Products of Cycles

Richard P. Stanley

M.I.T.
Had Elegant Research Breakthroughs Which Include Lovely Formulas
$S_n$: permutations of $1, 2, \ldots, n$
\( \mathcal{S}_n \): permutations of 1, 2, \ldots, n

Let \( n \geq 2 \). Choose \( w \in \mathcal{S}_n \) (uniform distribution). What is the probability \( \rho_2(n) \) that 1, 2 are in the same cycle of \( w \)?
The “fundamental bijection”

Write $w$ as a product of disjoint cycles, least element of each cycle first, decreasing order of least elements:

$$(6, 8)(4)(2, 7, 3)(1, 5).$$
Write $\omega$ as a product of disjoint cycles, least element of each cycle first, decreasing order of least elements:

$$(6, 8)(4)(2, 7, 3)(1, 5).$$

Remove parentheses, obtaining $\hat{\omega} \in S_n$

(one-line form):

$68427315$. 

The “fundamental bijection”
Write $w$ as a product of disjoint cycles, least element of each cycle first, decreasing order of least elements:

$$(6, 8)(4)(2, 7, 3)(1, 5).$$

Remove parentheses, obtaining $\hat{w} \in \mathfrak{S}_n$ (one-line form):

$$68427315.$$ 

The map $f : \mathfrak{S}_n \to \mathfrak{S}_n, f(w) = \hat{w}$, is a bijection (Foata).
Answer to question

\[ w = (6, 8)(4)(2, 7, 3)(1, 5) \]
\[ \hat{w} = 68427315 \]
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Note. 1 and 2 are in the same cycle of \( w \)
\[ \iff 1 \text{ precedes } 2 \text{ in } \hat{w}. \]
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\[ \hat{w} = 68427315 \]

**Note.** 1 and 2 are in the same cycle of \( w \)
\[ \Leftrightarrow \text{1 precedes 2 in } \hat{w}. \]

\[ \Rightarrow \textbf{Theorem. } \rho_2(n) = 1/2 \]
Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of $m$, i.e., $\alpha_i \geq 1$, $\sum \alpha_i = m$.

Let $n \geq m$. Define $w \in \mathfrak{S}_n$ to be $\alpha$-separated if $1, 2, \ldots, \alpha_1$ are in the same cycle $C_1$ of $w$, $\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_1 + \alpha_2$ are in the same cycle $C_2 \neq C_1$ of $w$, etc.
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Let $n \geq m$. Define $w \in S_n$ to be $\alpha$-separated if $1, 2, \ldots, \alpha_1$ are in the same cycle $C_1$ of $w$, $\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_1 + \alpha_2$ are in the same cycle $C_2 \neq C_1$ of $w$, etc.

**Example.** $w = (1, 2, 10)(3, 12, 7)(4, 6, 5, 9)(8, 11)$ is $(2, 1, 2)$-separated.
Generalization of $\rho_2(n) = 1/2$

Let $\rho_{\alpha}(n)$ be the probability that a random permutation $w \in S_n$ is $\alpha$-separated, $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\sum \alpha_i = m$. 
Generalization of $\rho_2(n) = 1/2$

Let $\rho_{\alpha}(n)$ be the probability that a random permutation $w \in S_n$ is $\alpha$-separated, $\alpha = (\alpha_1, \ldots, \alpha_k), \sum \alpha_i = m$.

Similar argument gives:

**Theorem.**

$$\rho_{\alpha}(n) = \frac{(\alpha_1 - 1)! \cdots (\alpha_k - 1)!}{m!}.$$
Conjecture (Bóna). Let $u, v$ be random $n$-cycles in $\mathcal{S}_n$, $n$ odd. The probability $\pi_2(n)$ that $uv$ is (2)-separated (i.e., 1 and 2 appear in the same cycle of $uv$) is $1/2$. 
Conjecture (Bóna). Let \( u, v \) be random \( n \)-cycles in \( \mathfrak{S}_n \), \( n \) odd. The probability \( \pi_2(n) \) that \( uv \) is \((2)\)-separated (i.e., 1 and 2 appear in the same cycle of \( uv \)) is 1/2.

Corollary. Probability that \( uv \) is \((1,1)\)-separated:

\[
\pi_{(1,1)}(n) = 1 - \frac{1}{2} = \frac{1}{2}.
\]
$n = 3$ and even $n$

Example $(n = 3)$.

$$(1, 2, 3)(1, 3, 2) = (1)(2)(3) : (1, 1) \text{ – separated}$$

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What about $n$ even?

Probability $\pi_2(n)$ that $uv$ is (2)-separated:

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_2(n)$</td>
<td>0</td>
<td>7/18</td>
<td>9/20</td>
<td>33/70</td>
<td>13/27</td>
</tr>
</tbody>
</table>
Theorem. We have

\[ \pi_2(n) = \begin{cases} 
  \frac{1}{2}, & n \text{ odd} \\
  \frac{1}{2} - \frac{2}{(n-1)(n+2)}, & n \text{ even.} 
\end{cases} \]
Let $w \in S_n$ have cycle type $\lambda \vdash n$, i.e.,

$$\lambda = (\lambda_1, \lambda_2, \ldots), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \sum \lambda_i = n,$$

cycle lengths $\lambda_i > 0$. 
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cycle lengths $\lambda_i > 0$.

$$\text{type}((1, 3)(2, 9, 5, 4)(7)(6, 8)) = (4, 2, 2, 1)$$
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$$q_\lambda = \frac{\sum \binom{\lambda_i}{2}}{\binom{n}{2}} = \frac{\sum \lambda_i(\lambda_i - 1)}{n(n - 1)}.$$
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q_\lambda = \frac{\sum \binom{\lambda_i}{2}}{\binom{n}{2}} = \frac{\sum \lambda_i(\lambda_i - 1)}{n(n - 1)}.
\]

E.g., $q_{(1,1,\ldots,1)} = 0$. 
Let $a_\lambda$ be the number of pairs $(u, v)$ of $n$-cycles in $\mathfrak{S}_n$ for which $uv$ has type $\lambda$ (a connection coefficient).
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E.g., $a_{(1,1,1)} = a_3 = 2$, $a_{(2,1)} = 0$.

Easy: $\pi_2(n) = \frac{1}{(n-1)!^2} \sum_{\lambda \vdash n} a_\lambda q_\lambda$. 
Let $n!/z_\lambda = \# \{ w \in \mathfrak{S}_n : \text{type}(w) = \lambda \}$. E.g.,

\[
\frac{n!}{z(1,1,\ldots,1)} = 1, \quad \frac{n!}{z(n)} = (n - 1)!. \\
\]

**Lemma** (Boccara, 1980).

\[
a_\lambda = \frac{n!(n-1)!}{z_\lambda} \int_0^1 \prod_i (x^{\lambda_i} - (x-1)^{\lambda_i}) \, dx. \\
\]
A “formula” for $\pi_2(n)$

$$\pi_2(n) = \frac{1}{(n-1)!^2} \sum_{\lambda \vdash n} \frac{n!}{z_{\lambda}} \left( \sum_i \frac{\lambda_i(\lambda_i - 1)}{n(n-1)} \right) \cdot (n-1)! \int_0^1 \prod_i (x^{\lambda_i} - (x-1)^{\lambda_i}) \, dx$$

$$= \frac{1}{n-1} \sum_{\lambda \vdash n} z_{\lambda}^{-1} \left( \sum_i \lambda_i(\lambda_i - 1) \right) \cdot \int_0^1 \prod_i (x^{\lambda_i} - (x-1)^{\lambda_i}) \, dx.$$
The exponential formula

How to extract information?
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**Answer:** generating functions.
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**Answer:** generating functions.

Let \( p_r(x) = x_1^r + x_2^r + \cdots \),

\[
p_\lambda(x) = p_{\lambda_1}(x)p_{\lambda_2}(x) \cdots.
\]

“Exponential formula, permutation version”

\[
\exp \sum_{r \geq 1} \frac{1}{r} p_r(x) = \sum_{\lambda} z_{\lambda}^{-1} p_\lambda(x).
\]
The “bad” factor

\[
\exp \sum_{m \geq 1} \frac{1}{m} p_m(x) = \sum_{\lambda} z_\lambda^{-1} p_\lambda(x).
\]
The “bad” factor

\[ \exp \sum_{m \geq 1} \frac{1}{m} p_m(x) = \sum_{\lambda} z^{-1}_\lambda p_\lambda(x). \]

Compare

\[ \pi_2(n) = \frac{1}{n - 1} \sum_{\lambda|n} z^{-1}_\lambda \left( \sum_i \lambda_i(\lambda_i - 1) \right) \]

\[ \cdot \int_0^1 \prod_i (x^\lambda_i - (x - 1)^\lambda_i) \, dx. \]
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\[ \exp \sum_{m \geq 1} \frac{1}{m} p_m(x) = \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x). \]

Compare

\[ \pi_2(n) = \frac{1}{n-1} \sum_{\lambda \vdash n} z_{\lambda}^{-1} \left( \sum_{i} \lambda_i (\lambda_i - 1) \right) \]

\[ \cdot \int_0^1 \prod_i \left( x^{\lambda_i} - (x - 1)^{\lambda_i} \right) dx. \]

Bad: \[ \sum \lambda_i (\lambda_i - 1) \]
A trick

**Straightforward:** Let \( \ell(\lambda) = \) number of parts.

\[
2^{-\ell(\lambda)+1} \left( \frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial a \partial b} \right) p_\lambda(a, b)|_{a=b=1} = \sum \lambda_i (\lambda_i - 1).
\]
A trick

**Straightforward:** Let $\ell(\lambda) =$ number of parts.

$$2^{-\ell(\lambda)+1} \left( \frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial a \partial b} \right) p_{\lambda}(a, b)|_{a=b=1} = \sum \lambda_i(\lambda_i - 1).$$

Exponential formula gives:

$$\sum (n - 1) \pi_2(n) t^n = 2 \int_0^1 \left( \frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial a \partial b} \right) \exp \left[ \sum_{k \geq 1} \frac{1}{k} \left( \frac{a^k + b^k}{2} \right) (x^k - (x - 1)^k) t^k \right] \bigg|_{a=b=1} dx.$$
Miraculous integral

Get:

$$\sum (n - 1) \pi_2(n) t^n = \int_0^1 \frac{t^2(1 - 2x - 2tx + 2tx^2)}{(1 - t(x - 1))(1 - tx)^3} dx$$

$$= \frac{1}{t^2} \log(1 - t^2) + \frac{3}{2} + \frac{-\frac{1}{2} + t}{(1 - t)^2}$$
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(coefficient of \(t^n\))/(\(n - 1\)):

\[
\pi_2(n) = \begin{cases} 
\frac{1}{2}, & n \text{ odd} \\
\frac{1}{2} - \frac{2}{(n-1)(n+2)}, & n \text{ even.}
\end{cases}
\]
$\pi_\alpha(n) = \text{probability that } uv \text{ is } \alpha\text{-separated for random } n\text{-cycles } u, v$
$\pi_\alpha(n)$ = probability that $uv$ is $\alpha$-separated for random $n$-cycles $u, v$

Some simple relations hold, e.g.,

$$\pi_3(n) = \pi_4(n) + \pi_{3,1}(n).$$
\( \pi_\alpha(n) \) = probability that \( uv \) is \( \alpha \)-separated for random \( n \)-cycles \( u, v \)

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\]

Previous technique for \( \pi_2(n) \) extends to \( \pi_\alpha(n) \).
Theorem. Let $n \geq m \geq 2$. Then $\pi_{(1^m)}(n)$ is given by

$$
\begin{cases}
\frac{1}{m!}, & n - m \text{ odd} \\
\frac{1}{m!} + \frac{2}{(m - 2)! (n - m + 1)(n + m)}, & n - m \text{ even}
\end{cases}
$$
Recall: $\rho_\alpha(n) = \text{probability that a random permutation } w \in \mathfrak{S}_n \text{ is } \alpha\text{-separated}
= (\alpha_1 - 1)! \cdots (\alpha_j - 1)!/m!$. 
A general result

Recall: \( \rho_\alpha(n) \) = probability that a random permutation \( w \in \mathfrak{S}_n \) is \( \alpha \)-separated
\( = (\alpha_1 - 1)! \cdots (\alpha_j - 1)! / m! \).

**Theorem.** Let \( \alpha \) be a composition. Then there exist rational functions \( R_\alpha(n) \) and \( S_\alpha(n) \) of \( n \) such that for \( n \) sufficiently large,

\[
\pi_\alpha(n) = \begin{cases} 
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**Theorem.** Let $\alpha$ be a composition. Then there exist rational functions $R_\alpha(n)$ and $S_\alpha(n)$ of $n$ such that for $n$ sufficiently large,

$$\pi_\alpha(n) = \begin{cases} 
R_\alpha(n), & n \text{ even} \\
S_\alpha(n), & n \text{ odd}.
\end{cases}$$

Moreover, $\pi_\alpha(n) = \rho_\alpha(n) + O(1/n)$. 
\[ \pi(2,2,2) = \begin{cases} 
\frac{1}{720} - \frac{n^2 + n - 32}{20(n-3)(n+4)(n-5)(n+6)}, & n \text{ even} \\
\frac{1}{720} - \frac{n^2 + n - 26}{20(n-2)(n+3)(n-4)(n+5)}, & n \text{ odd} 
\end{cases} \]
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\[ \pi(4,2) = \begin{cases} \frac{1}{120} - \frac{n^4 + 2n^3 - 38n^2 - 39n + 234}{5(n-1)(n+2)(n-3)(n+4)(n-5)(n+6)}, & n \text{ even} \\ \frac{1}{120} - \frac{3n^2 + 3n - 58}{10(n-2)(n+3)(n-4)(n+5)}, & n \text{ odd} \end{cases} \]
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Obvious conjecture for denominators and degree of “error term.”
The function $\sigma_\alpha(n)$

E.g., $\sigma_{3211}(n) = \text{probability that no cycle of a product } uv \text{ of two random } n\text{-cycles } u, v \in S_n \text{ contains elements from two (or more) of the sets } \{1, 2, 3\}, \{4, 5\}, \{6\}, \{7\}.
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$$\sigma_{32}(n) = \pi_{32}(n) + 3\pi_{221}(n) + \pi_{311}(n)$$
$$+4\pi_{2111}(n) + \pi_{11111}(n).$$
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\sigma_{32}(n) = \pi_{32}(n) + 3\pi_{221}(n) + \pi_{311}(n) \\
+ 4\pi_{2111}(n) + \pi_{11111}(n).
\]

Möbius inversion on $\Pi_5$ gives:

\[
\pi_{32}(n) = \sigma_{32}(n) - 3\sigma_{221}(n) - \sigma_{311}(n) \\
+ 5\sigma_{2111}(n) - 2\sigma_{11111}(n).
\]
Some data

\( n \text{ even } \Rightarrow \)

\[
\sigma_{31}(n) = \frac{1}{4} + \frac{n^2 + n - 8}{(n - 1)(n + 2)(n - 3)(n + 4)}
\]

\[
\sigma_{22}(n) = \frac{2}{3} \left( \frac{1}{4} + \frac{n^2 + n - 8}{(n - 1)(n + 2)(n - 3)(n + 4)} \right)
\]

\( n \text{ odd } \Rightarrow \sigma_{31}(n) = \frac{1}{4} + \frac{1}{(n - 2)(n + 3)} \)

\[
\sigma_{22}(n) = \frac{2}{3} \left( \frac{1}{4} + \frac{1}{(n - 2)(n + 3)} \right).
\]
A conjecture

Conjecture. Let $\alpha$ and $\beta$ be compositions of $m$ with the same number $k$ of parts. Then

$$\frac{\sigma_\alpha(n)}{\prod \alpha_i!} = \frac{\sigma_\beta(n)}{\prod \beta_i!}. $$
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Implies all previous conjectures.
Olivier Bernardi and Alejandro Morales, 2011: conjecture is true.
Bernardi-Morales

Olivier Bernardi and Alejandro Morales, 2011: conjecture is true.

Moreover, for \( \alpha \) a composition of \( m \) with \( k \) parts,

\[
\sigma_\alpha(n) = \frac{1}{\prod \alpha_i! \cdot (n - 1)_{m-1}} \left[ \sum_{j=0}^{m-k} (-1)^j \frac{(m-k)}{k} \frac{(n+j+1)}{m} \frac{(n+k+j)}{(j+1)(n+k+j)} + \frac{(-1)^{n-m}(n-1)}{(m-k+1)(n+m)} \right]
\]
conjecture is true. Moreover, for $\alpha$ a composition of $m$ with $k$ parts,

$$\sigma_\alpha(n) = \frac{1}{\prod \alpha_i! \cdot (n - 1)_{m-1}}$$

$$\sum_{j=0}^{m-k} (-1)^j \frac{(m-k)}{k} \frac{(n+j+1)}{m} \frac{(n+k+j)}{(j+1)(n+k+j)} + \frac{(-1)^{n-m}(n-1)}{(m-k+1)(n+m)}$$

Determines $\sigma_\alpha(n)$ and $\pi_\alpha(n)$ for all $\alpha$. 
A basic bijection

Proof by Bernardi-Morales begins with a standard bijection between products \( uv = n \)-cycle and \textbf{bipartite unicellular edge-labelled maps} on an (orientable) surface.
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\(uv = n\)-cycle and **bipartite unicellular edge-labelled maps** on an (orientable) surface.

Genus \(g\) of surface given by

\[
g = \frac{1}{2}(n + 1 - \kappa(u) - \kappa(v)),
\]

where \(\kappa\) denotes number of cycles.
An example for $g = 1$

$$(1, 2, 3, 4)(5)(6, 7)(8, 9)(10)(11) \cdot (1, 7, 8)(2, 5, 6)(3, 11, 10)(4, 9) = (1, 5, 6, 8, 4, 7, 2, 11, 10, 3, 9)$$
Tree-rooted maps

There is a (difficult) bijection with bipartite tree-rooted maps.
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*tree-rooted maps*. 
Tree-rooted maps

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How can we generalize the product $uv$ of two $n$-cycles?
Generalizations

How can we generalize the product $uv$ of two $n$-cycles?

Most successful generalization: product of $n$-cycle and $(n - j)$-cycle.
Let $\lambda \vdash n$, $0 \leq j < n$. Let $a_{\lambda,j}$ be the number of pairs $(u, v) \in S_n \times S_n$ for which $u$ is an $n$-cycle, $v$ is an $(n - j)$-cycle, and $uv$ has type $\lambda$. 
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Theorem (Boccara).

$$a_{\lambda,j} = \frac{n!(n - j - 1)!}{z_{\lambda,j} j!} \int_0^1 \frac{d^j}{dx^j} \prod_i (x^{\lambda_i} - (x - 1)^{\lambda_i}) \, dx.$$
The case $j = 1$

\[
\alpha_{\lambda,1} = \frac{n!(n-2)!}{z_{\lambda}} \int_0^1 \frac{d}{dx} \prod_i (x^{\lambda_i} - (x - 1)^{\lambda_i}) \, dx
\]

\[
= \begin{cases} 
\frac{2n!(n-2)!}{z_{\lambda}}, & \lambda \text{ odd type} \\
0, & \lambda \text{ even type.}
\end{cases}
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The case $j = 1$

\[ \alpha_{\lambda,1} = \frac{n!(n-2)!}{z_\lambda} \int_0^1 \frac{d}{dx} \prod_i \left( x^{\lambda_i} - (x-1)^{\lambda_i} \right) dx \]

\[ = \begin{cases} \frac{2n!(n-2)!}{z_\lambda}, & \lambda \text{ odd type} \\ 0, & \lambda \text{ even type} \end{cases} \]

In other words, if $u$ is an $n$-cycle and $v$ is an $(n-1)$-cycle, then $uv$ is equidistributed on odd permutations.
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In other words, if $u$ is an $n$-cycle and $v$ is an $(n-1)$-cycle, then $uv$ is equidistributed on odd permutations.

Bijective proof known (A. Machì, 1992).
Let $u \in \mathfrak{S}_n$ be a random $n$-cycle and $v \in \mathfrak{S}_n$ a random $(n-1)$-cycle. Let $\pi_{\alpha}(n, n - 1)$ be the probability that $uv$ is $\alpha$-separated.

**Theorem.** Let $\sum \alpha_i = m$. Then

$$
\pi_{\alpha}(n, n - 1) = \frac{(\alpha_1 - 1)! \cdots (\alpha_\ell - 1)!}{(m - 2)!} \times \left( \frac{1}{m(m - 1)} + (-1)^{n-m} \frac{1}{n(n - 1)} \right)
$$