The Visibility Arrangement and Line Shelling Arrangement of a Convex Polytope

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M.I.T.

Visible facets

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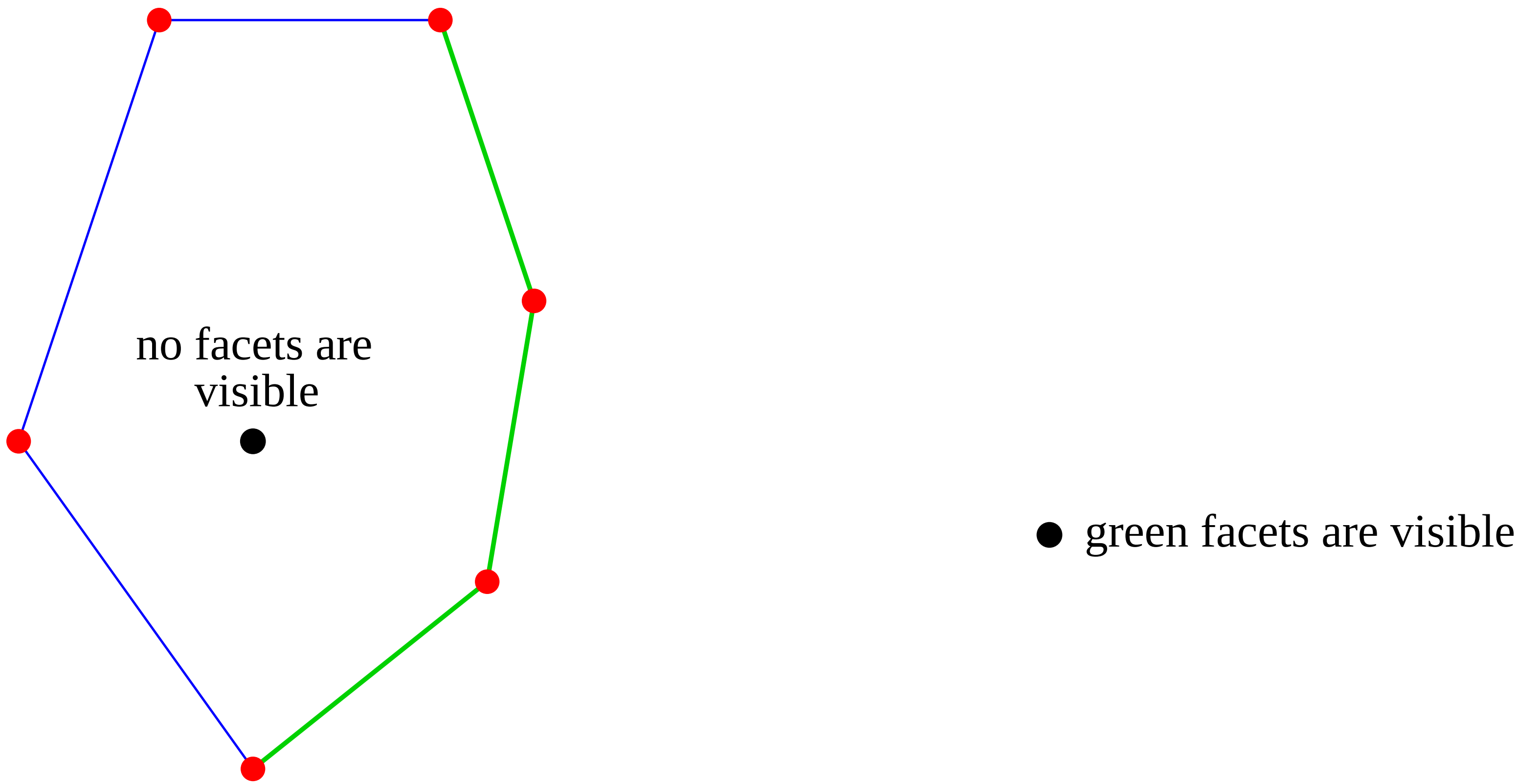

The visibility arrangement

aff(S): the affine span of a subset $S \subset \mathbb{R}^d$

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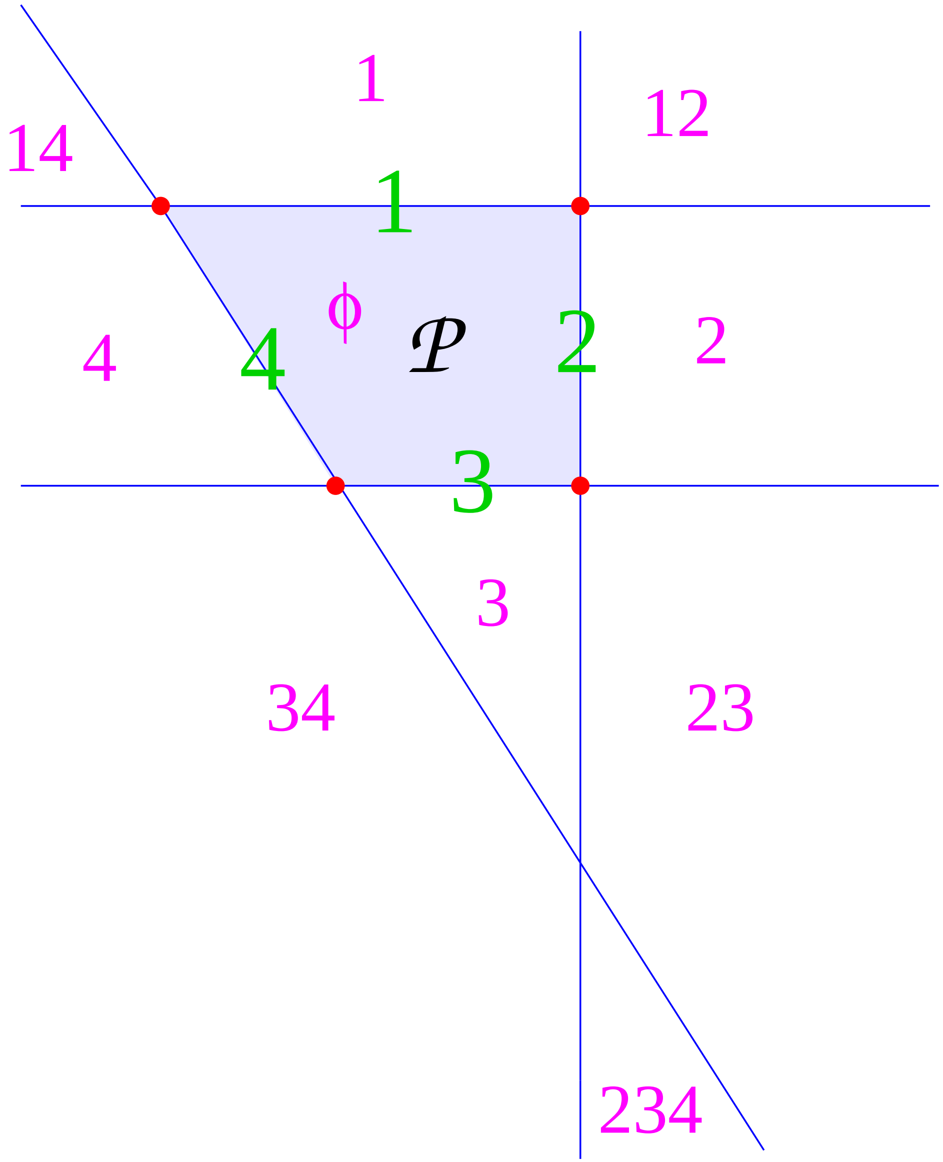
Regions of $\mathrm{vis}(\mathcal{P})$ correspond to sets of facets that are visible from some point $v \in \mathbb{R}^d$.

An example



Number of regions

$\boldsymbol{v}(\mathcal{P})$: number of regions of $\mathrm{vis}(\mathcal{P})$, i.e., the number of visibility sets of $\mathcal{P}$

$\boldsymbol{\chi_{\mathcal{A}}}(q)$: characteristic polynomial of the arrangement $\mathcal{A}$

Zaslavsky's theorem. *Number of regions of $\mathcal{A}$ is $(-1)^d \chi_{\mathcal{A}}(-1)$.*

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In general, $v(\mathcal{P})$ and $\chi_{\mathrm{vis}(\mathcal{P})}(q)$ are hard to compute.

A simple example

$\mathcal{P}_n = n$-cube

$$\chi_{\mathrm{vis}(\mathcal{P}_n)}(q) = (q-2)^n$$

$$v(\mathcal{P}_n) = 3^n$$

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For any facet F, can see either F, $-F$, or neither.

Order polytopes

$\boldsymbol{P} = \{t_1, \ldots, t_d\}$: a poset (partially ordered set)

Order polytope of P:

$$\mathcal{O}(P) =$$
$$\{(x_1, \ldots, x_d) \in \mathbb{R}^d \,:\, 0 \leq x_i \leq x_j \leq 1 \text{ if } t_i \leq t_j\}$$

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$\chi_{\mathrm{vis}(\mathcal{O}(P))}(q)$ can be described in terms of "generalized chromatic polynomials" (later, if time), but there is a curious special case.

Rank one posets

Suppose that P has rank at most one (no three-element chains).

$\boldsymbol{H}(P) =$ Hasse diagram of P, with vertex set $\boldsymbol{V}$

For $W \subseteq V$, let $\boldsymbol{H_W} =$ restriction of H to W

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Theorem.

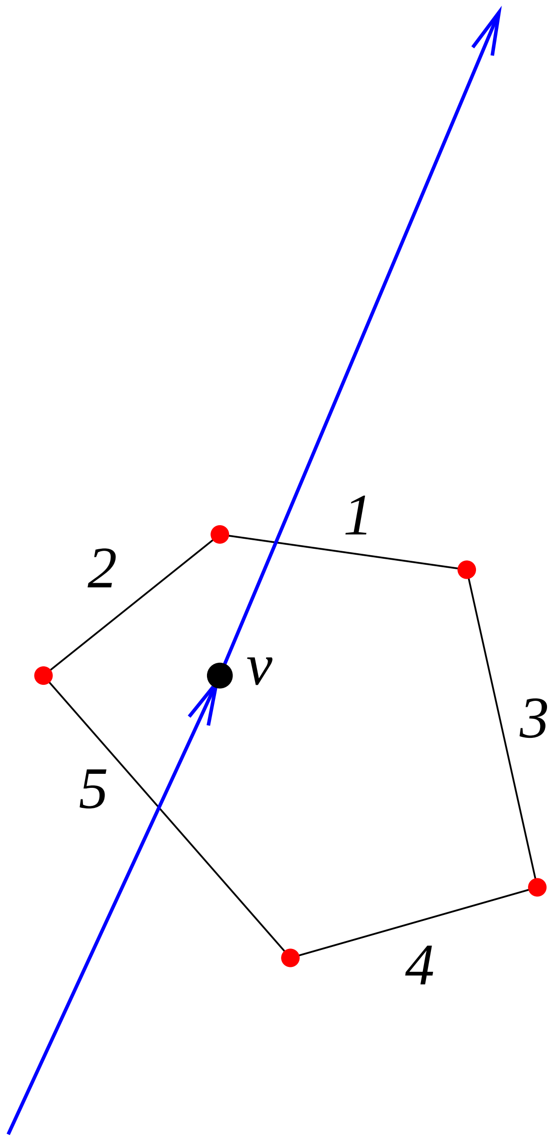
$$v(\mathcal{O}(P)) = (-1)^{\#P} \sum_{W \subseteq V} \chi_{H_W}(-3)$$

Line shellings

Let $v \in \mathbf{int}(\mathcal{P})$ (interior of $\mathcal{P}$)

Line shelling based at v: let L be a directed line from v. Let $F_1, F_2, \ldots, F_k$ be the order in which facets become visible along L, followed by the order in which they become invisible from ∞ along the other half of L. Assume L is sufficiently generic so that no two facets become visible or invisible at the same time.

Example of a line shelling



The line shelling arrangment

$\mathbf{ls}(\boldsymbol{\mathcal{P}}, \boldsymbol{v})$: hyperplanes are

- affine span of v with aff$(F_1) \cap \mathrm{aff}(F_2) \neq \emptyset$, where F_1, F_2 are distinct facets
- if aff$(F_1) \cap \mathrm{aff}(F_2) = \emptyset$, then the hyperplane through v parallel to F_1, F_2

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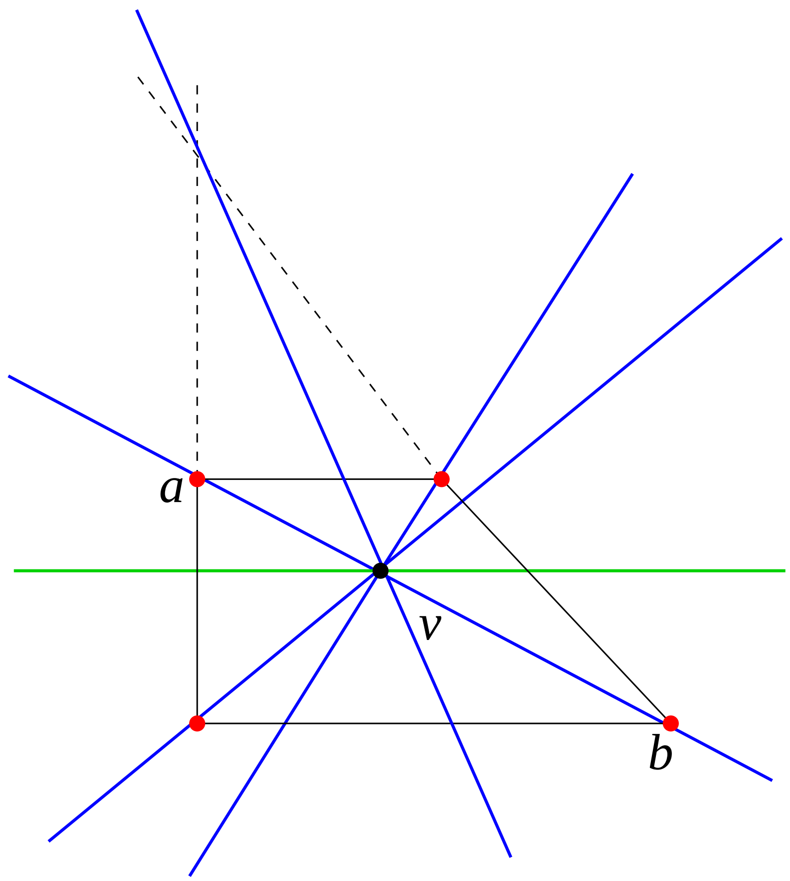
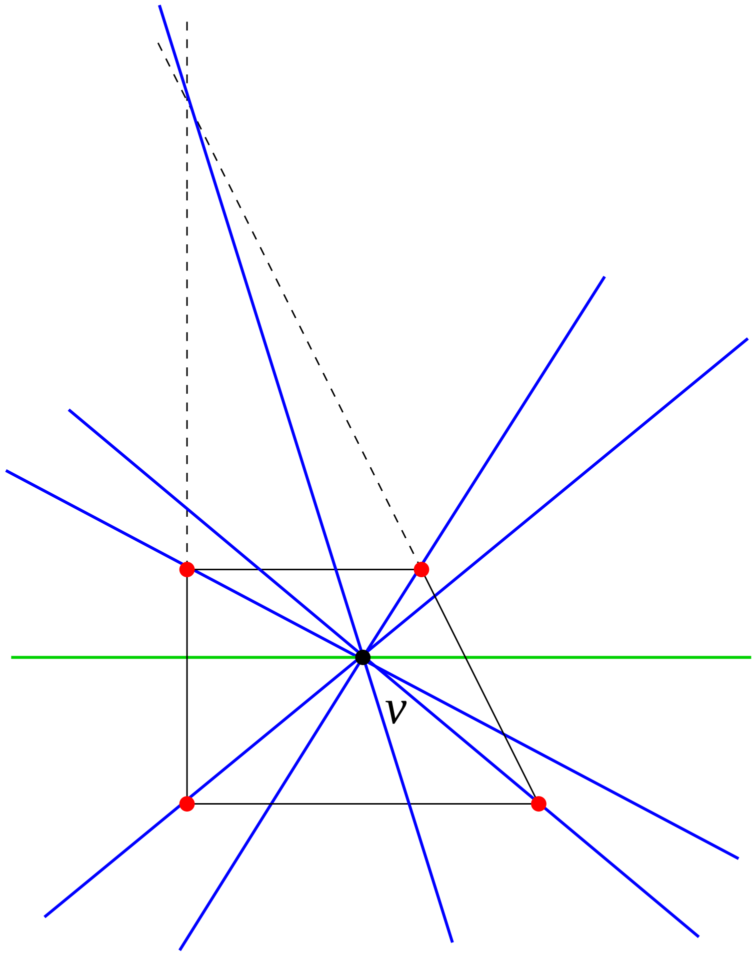
Line shellings at v are in bijection with regions of $\mathrm{ls}(\mathcal{P}, v)$.

A nongeneric example



v is not generic: $\overline{av} = \overline{bv}$ (10 line shellings at v)

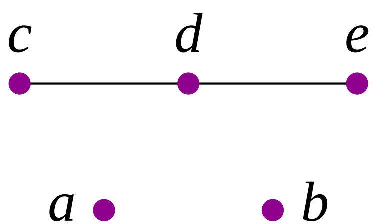
A generic example



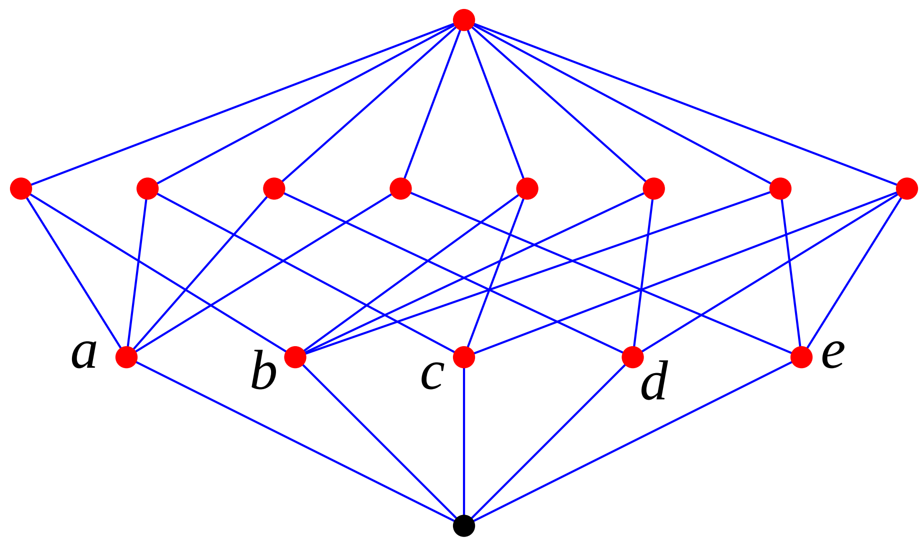
One hyperplane for every pair of facets (12 line shellings at v)

Lattice of flats

L: lattice of flats of a matroid, e.g., the intersection poset of a central hyperplane arrangement



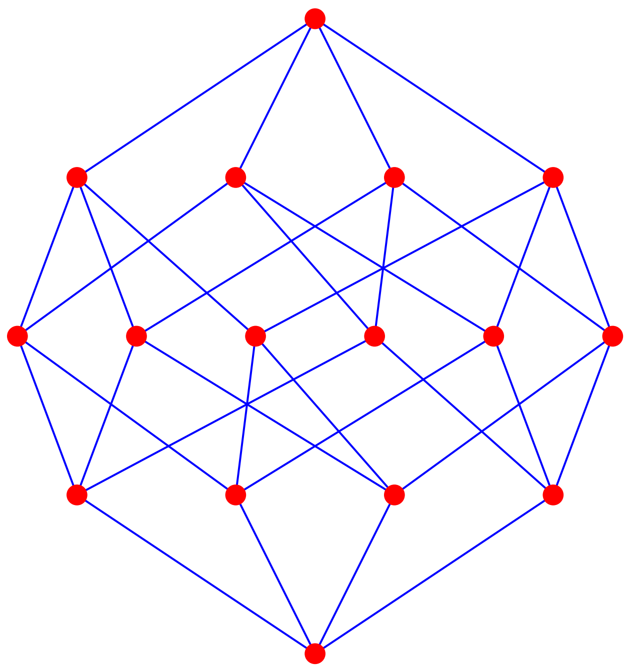
a matroid
(affine diagram)



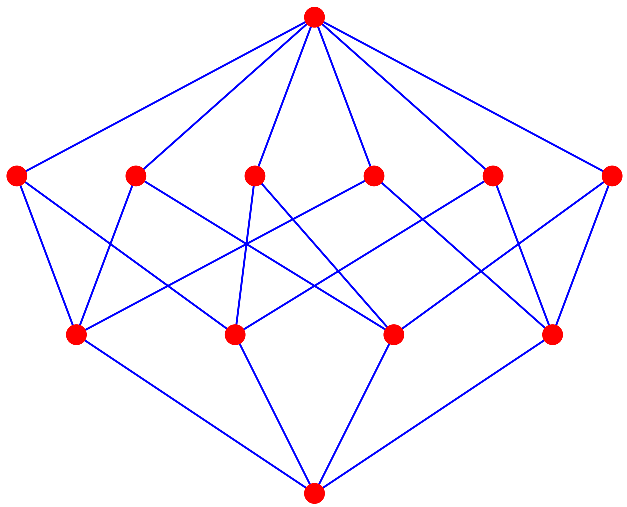
lattice of flats

Upper truncation

$T^k(L)$: L with top k levels (excluding the maximum element) removed, called the kth **truncation** of L.



lattice L of flats of four independent points



$T^1(L)$

Upper truncation (cont.)

$T^k(L)$ is still the lattice of flats of a matroid, i.e., a **geometric lattice** (easy).

Lower truncation

What if we remove the bottom k levels of L (excluding the minimal element)? Not a geometric lattice if rank is at least three.

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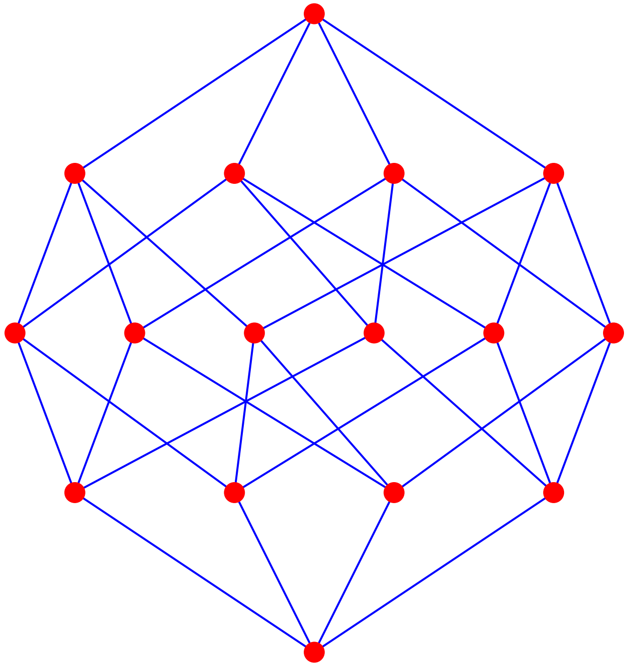
Want to “fill in” the kth lower truncation with as many new elements as possible without adding new elements of rank one, increasing the rank of L, or altering the partial order relation of L.

Lower truncation is “bad”



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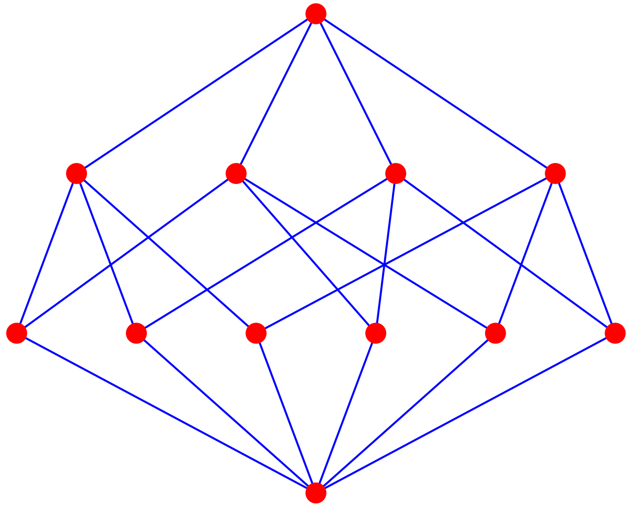
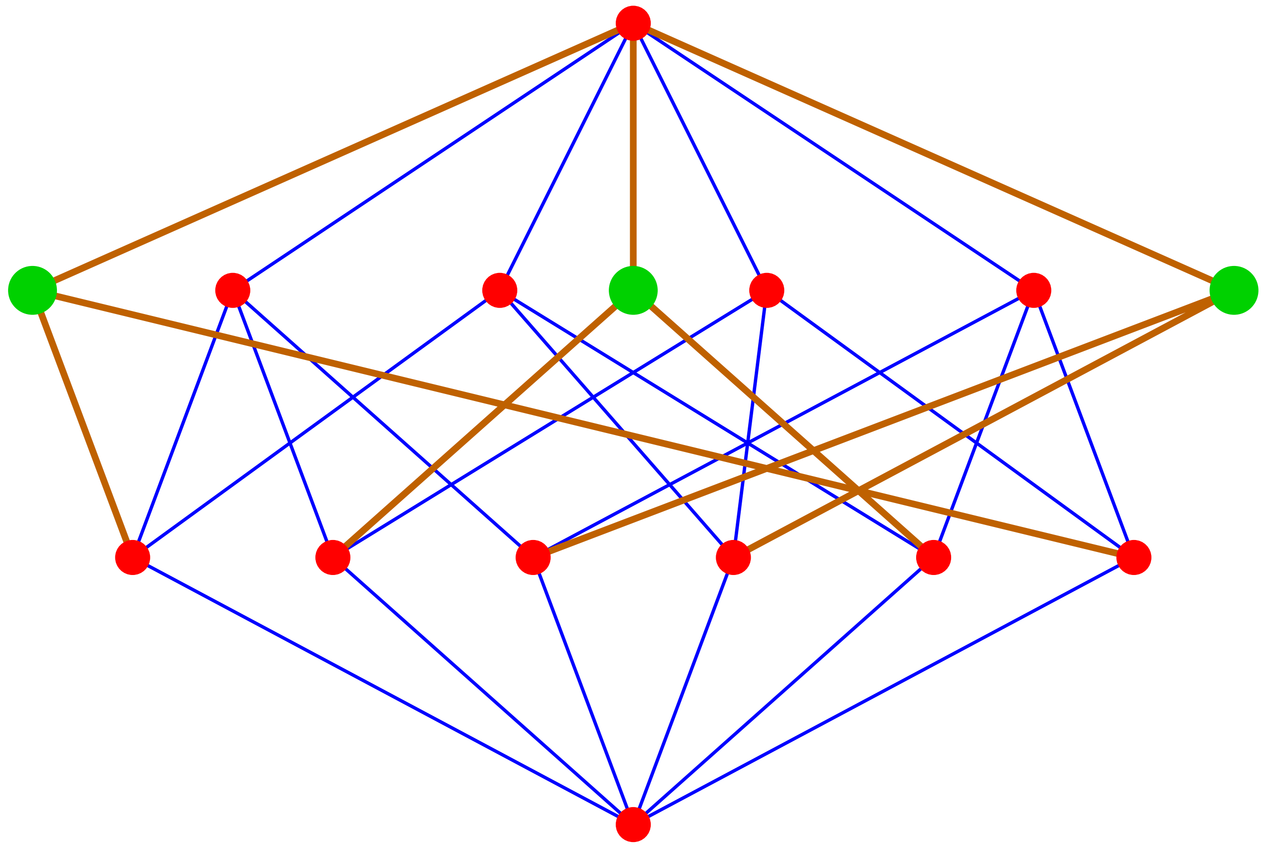


not a geometric lattice

An example of “filling in”



$D_1(B_4)$

The Dilworth truncation

Matroidal definition: Let M be a matroid on a set E of rank n, and let $1 \leq k < n$. The **kth Dilworth truncation** $D_k(M)$ has ground set $\binom{E}{k+1}$, and independent sets

$$\mathcal{I} = \left\{ I \subseteq \binom{E}{k+1} \;:\; \operatorname{rank}_M \left(\bigcup_{p \in I'} p \right) \geq \#I' + k, \right.$$

$$\left. \forall \emptyset \neq I' \subseteq I \right\}.$$

First Dilworth truncation of B_n

$L = \boldsymbol{B_n}$, the boolean algebra of rank n (lattice of flats of the matroid $\boldsymbol{F_n}$ of n independent points)

$D_1(B_n)$ is a geometric lattice whose atoms are the 2-element subsets of an n-set.

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$D_1(F_n)$ is the **braid arrangement** $x_i = x_j$, $1 \leq i < j \leq n$

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Number of bases of $D_1(B_n)$ equals n^{n-2}.

Second Dilworth completion of B_n

Matroid is on the set $\binom{[n]}{3}$

A set S of triangles is an independent set if for any $\emptyset \neq T \subseteq S$, the total number of vertices of triangles in T is at least $\#T + 2$.

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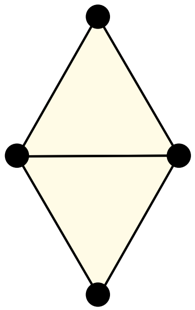
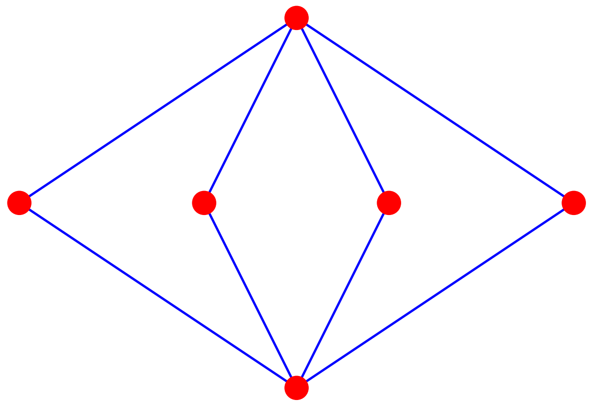
Note. If instead $\binom{[n]}{2}$ and total number of vertices of edges in T is at least $\#T + 1$, then we get a forest.

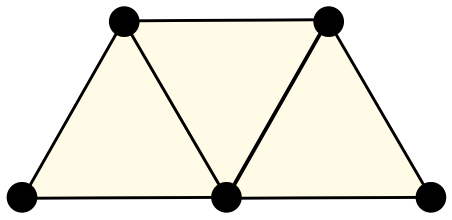
Bases of $D_2(B_4)$

$D_2(B_4)$: every pair of triangles is a basis (two triangles use four vertices)

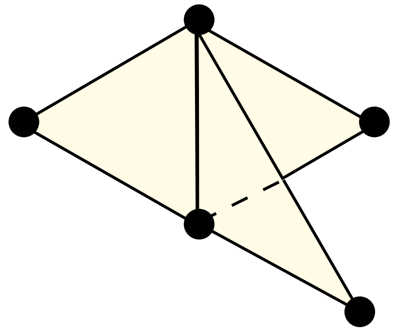




Bases of $D_2(B_5)$



60



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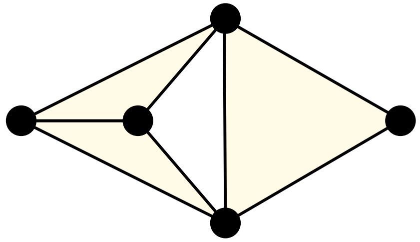

30

100 bases in all

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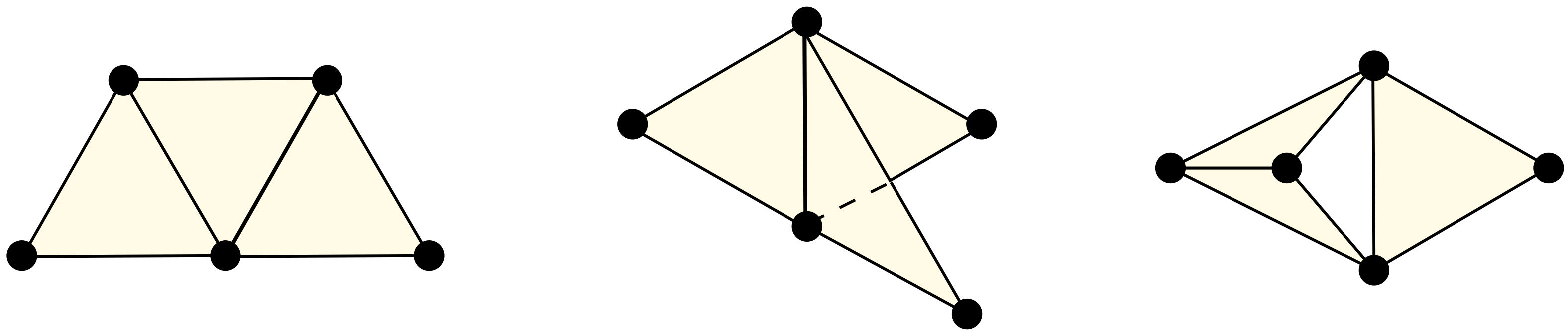
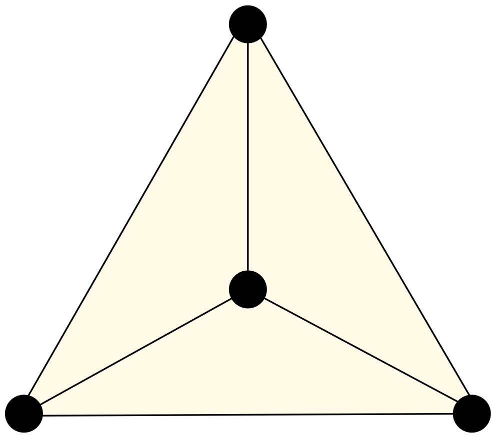


60 10 30

100 bases in all

bad



Some data

$$
\begin{aligned}
\chi_{D_2(B_5)}(q) &= q^2(q-1)(q^2-9q+21),\ r=62 \\
\chi_{D_2(B_6)}(q) &= q^2(q-1)(q^3-19q^2+126q-300),
\end{aligned}
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r = 892 = 2^2 \cdot 223
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$$b(7)=191436=2^2\cdot 3\cdot 7\cdot 43\cdot 53$$

Rank four

$\boldsymbol{L}$: geometric lattice of rank four

$\boldsymbol{\rho_2}$: number of elements of rank two

$\boldsymbol{L_3}$: set of elements of rank three

$\boldsymbol{c(t)}$: number or elements covering $t \in L$

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Theorem.

$$\chi_{D_1(L)}(q) = q^3 - \rho_2\, q^2 + \left[\binom{\rho_2}{2} - \sum_{t \in L_3} \binom{c(t)-1}{2}\right] q$$
$$+ \sum_{t \in L_3} \binom{c(t)-1}{2} - \binom{\rho_2 - 1}{2}$$

Back to $\mathrm{vis}(\mathcal{P})$ and $\mathrm{ls}(\mathcal{P}, v)$

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Proof omitted here, but is straightforward.

The n-cube

Let $\mathcal{P}$ be an n-cube. Can one describe in a reasonable way $L_{\mathrm{ls}(\mathcal{P},v)}$ and/or $\chi_{\mathrm{ls}(\mathcal{P},v)}(q)$?

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Let $\mathcal{P}$ have vertices $(a_1, \ldots, a_n)$, $a_i = 0, 1$. If $v = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, then $\mathrm{ls}(\mathcal{P}, v)$ is isomorphic to the Coxeter arrangement of type B_n, with

$$\begin{aligned} \chi_{\mathrm{ls}(\mathcal{P},v)}(q) &= (q-1)(q-3)\cdots(q-(2n-1)) \\ r(\mathrm{ls}(\mathcal{P},v)) &= 2^n\, n!. \end{aligned}$$

The 3-cube

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Let v be generic. Then

$$\chi(q) = q(q-1)(q^2 - 14q + 53), \quad r = 136 = 2^3 \cdot 17.$$

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Total number of line shellings of the 3-cube is 288. Total number of shellings is 480.

Three asides

1. Let $f(n)$ be the total number of shellings of the n-cube. Then

$$\sum_{n\geq 1} f(n)\frac{x^n}{n!} = 1 - \frac{1}{\sum_{n\geq 0}(2n)!\frac{x^n}{n!}}.$$

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2. Total number of line shellings of the n-cube is $2^n n!^2$.

3. **Every** shelling of the n-cube C_n can be realized as a line shelling of a polytope combinatorially equivalent to C_n (**M. Develin**).

Two consequences

- The number of line shellings from a generic $v \in \mathrm{int}(\mathcal{P})$ depends only on which sets of facet normals of $\mathcal{P}$ are linearly independent, i.e., matroid structure of $\mathrm{vis}(\mathcal{P})$.

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- The number of line shellings from a generic $v \in \mathrm{int}(\mathcal{P})$ depends only on which sets of facet normals of $\mathcal{P}$ are linearly independent, i.e., matroid structure of $\mathrm{vis}(\mathcal{P})$.

Recall **Minkowski's theorem**: There exists a convex d-polytope with outward facet normals $v_1, \ldots, v_m$ and corresponding facet $(d-1)$-dimensional volumes $c_1, \ldots, c_m$ if and only if the v_i's span a d-dimensional space and

$$\sum c_i v_i = 0.$$

Second consequence

- $\mathcal{P}$: d-polytope with m facets, $v \in \mathrm{int}(\mathcal{P})$

  $c(n,k)$: signless Stirling number of first kind (number of $w \in \mathfrak{S}_n$ with k cycles)

  Then

$$\mathrm{ls}(\mathcal{P}, v) \leq 2(c(m, m-d+1) + c(m, m-d+3) + c(m, m-d+5) + \cdots)$$

  (best possible).

Many further directions

Valid hyperplane orderings. We can extend the result

$$L_{\mathrm{ls}(\mathcal{P},v)} \cong D_1\big(L_{\mathrm{vis}(\mathcal{P})}\big)$$

to **any** (hyperplane) arrangement.

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$\mathcal{A}$: any (finite) arrangement in $\mathbb{R}^n$

v: any point not on any $H \in \mathcal{A}$

L: sufficiently generic directed line through v

Valid orderings

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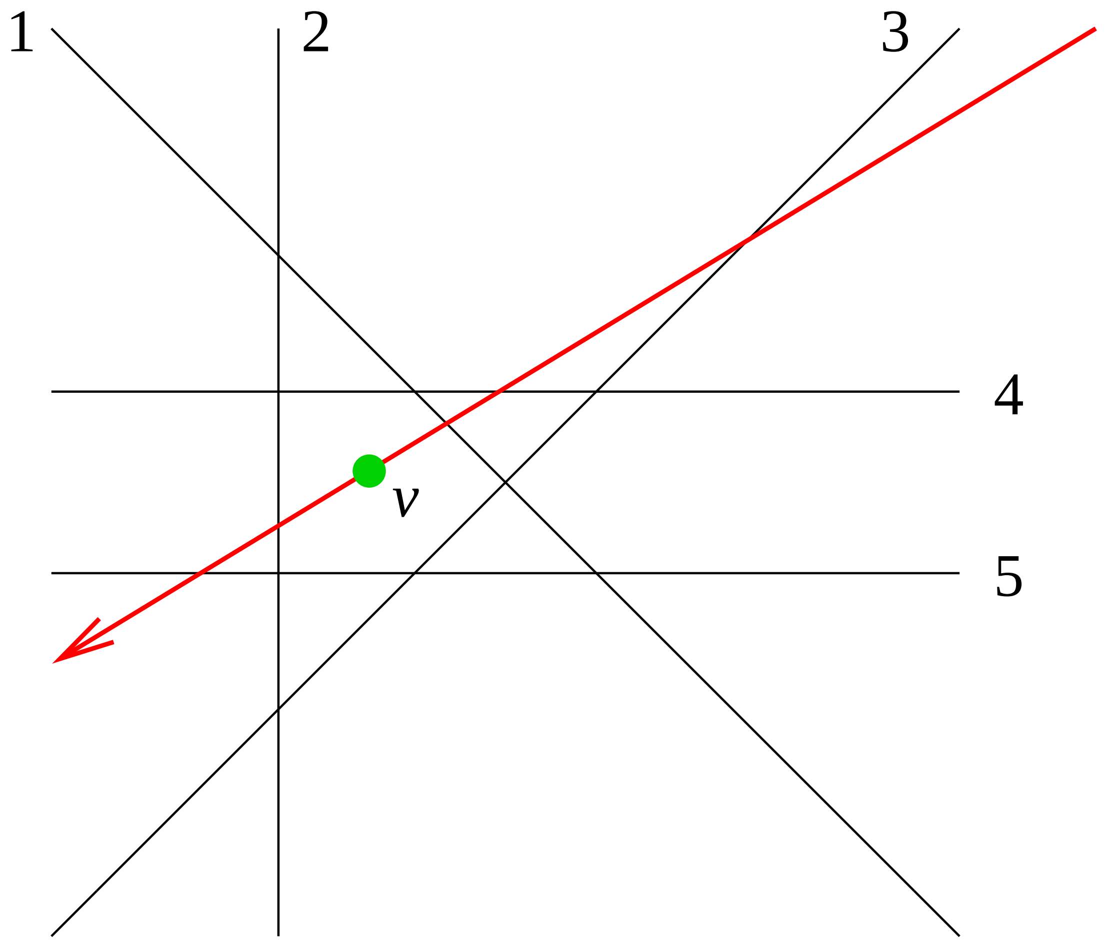
Call this a **valid ordering** of $(\mathcal{A}, v)$.

An example



valid ordering: 3, 4, 1, 2, 5

The valid ordering arrangment

VO$(\mathcal{A}, v)$: hyperplanes through v and every intersection of two hyperplanes in $\mathcal{A}$, together with all hyperplanes through v parallel to (at least) two hyperplanes of $\mathcal{A}$

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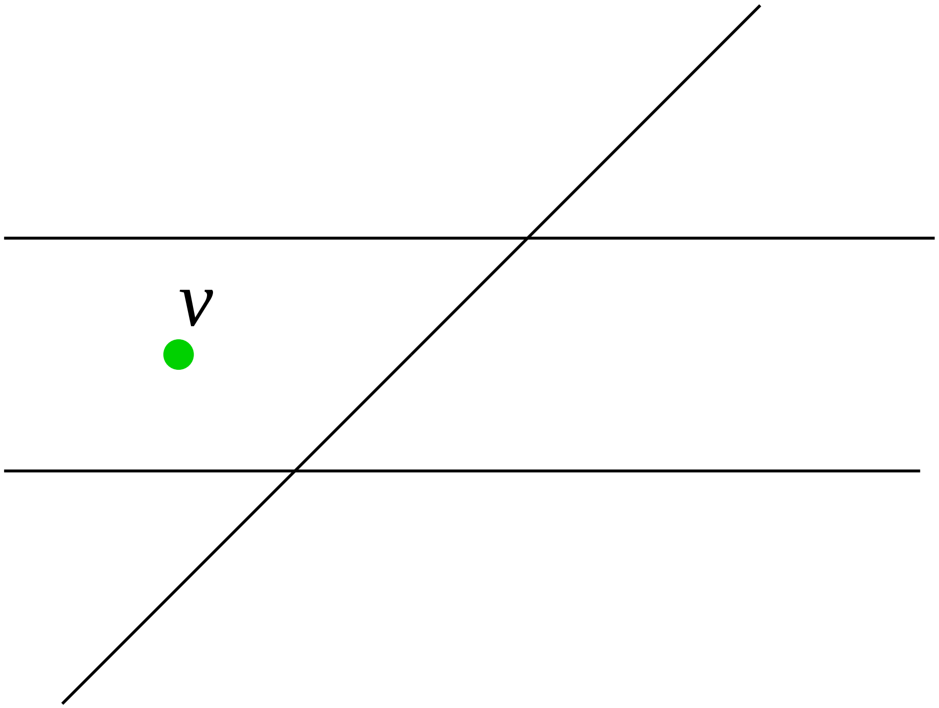

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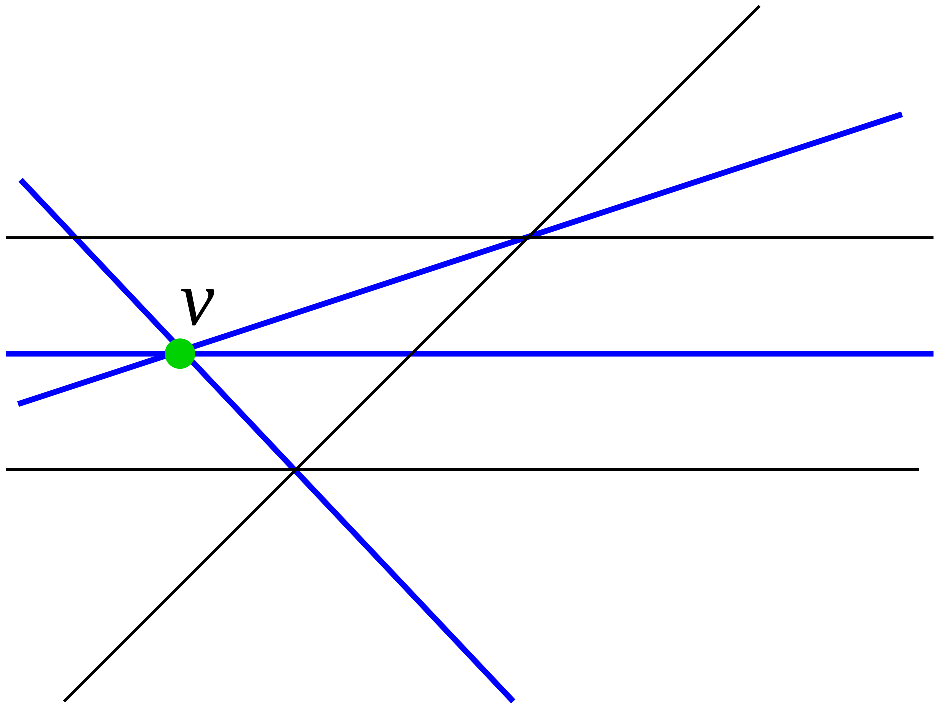

The Dilworth truncation of $\mathcal{A}$

The regions of $\mathrm{vo}(\mathcal{A}, v)$ correspond to valid orderings of hyperplanes by lines through v (easy).

Theorem. *Let v be generic. Then*

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The Dilworth truncation of $\mathcal{A}$

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Note that right-hand side is independent of v.

m-planes

Rather than a line through v, pick an m-plane P through m generic points $v_1, \ldots, v_m$. For "sufficiently generic" P, get a "full-sized" induced arrangement

$$\boldsymbol{\mathcal{A}_P} = \{H \cap P \,:\, H \in \mathcal{A}\}$$

in P.

Define $\mathbf{vo}(\boldsymbol{\mathcal{A}}; \boldsymbol{v_1, \ldots, v_m})$ to consist of all hyperplanes passing through $v_1, \ldots, v_m$ and every intersection of $m+1$ hyperplanes of $\mathcal{A}$ (including "intersections at ∞").

mth Dilworth truncation

Theorem. *If $v_1, \ldots, v_m$ are generic, then*

$$\mathrm{vo}(\mathcal{A}(v_1, \ldots, v_m)) \cong D_m(\mathcal{A}).$$

mth Dilworth truncation

Theorem. *If* $v_1, \ldots, v_m$ *are generic, then*

$$\mathrm{vo}(\mathcal{A}(v_1, \ldots, v_m)) \cong D_m(\mathcal{A}).$$

Proof is straightforward.

Non-generic base points

For simplicity, consider only the original case $m = 1$. Recall:

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What if v is not generic?

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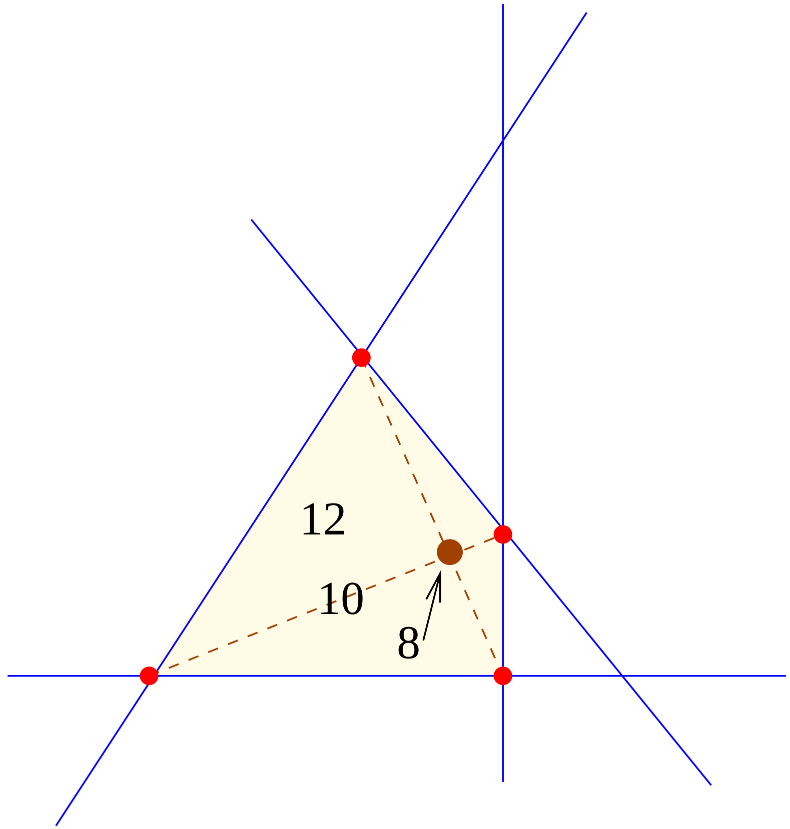
What if v is not generic?

Then we get “smaller” arrangements than the generic case.

We obtain a polyhedral subdivision of $\mathbb{R}^n$ depending on which arrangement corresponds to v.

An example



Numbers are number of line shellings from points in the interior of the face.

Order polytopes redux

Recall:

$$\mathcal{O}(P) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \,:\, x_i \leq x_j \text{ if } t_i \leq t_j\}$$

We will relate $\chi_{\mathrm{vis}(\mathcal{O}(P))}(q)$ to "generalized chromatic polynomials."

Generalized chromatic polynomials

$\boldsymbol{G}$: finite graph with vertex set $\boldsymbol{V}$

$\mathbb{P} = \{1, 2, 3, \dots\}$

$\boldsymbol{\sigma} : V \to 2^{\mathbb{P}}$ such that $\sigma(v) < \infty, \ \forall v \in V$

$\boldsymbol{\chi_{G,\sigma}(q)}$, $q \in \mathbb{P}$: number of proper colorings $f : V \to \{1, 2, \dots, q\}$ such that

$$f(v) \notin \sigma(v), \ \forall v \in V$$

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Each f is a **list coloring**, but the definition of $\chi_{G,\sigma}(q)$ seems to be new.

The arrangement $\mathcal{A}_{G,\sigma}$

$\boldsymbol{d} = \#V = \#\{v_1, \ldots, v_d\}$

$\mathcal{A}_{G,\sigma}$: the arrangement in $\mathbb{R}^d$ given by

$$\begin{aligned} x_i &= x_j, \text{ if } v_i v_j \text{ is an edge} \\ x_i &= \alpha_j, \text{if } \alpha_j \in \sigma(i) \end{aligned}$$

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Theorem (easy). $\chi_{\mathcal{A}_{G,\sigma}} = \chi_{G,\sigma}(q)$ for $q \gg 0$

Consequences

Since $\chi_{G,\sigma}(q)$ is the characteristic polynomial of a hyperplane arrangement, it has such properties as a **deletion-contraction recurrence**, **broken circuit theorem**, Tutte polynomial, etc.

$\operatorname{vis}(\mathcal{O}(P))$ and $\mathcal{A}_{H,\sigma}$

Theorem (easy). *Let H be the Hasse diagram of P, considered as a graph. Define* $\boldsymbol{\sigma} : H \to \mathbb{P}$ *by*

$$\sigma(v) = \begin{cases} \{1,2\}, & v = \text{ isolated point} \\ \{1\}, & v \text{ minimal, not maximal} \\ \{2\}, & v \text{ maximal, not minimal} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\operatorname{vis}(\mathcal{O}(P)) = \mathcal{A}_{H,\sigma}$.

Supersolvable and free

Recall that the following three properties are equivalent for the usual graphic arrangement $\mathcal{A}_G$.

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- $\mathcal{A}_G$ is **supersolvable** (not defined here).
- $\mathcal{A}_G$ is **free** in the sense of Terao (not defined here).
- G is a **chordal** graph, i.e., can order vertices $v_1, \ldots, v_d$ so that v_{i+1} connects to previous vertices along a clique. (Numerous other characterizations.)

Generalize to (G, σ)

Theorem (easy). *Suppose that we can order the vertices of G as $v_1, \ldots, v_p$ such that:*

- v_{i+1} *connects to previous vertices along a clique (so G is chordal).*
- *If $i < j$ and v_i is adjacent to v_j, then $\sigma(v_j) \subseteq \sigma(v_i)$.*

Then $\mathcal{A}_{G,\sigma}$ is supersolvable.

Open questions

- Is this sufficient condition for supersolvability also necessary?
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- Is it necessary for freeness? (In general, supersolvable $\Rightarrow$ free.)
- Are there characterizations of supersolvable arrangements $\mathcal{A}_{G,\sigma}$ analogous to the known characterizations of supersolvable $\mathcal{A}_G$?

The last slide

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