac{1}{k}\left(\frac{a^k+b^k}{2}\right)(x^k-(x-1)^k)t^k\right]\Bigg|_{a=b=1}dx.$$

Miraculous integral

Get:

$$\sum(n-1)\pi_2(n)t^n = \int_0^1 \frac{t^2(1-2x-2tx+2tx^2)}{(1-t(x-1))(1-tx)^3}dx$$

$$= \frac{1}{t^2}\log(1-t^2) + \frac{3}{2} + \frac{-\frac{1}{2}+t}{(1-t)^2}$$

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(coefficient of t^n)$/(n-1)$:

$$\pi_2(n) = \begin{cases} \frac{1}{2}, & n \text{ odd} \\ \frac{1}{2} - \frac{2}{(n-1)(n+2)}, & n \text{ even.} \end{cases}$$

Generalizations, with R. Du (杜若霞)

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Some simple relations hold, e.g.,

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Previous technique for $\pi_2(n)$ extends to $\pi_\alpha(n)$.

$\pi_{(1^m)}(n)$

Theorem. *Let* $n \geq m \geq 2$. *Then* $\pi_{(1^m)}(n)$ *is given by*

$$\begin{cases} \dfrac{1}{m!}, & n-m \text{ odd} \\ \dfrac{1}{m!} + \dfrac{2}{(m-2)!(n-m+1)(n+m)}, & n-m \text{ even} \end{cases}$$

A general result

Recall: $\boldsymbol{\rho_\alpha(n)}$ $=$ probability that a random permutation $w \in \mathfrak{S}_n$ is α-separated $= (\alpha_1 - 1)! \cdots (\alpha_j - 1)!/m!$.

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Theorem. *Let α be a composition. Then there exist rational functions $R_\alpha(n)$ and $S_\alpha(n)$ of n such that for n sufficiently large,*

$$\pi_\alpha(n) = \begin{cases} R_\alpha(n), \; n \text{ even} \\ S_\alpha(n), \; n \text{ odd.} \end{cases}$$

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Moreover, $\pi_\alpha(n) = \rho_\alpha(n) + O(1/n)$.

Not the whole story

$$\pi_{(2,2,2)} = \begin{cases} \frac{1}{720} - \frac{n^2+n-32}{20(n-3)(n+4)(n-5)(n+6)}, & n \text{ even} \\ \frac{1}{720} - \frac{n^2+n-26}{20(n-2)(n+3)(n-4)(n+5)}, & n \text{ odd} \end{cases}$$

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$$\pi_{(4,2)} = \begin{cases} \frac{1}{120} - \frac{n^4+2n^3-38n^2-39n+234}{5(n-1)(n+2)(n-3)(n+4)(n-5)(n+6)}, & n \text{ even} \\ \frac{1}{120} - \frac{3n^2+3n-58}{10(n-2)(n+3)(n-4)(n+5)}, & n \text{ odd} \end{cases}$$

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Obvious conjecture for denominators and degree of “error term.”

The function $\sigma_\alpha(n)$

E.g., $\boldsymbol{\sigma_{3211}(n)}$ = probability that no cycle of a product uv of two random n-cycles $u, v \in \mathfrak{S}_n$ contains elements from two (or more) of the sets $\{1,2,3\}$, $\{4,5\}$, $\{6\}$, $\{7\}$.

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$$\begin{aligned}\sigma_{32}(n) &= \pi_{32}(n) + 3\pi_{221}(n) + \pi_{311}(n) \\ &\quad +4\pi_{2111}(n) + \pi_{11111}(n).\end{aligned}$$

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Möbius inversion on Π_5 gives:

$$\begin{aligned} \pi_{32}(n) &= \sigma_{32}(n) - 3\sigma_{221}(n) - \sigma_{311}(n) \\ &\quad +5\sigma_{2111}(n) - 2\sigma_{11111}(n). \end{aligned}$$

Some data

n even $\Rightarrow$

$$\sigma_{31}(n) = \frac{1}{4} + \frac{n^2+n-8}{(n-1)(n+2)(n-3)(n+4)}$$

$$\sigma_{22}(n) = \frac{2}{3}\left(\frac{1}{4} + \frac{n^2+n-8}{(n-1)(n+2)(n-3)(n+4)}\right)$$

$$n \text{ odd } \Rightarrow \sigma_{31}(n) = \frac{1}{4} + \frac{1}{(n-2)(n+3)}$$

$$\sigma_{22}(n) = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{(n-2)(n+3)}\right).$$

A conjecture

Conjecture. Let α and β be compositions of m with the same number k of parts. Then

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Implies all previous conjectures.

Bernardi-Morales

Olivier Bernardi and Alejandro Morales, 2011: conjecture is true.

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Moreover, for α a composition of m with k parts,

$$\sigma_\alpha(n) = \frac{1}{\prod \alpha_i! \cdot (n-1)_{m-1}}$$

$$\left[\sum_{j=0}^{m-k} (-1)^j \frac{\binom{m-k}{k}\binom{n+j+1}{m}}{(j+1)\binom{n+k+j}{j+1}} + \frac{(-1)^{n-m}\binom{n-1}{k-2}}{(m-k+1)\binom{n+m}{m-k+1}}\right]$$

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Determines $\sigma_\alpha(n)$ and $\pi_\alpha(n)$ for all α.

A basic bijection

Proof by Bernardi-Morales begins with a standard bijection between products $uv = n$-cycle and **bipartite unicellular edge-labelled maps** on an (orientable) surface.

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Genus $\boldsymbol{g}$ of surface given by

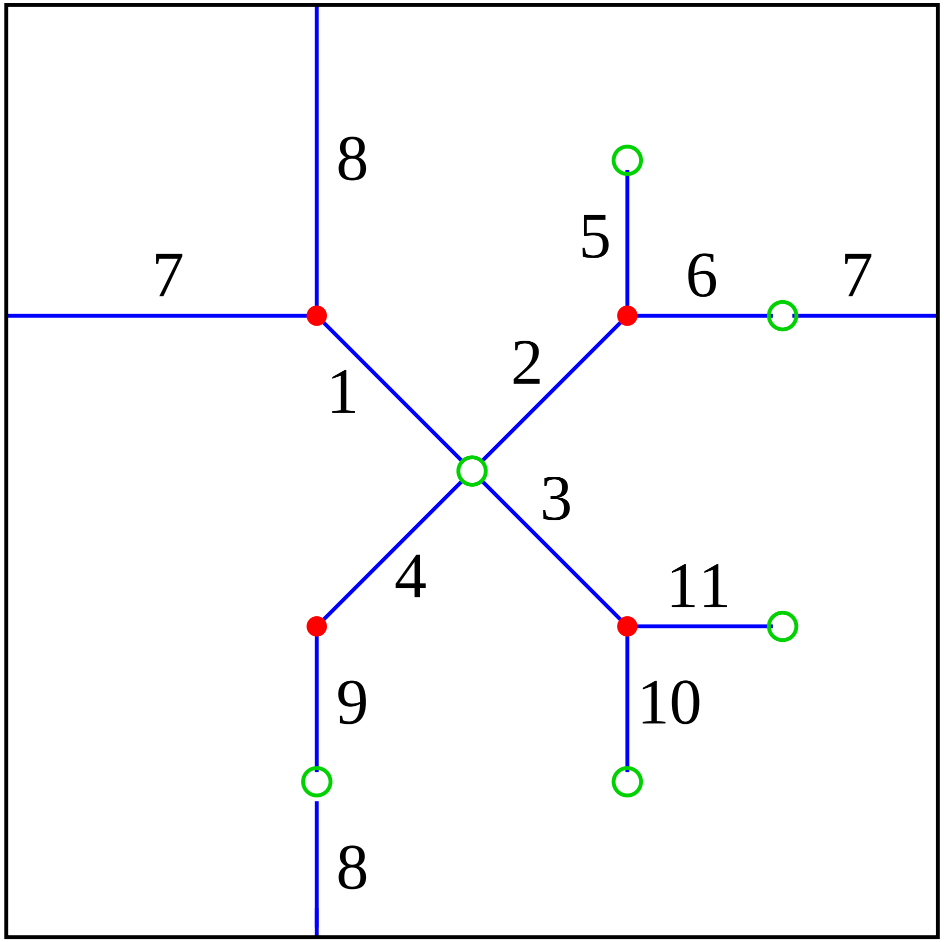
$$g = \frac{1}{2}(n + 1 - \kappa(u) - \kappa(v)),$$

where $\boldsymbol{\kappa}$ denotes number of cycles.

An example for $g = 1$

$$(1,2,3,4)(5)(6,7)(8,9)(10)(11) \cdot (1,7,8)(2,5,6)(3,11,10)(4,9)$$

$$= (1,5,6,8,4,7,2,11,10,3,9)$$



Tree-rooted maps

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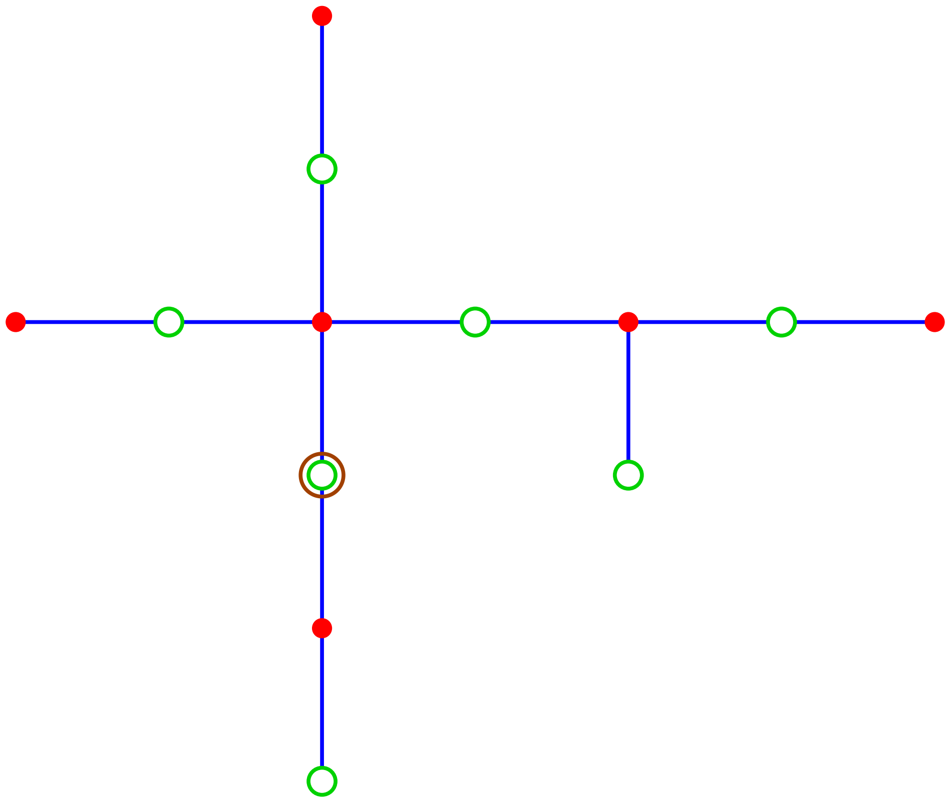

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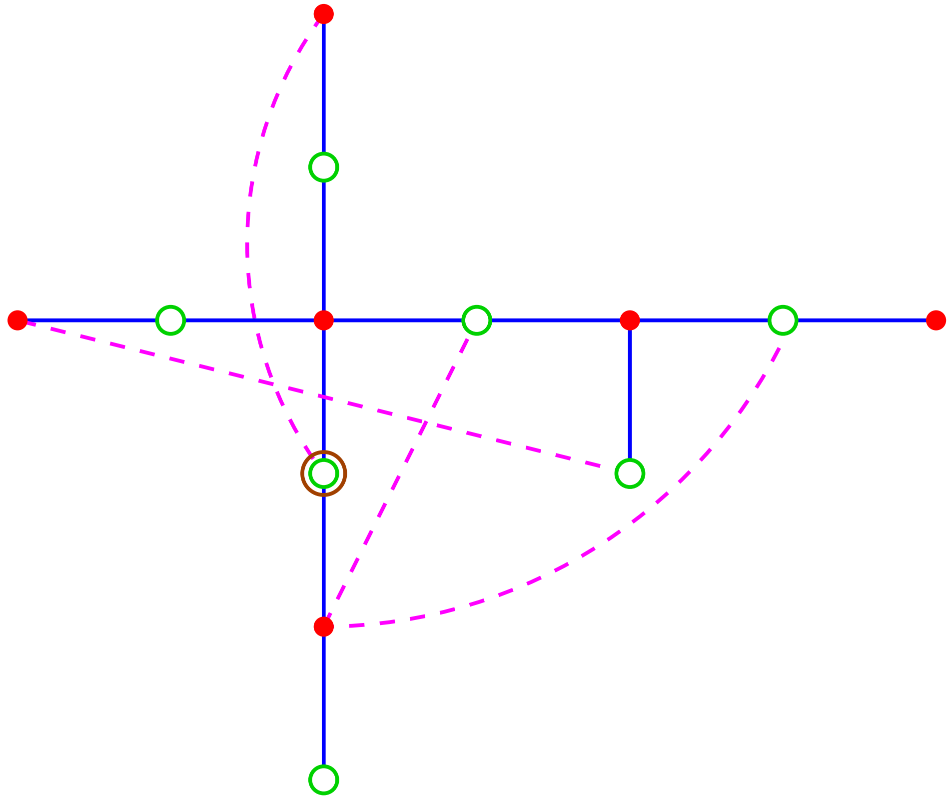

Generalizations

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Most successful generalization: product of n-cycle and $(n-j)$-cycle.

n-cycle times $(n-j)$-cycle

Let $\boldsymbol{\lambda} \vdash n$, $0 \leq j < n$. Let $\boldsymbol{a_{\lambda,j}}$ be the number of pairs $(u,v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ for which u is an n-cycle, v is an $(n-j)$-cycle, and uv has type λ.

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Theorem (Boccara).

$$a_{\lambda,j} = \frac{n!(n-j-1)!}{z_\lambda\, j!} \int_0^1 \frac{d^j}{dx^j} \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx.$$

The case $j = 1$

$$\alpha_{\lambda,1} = \frac{n!(n-2)!}{z_\lambda} \int_0^1 \frac{d}{dx} \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx$$

$$= \begin{cases} \frac{2n!(n-2)!}{z_\lambda}, & \lambda \text{ odd type} \\ 0, & \lambda \text{ even type.} \end{cases}$$

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In other words, if u is an n-cycle and v is an $(n-1)$-cycle, then uv is equidistributed on odd permutations.

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$$\alpha_{\lambda,1} = \frac{n!(n-2)!}{z_\lambda} \int_0^1 \frac{d}{dx} \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx$$

$$= \begin{cases} \frac{2n!(n-2)!}{z_\lambda}, & \lambda \text{ odd type} \\ 0, & \lambda \text{ even type.} \end{cases}$$

In other words, if u is an n-cycle and v is an $(n-1)$-cycle, then uv is equidistributed on odd permutations.

Bijective proof known (**A. Machì**, 1992).

Explicit formula

Let $\boldsymbol{u} \in \mathfrak{S}_n$ be a random n-cycle and $\boldsymbol{v} \in \mathfrak{S}_n$ a random $(n-1)$-cycle. Let $\boldsymbol{\pi_\alpha(n, n-1)}$ be the probability that uv is α-separated.

Theorem. Let $\sum \alpha_i = \boldsymbol{m}$. Then

$$\pi_\alpha(n, n-1) = \frac{(\alpha_1 - 1)! \cdots (\alpha_\ell - 1)!}{(m-2)!}$$

$$\times \left(\frac{1}{m(m-1)} + (-1)^{n-m} \frac{1}{n(n-1)} \right)$$


That's all Folks!