The Valid Order Arrangement of a Real Hyperplane Arrangement

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M.I.T.

Valid orderings

(real) arrangement: a set of hyperplanes in $\mathbb{R}^n$

$\mathcal{A}$: a **finite** arrangement in $\mathbb{R}^n$

p: any point not on any $H \in \mathcal{A}$

L: sufficiently generic directed line through p

Valid orderings

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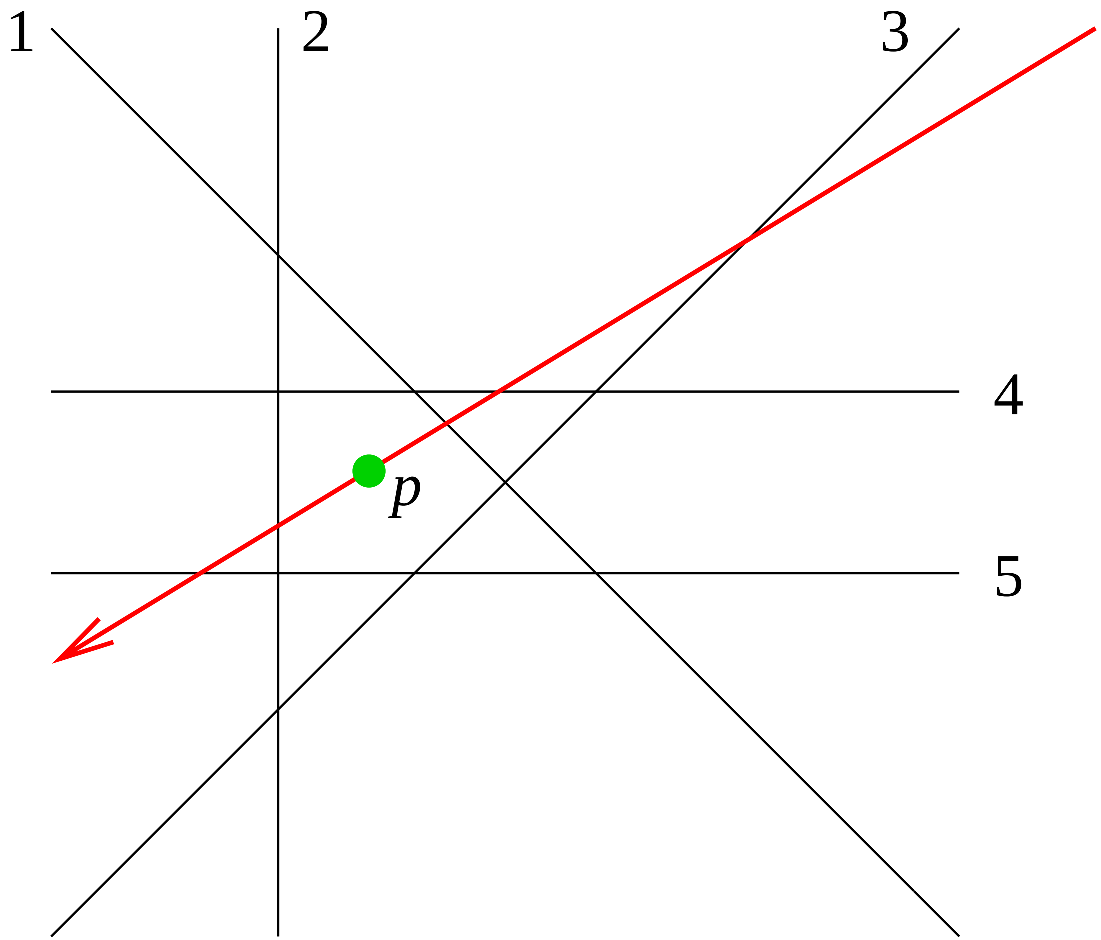
Call this a **valid ordering** of $(\mathcal{A}, p)$.

An example



valid ordering: 3, 4, 1, 2, 5

The valid order arrangment

vo$(\mathcal{A}, p)$: hyperplanes are

- affine span of p with $H_1 \cap H_2 \neq \emptyset$, where H_1, H_2 are distinct hyperplanes
- if $H_1 \cap H_2 = \emptyset$, then the hyperplane through p parallel to H_1, H_2

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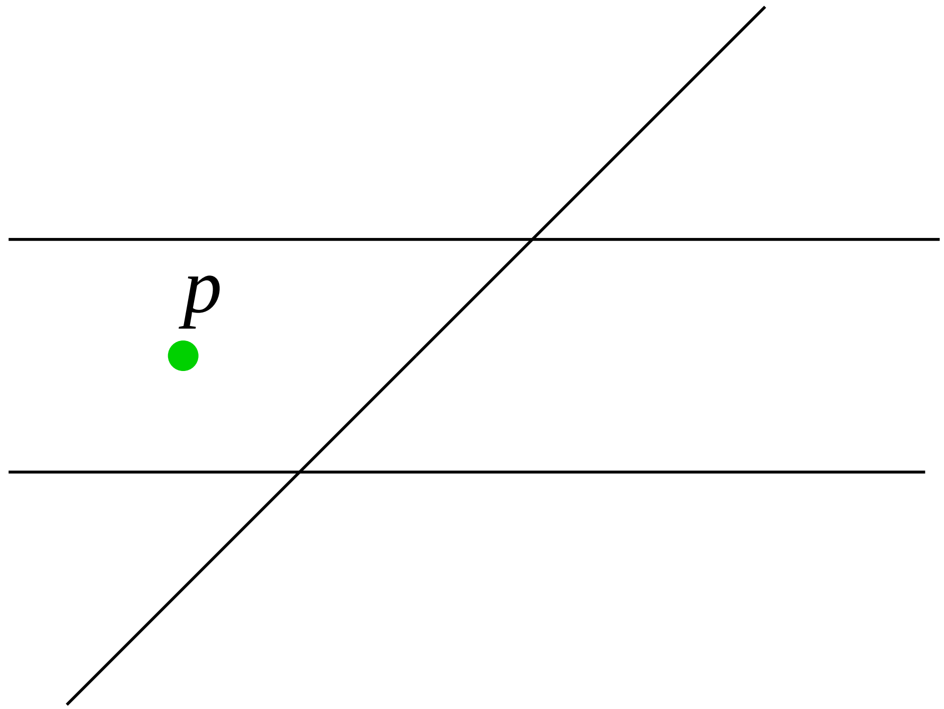

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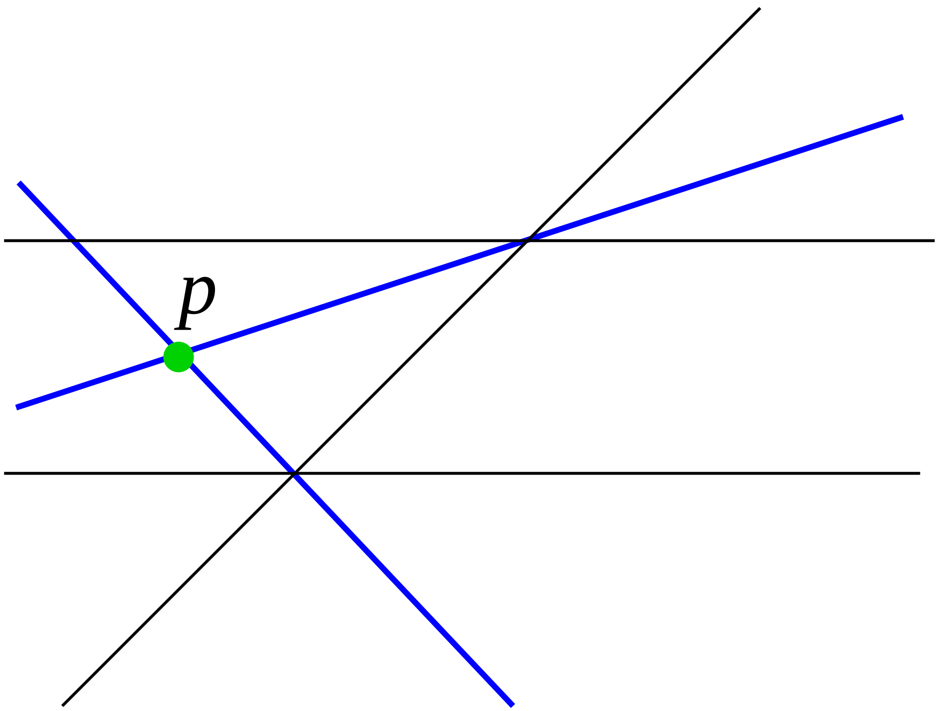

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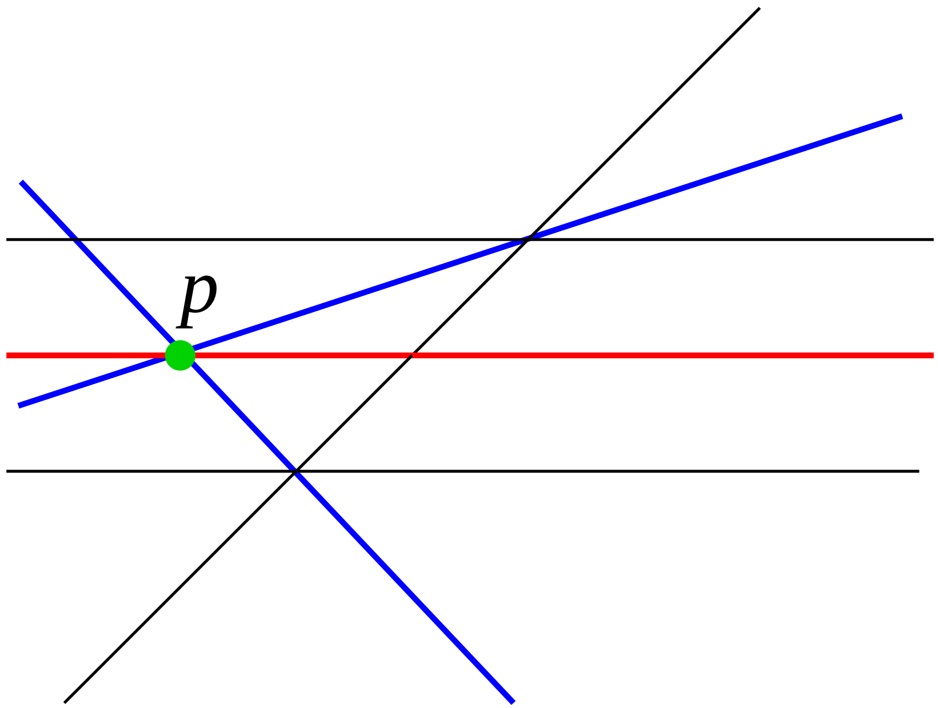

Regions of $\mathrm{vo}(\mathcal{A})$

region of $\mathcal{A}$: connected component of $\mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H$.

The regions of $\mathrm{vo}(\mathcal{A}, p)$ correspond to valid orderings of hyperplanes by lines through p (easy).

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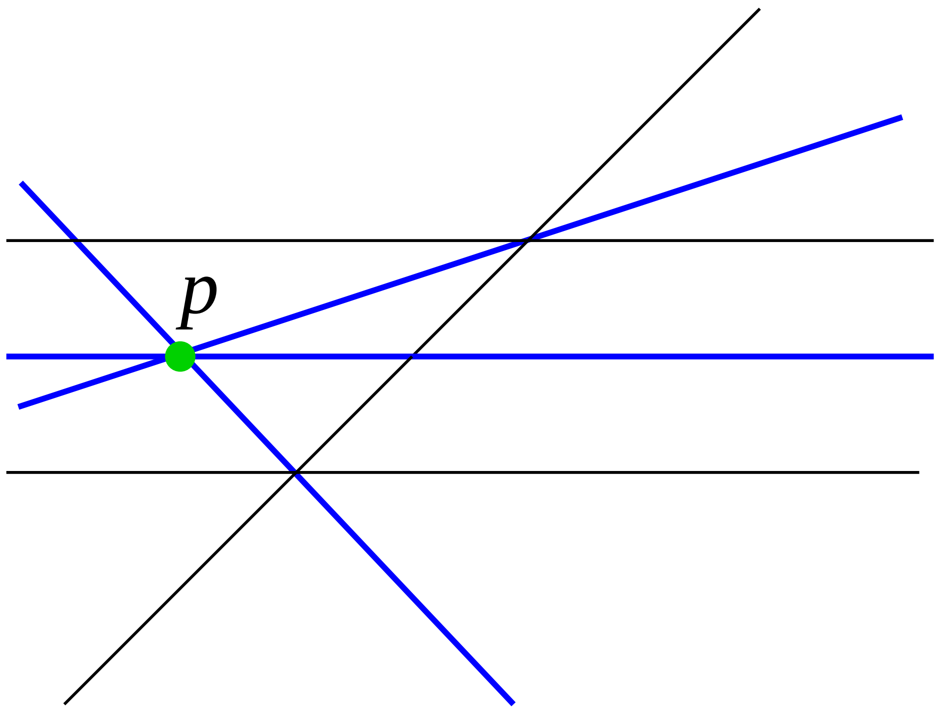

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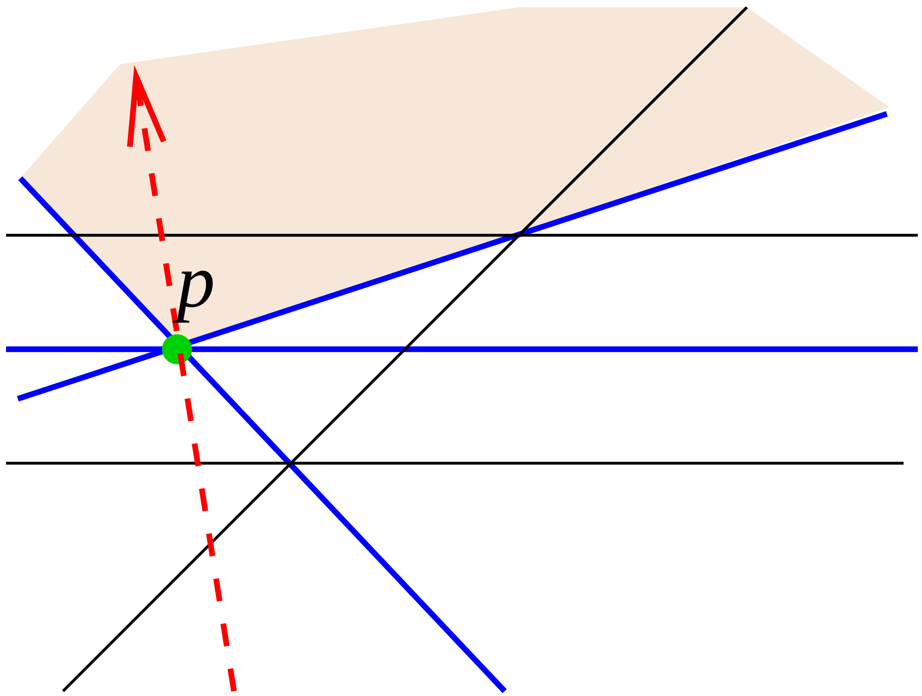

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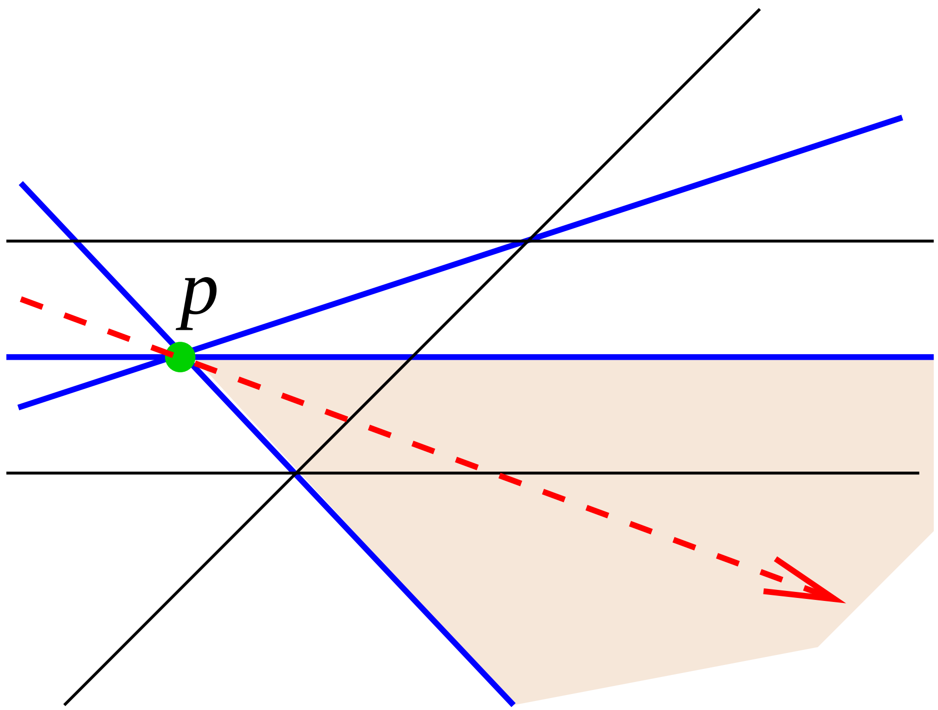

A special case

$\mathcal{P}$: a d-dimensional convex polytope in $\mathbb{R}^d$

Certain facets of $\mathcal{P}$ are visible from points $p \in \mathbb{R}^d$

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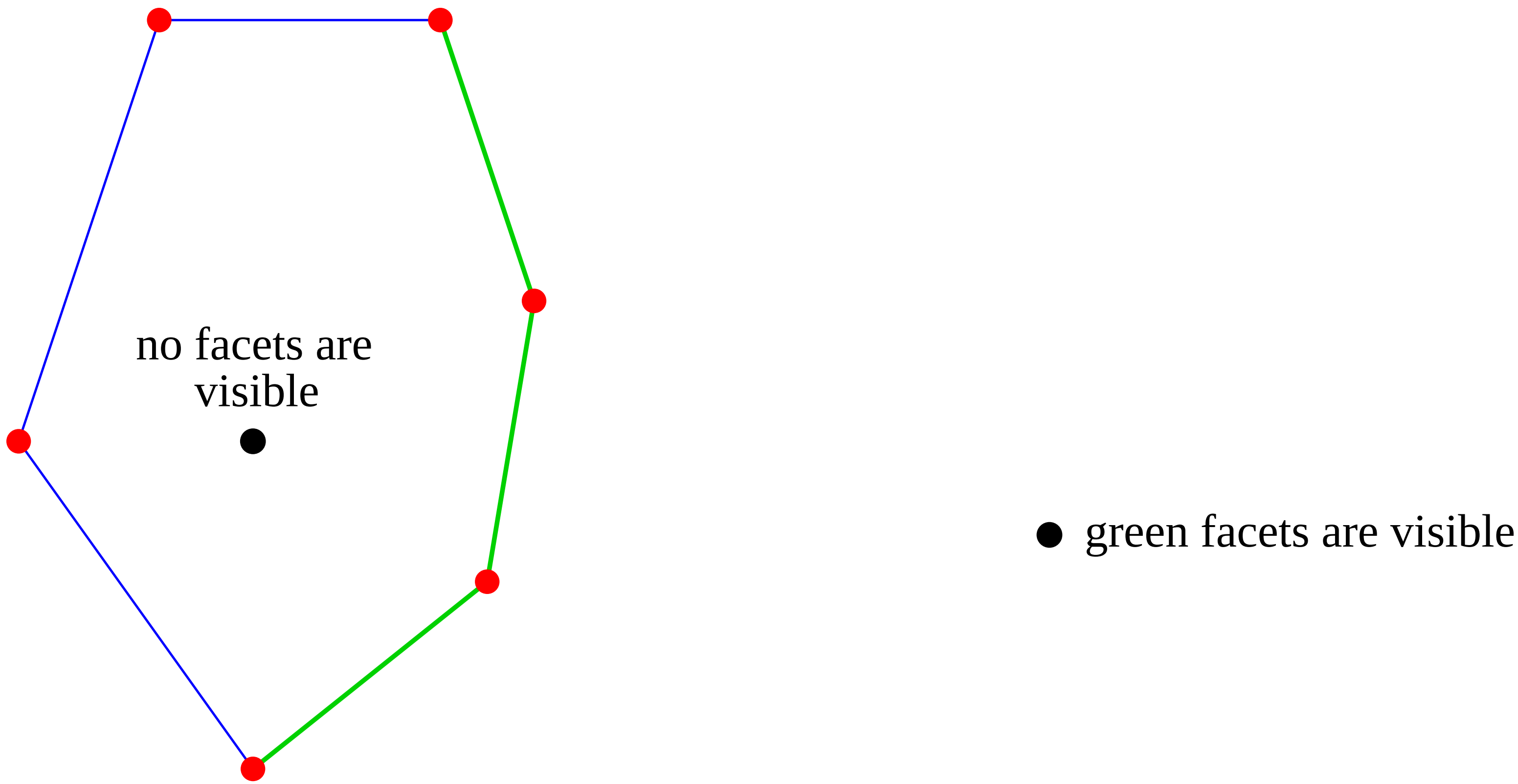

The visibility arrangement

aff(S): the affine span of a subset $S \subset \mathbb{R}^d$

visibility arrangement:

$$\mathbf{vis}(\mathcal{P}) = \{\mathrm{aff}(F)\ :\ F \text{ is a facet of } \mathcal{P}\}$$

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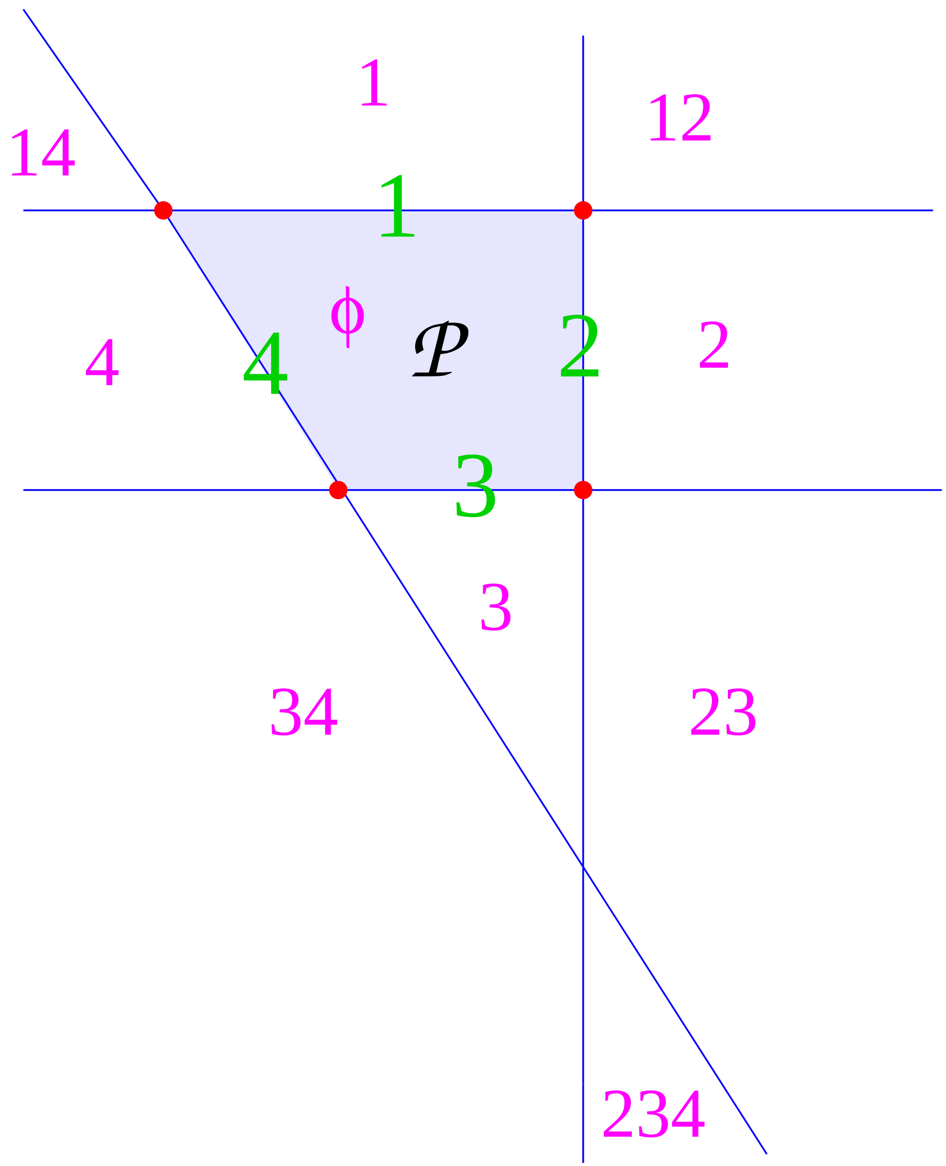
Regions of $\mathrm{vis}(\mathcal{P})$ correspond to sets of facets that are visible from some point $p \in \mathbb{R}^d$.

An example



Number of regions

$\boldsymbol{v}(\mathcal{P})$: number of regions of $\mathrm{vis}(\mathcal{P})$, i.e., the number of visibility sets of $\mathcal{P}$

$\boldsymbol{\chi_{\mathcal{A}}}(q)$: characteristic polynomial of the arrangement $\mathcal{A}$

Zaslavsky's theorem. *Number of regions of $\mathcal{A}$ is*

$$\boldsymbol{r(\mathcal{A})} = (-1)^d \chi_{\mathcal{A}}(-1).$$

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In general, $r(\mathcal{A})$ and $\chi_{\mathcal{A}}(q)$ are hard to compute.

A simple example

$\mathcal{P}_n = n$-cube

$$\chi_{\mathrm{vis}(\mathcal{P}_n)}(q) = (q-2)^n$$

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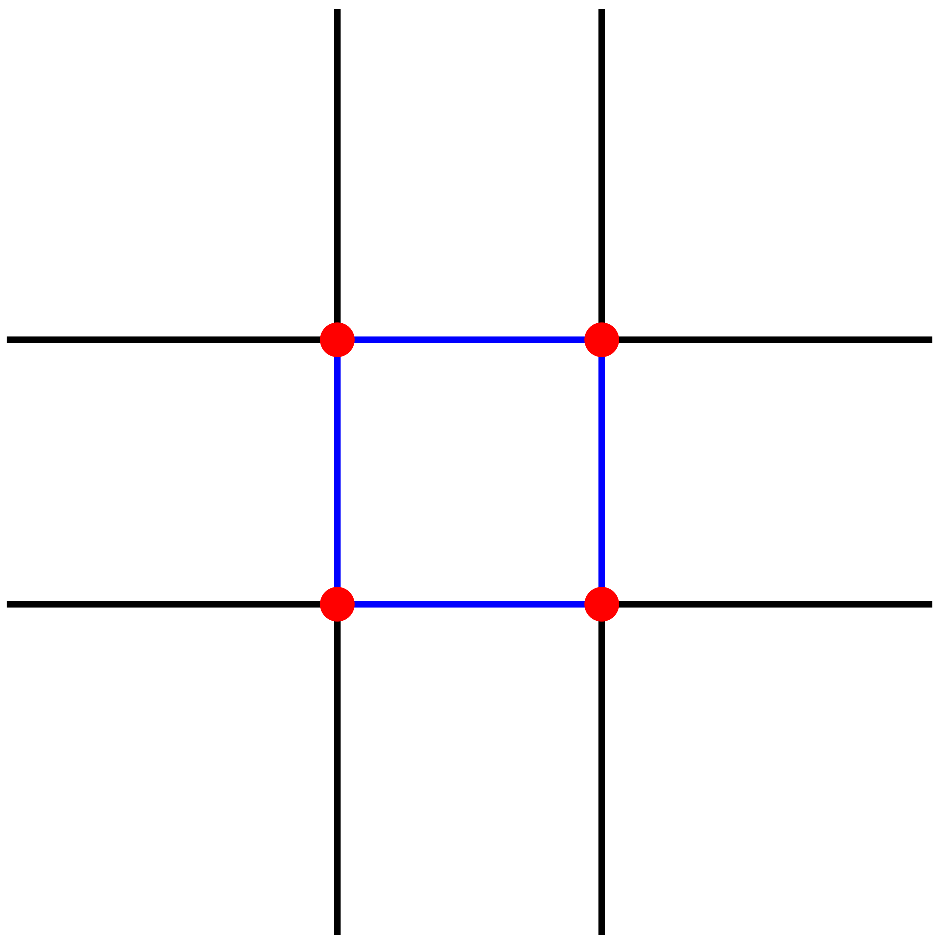
For any facet F, can see either F, $-F$, or neither.

The 2-cube

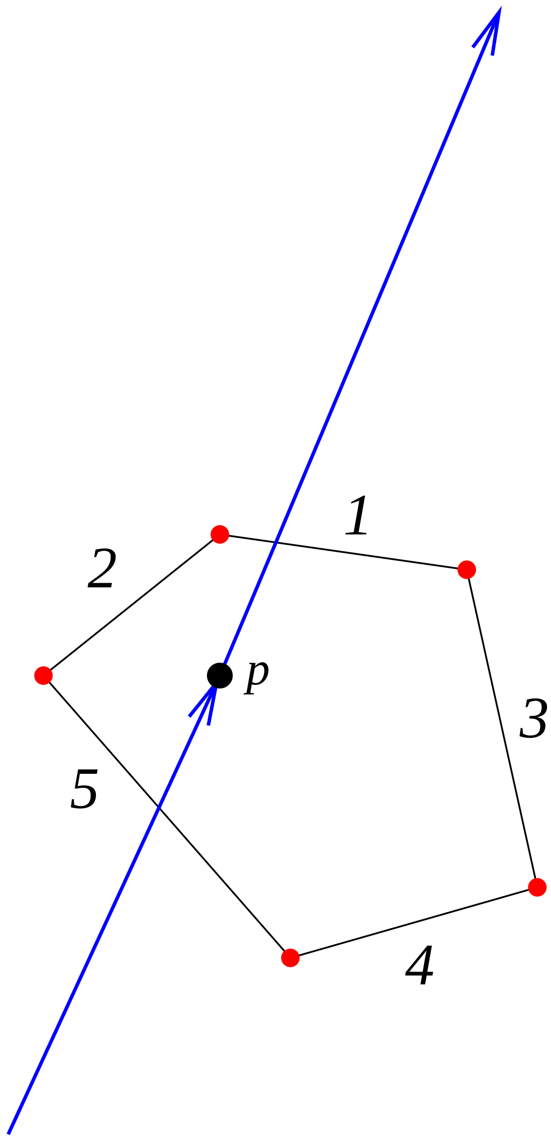


Line shellings

Let $p \in \textbf{int}(\mathcal{P})$ (interior of $\mathcal{P}$)

Line shelling based at p: let L be a directed line from p. Let $F_1, F_2, \ldots, F_k$ be the order in which facets become visible along L, followed by the order in which they become invisible from ∞ along the other half of L. Assume L is sufficiently generic so that no two facets become visible or invisible at the same time.

Example of a line shelling



The line shelling arrangment

$\mathbf{ls}(\boldsymbol{\mathcal{P}}, \boldsymbol{p})$: hyperplanes are

- affine span of p with aff$(F_1) \cap \mathrm{aff}(F_2) \neq \emptyset$, where F_1, F_2 are distinct facets
- if aff$(F_1) \cap \mathrm{aff}(F_2) = \emptyset$, then the hyperplane through p parallel to F_1, F_2

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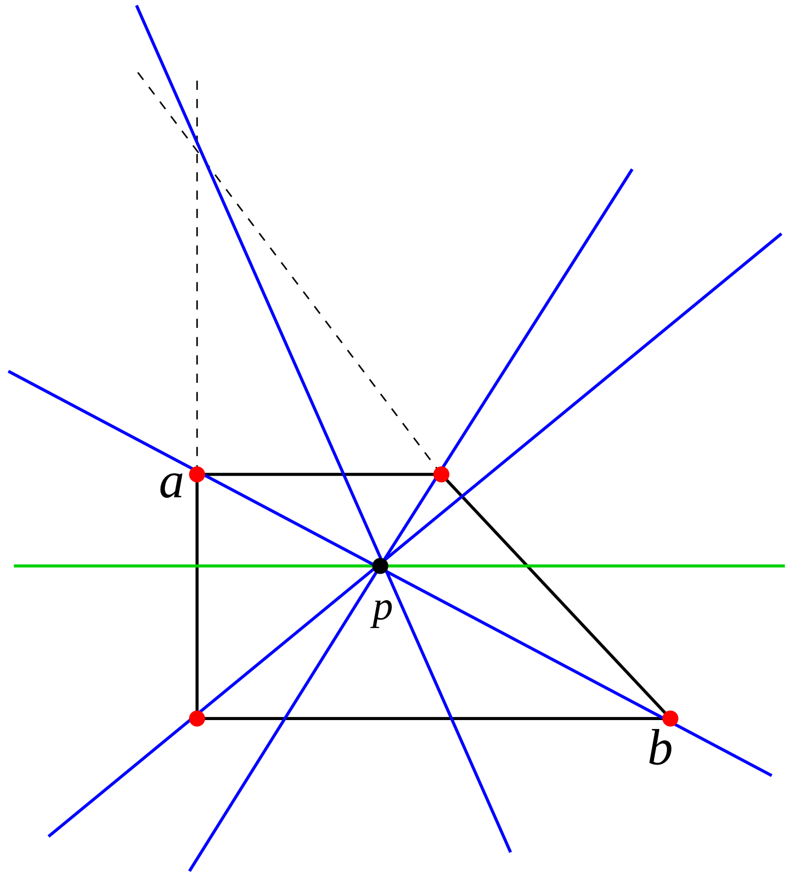
Line shellings at p are in bijection with regions of $\mathrm{ls}(\mathcal{P}, p)$.

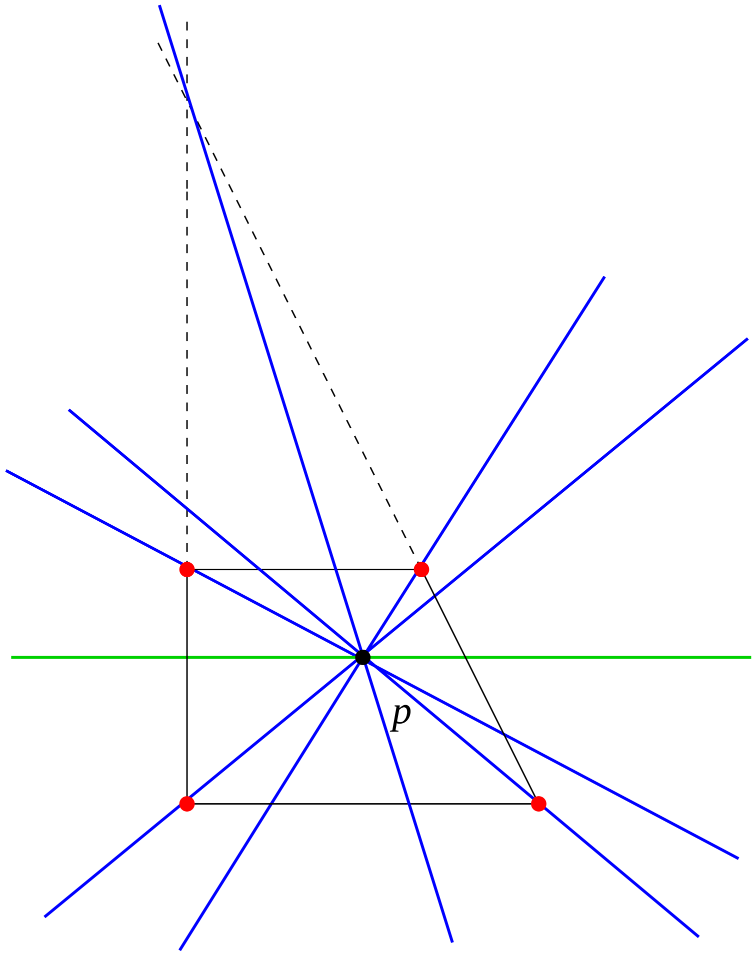
NOTE. $\boxed{\mathrm{ls}(\mathcal{P}, p) = \mathrm{vo}(\mathrm{vis}(\mathcal{P}), p)}$

A nongeneric example



p is not generic: $\overline{ap} = \overline{bp}$ (10 line shellings at p)

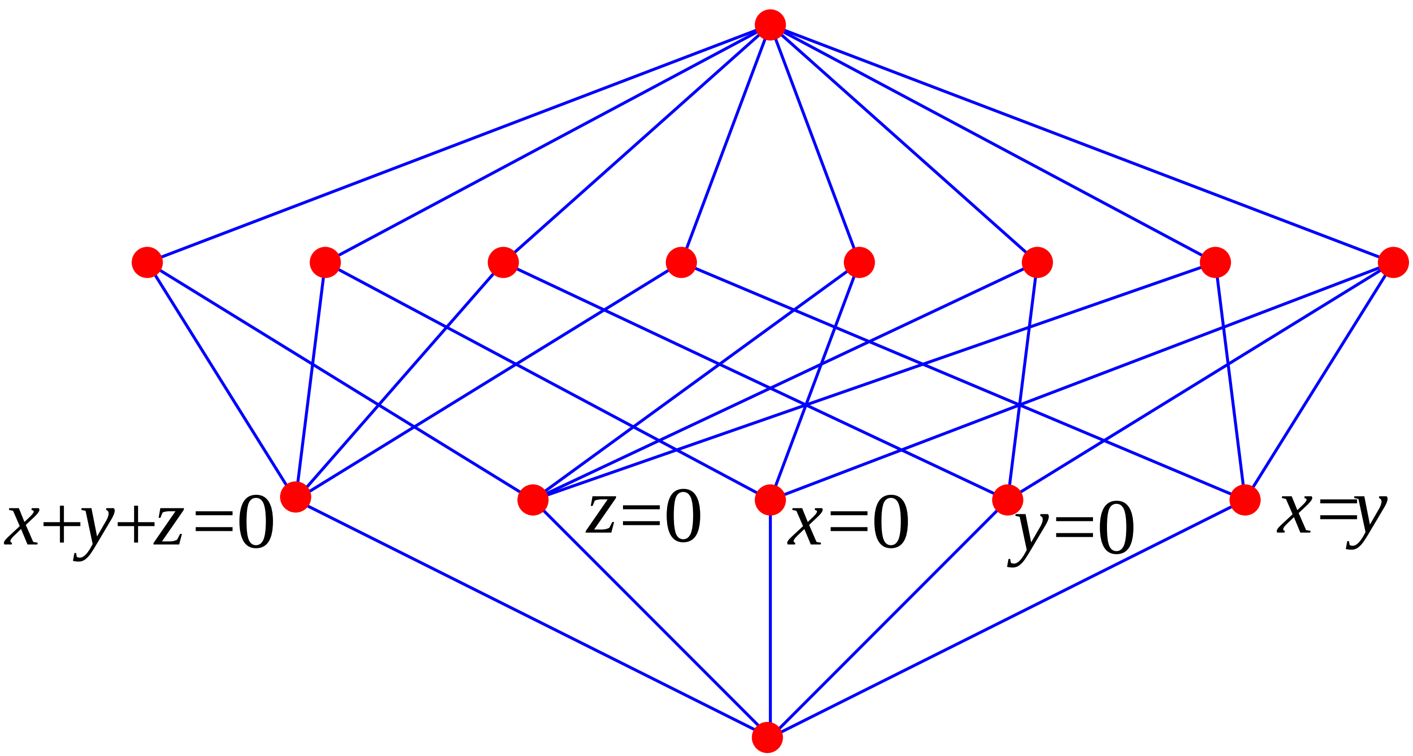
A generic example



One hyperplane for every pair of facets (12 line shellings at v)

Lattice of flats

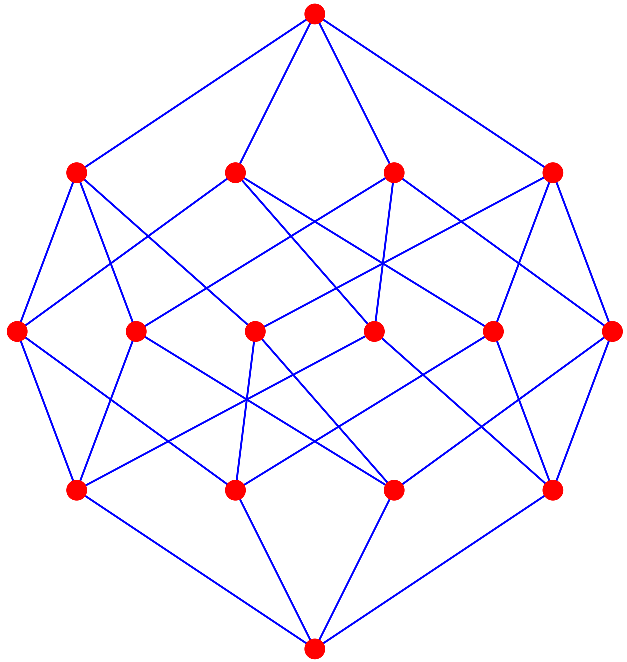
L: lattice of flats of a matroid, e.g., the intersection poset of a central hyperplane arrangement



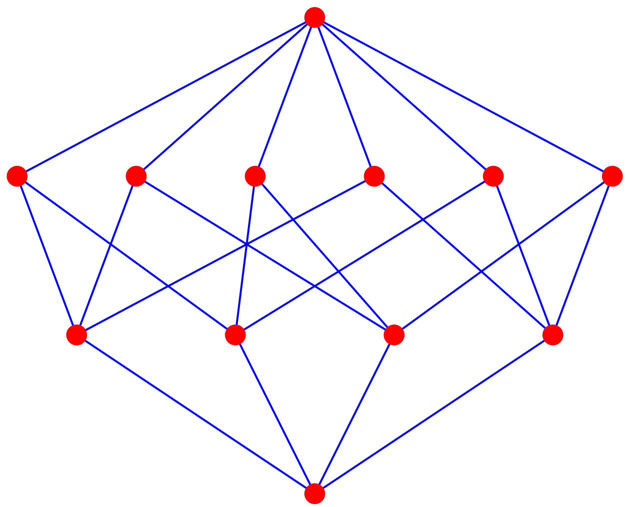
lattice of flats

Upper truncation

$T^k(L)$: L with top k levels (excluding the maximum element) removed, called the kth **truncation** of L.



lattice L of flats of four independent points



$T^1(L)$

Upper truncation (cont.)

$T^k(L)$ is still the lattice of flats of a matroid, i.e., a **geometric lattice** (easy).

Lower truncation

What if we remove the bottom k levels of L (excluding the minimal element)? Not a geometric lattice if rank is at least three.

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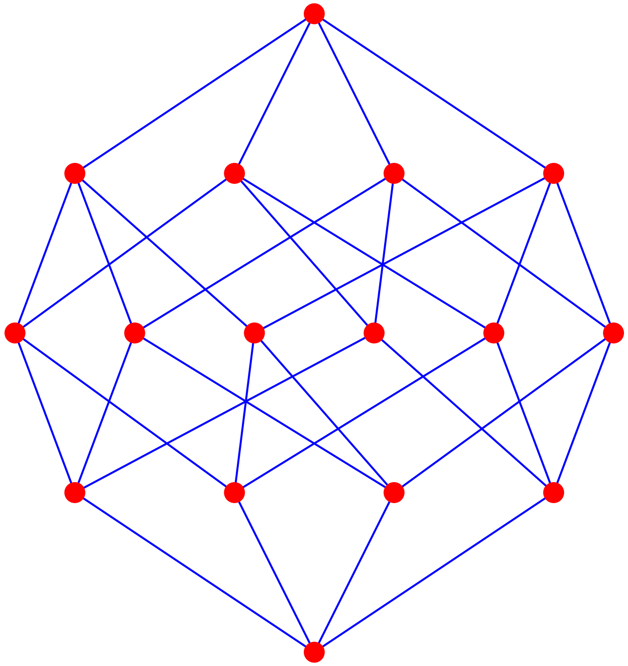
Want to "fill in" the kth lower truncation with as many new elements as possible without adding new elements of rank one, increasing the rank of L, or altering the partial order relation of L.

Lower truncation is “bad”



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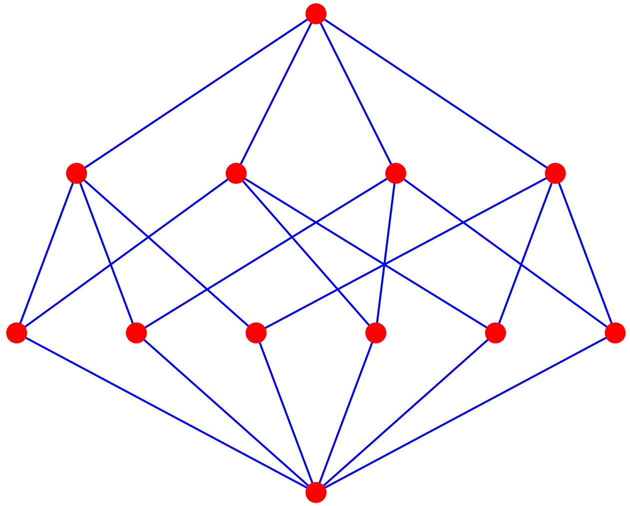
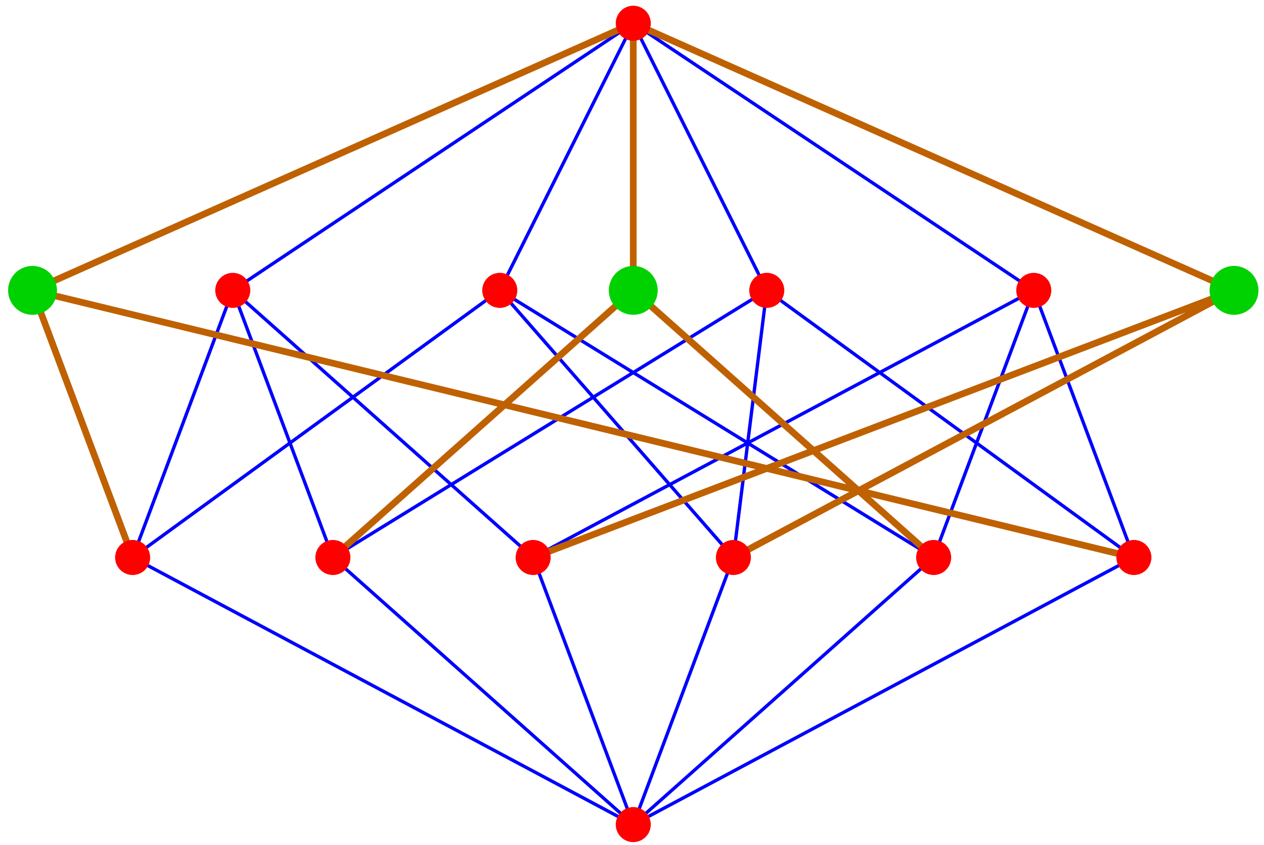


not a geometric lattice

An example of "filling in"



$D_1(B_4)$

The Dilworth truncation

Matroidal definition: Let M be a matroid on a set E of rank n, and let $1 \leq k < n$. The **kth Dilworth truncation** $D_k(M)$ has ground set $\binom{E}{k+1}$, and independent sets

$$\mathcal{I} = \left\{ I \subseteq \binom{E}{k+1} \;:\; \operatorname{rank}_M \left(\bigcup_{p \in I'} p \right) \geq \#I' + k, \right.$$

$$\left. \forall \emptyset \neq I' \subseteq I \right\}.$$

Geometric lattices

$D_k(M)$ “transfers” to $\boldsymbol{D_k(L)}$, where L is a geometric lattice.

rank$(L) = n \Rightarrow D_k(L)$ is a geometric lattice of rank $n-k$ whose atoms are the elements of L of rank $k+1$.

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Details not explained here.

First Dilworth truncation of B_n

$L = \boldsymbol{B_n}$, the boolean algebra of rank n (lattice of flats of the matroid $\boldsymbol{F_n}$ of n independent points)

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$D_1(B_n) = \boldsymbol{\Pi_n}$ (lattice of partitions of an n-set)

$D_1(F_n)$ is the **braid arrangement** $x_i = x_j$, $1 \leq i < j \leq n$

Back to valid orderings

$\mathcal{A}$: an arrangement in $\mathbb{R}^n$ with hyperplanes

$$\boldsymbol{v_i \cdot x = \alpha_i}, \quad 0 \neq v_i \in \mathbb{R}^n, \quad \alpha_i \in \mathbb{R}, \quad 1 \leq i \leq m.$$

semicone $\mathbf{sc}(\mathcal{A})$ of $\mathcal{A}$: arrangement in $\mathbb{R}^{n+1}$ (with new coordinate $\boldsymbol{y}$) with hyperplanes

$$\boldsymbol{v_i \cdot x = \alpha_i y}, \quad 1 \leq i \leq m.$$

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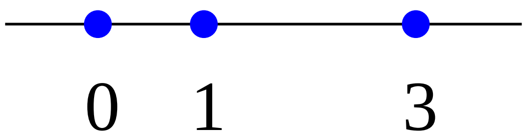
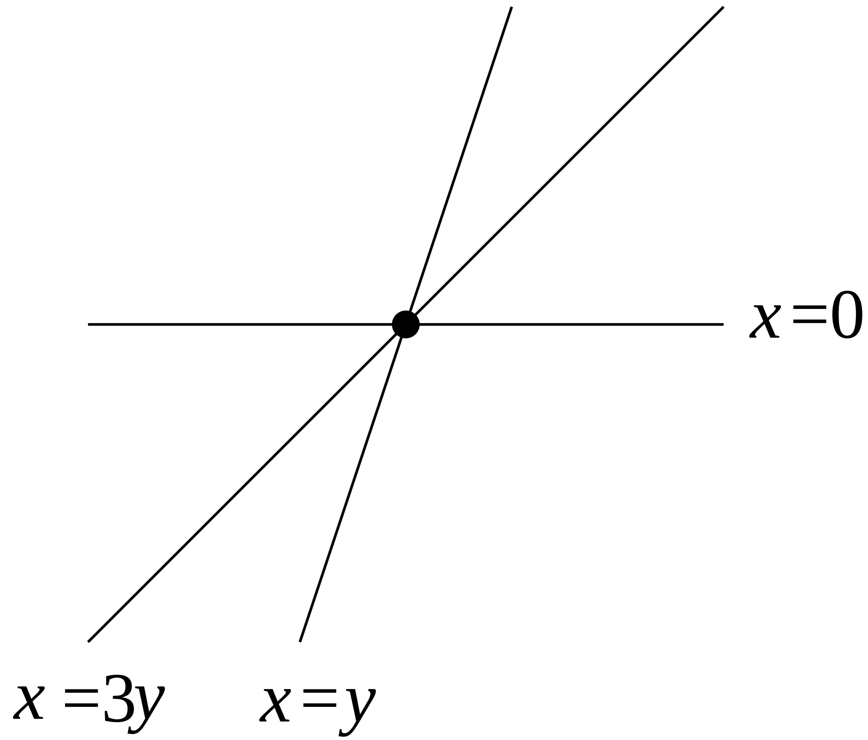
NOTE (for cognoscenti): do not confuse $\mathrm{sc}(\mathcal{A})$ with the **cone** $\boldsymbol{c}(\mathcal{A})$, which has the additional hyperplane $y = 0$.

Example of a semicone



A



sc(A)

Main result

Theorem. *Let p be generic. Then*

$$L_{\mathrm{vo}(\mathcal{A},p)} \cong D_1(L_{\mathrm{sc}(\mathcal{A})}).$$

In particular, when $\mathcal{A} = \mathrm{vis}(\mathcal{P})$ and $p \in \mathrm{int}(\mathcal{P})$ we have

$$L_{\mathrm{ls}(\mathcal{P},p)} \cong D_1(L_{\mathrm{sc}(\mathrm{vis}(\mathcal{P}))}).$$

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Proof omitted here, but straightforward.

The n-cube

Let $\mathcal{P}$ be an n-cube. Can one describe in a reasonable way $L_{\mathrm{ls}(\mathcal{P},p)}$ and/or $\chi_{\mathrm{ls}(\mathcal{P},p)}(q)$?

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Let $\mathcal{P}$ have vertices $(a_1, \ldots, a_n)$, $a_i = 0, 1$. If $p = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, then $\mathrm{ls}(\mathcal{P}, p)$ is isomorphic to the Coxeter arrangement of type B_n, with

$$\begin{aligned} \chi_{\mathrm{ls}(\mathcal{P},p)}(q) &= (q-1)(q-3)\cdots(q-(2n-1)) \\ r(\mathrm{ls}(\mathcal{P},p)) &= 2^n\, n!. \end{aligned}$$

The 3-cube

Let $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then

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Let p be generic. Then

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Total number of line shellings of the 3-cube is 288. Total number of shellings is 480.

Three asides

1. Let $f(n)$ be the total number of shellings of the n-cube. Then

$$\sum_{n\geq 1} f(n)\frac{x^n}{n!} = 1 - \frac{1}{\sum_{n\geq 0}(2n)!\frac{x^n}{n!}}.$$

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2. Total number of line shellings of the n-cube is $2^n n!^2$.

3. **Every** shelling of the n-cube C_n can be realized as a line shelling of a polytope combinatorially equivalent to C_n (**M. Develin**).

Two consequences

- The number of valid orderings from a generic p depends only on $L_{\mathcal{A}}$. In particular, it is independent of the region in which p lies.

Second consequence

- $\boldsymbol{\mathcal{A}}$: an arrangement in $\mathbb{R}^d$ with $\boldsymbol{m}$ hyperplanes

  $\boldsymbol{c(m,k)}$: signless Stirling number of first kind (number of $w \in \mathfrak{S}_m$ with k cycles)

  Then

  $$r(\mathrm{vo}(\mathcal{A}), p) \leq 2(c(m, m-d+1) + c(m, m-d+3)$$
  $$+c(m, m-d+5) + \cdots)$$

  (best possible). Can be achieved by $\mathcal{A} = \mathrm{vis}(\mathcal{P})$.

Non-generic base points

Recall:

$$L_{\mathrm{vo}(\mathcal{A},p)} \cong L_{D_1(\mathcal{A})}.$$

What if p is not generic?

Non-generic base points

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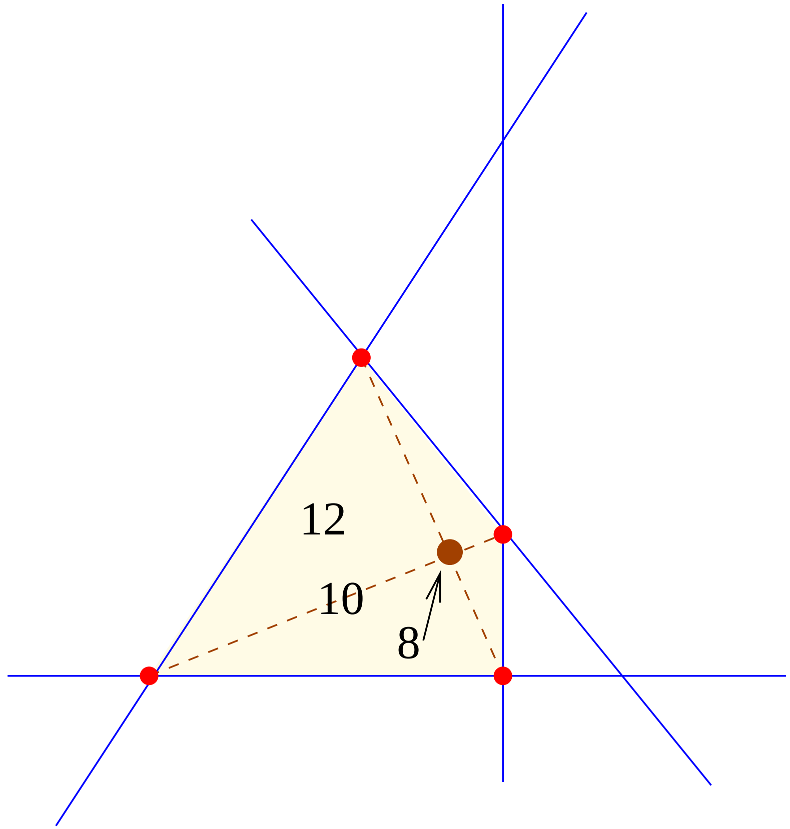
What if p is not generic?

Then we get “smaller” arrangements than the generic case.

We obtain a polyhedral subdivision of $\mathbb{R}^n$ depending on which arrangement corresponds to p.

An example



Numbers are number of line shellings from points in the interior of the face.

An example: order polytopes

$\boldsymbol{P} = \{t_1, \ldots, t_d\}$: a poset (partially ordered set)

Order polytope of P:

$$\mathcal{O}(P) =$$

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$\chi_{\mathrm{vis}(\mathcal{O}(P))}(q)$ can be described in terms of "generalized chromatic polynomials."

Generalized chromatic polynomials

$\boldsymbol{G}$: finite graph with vertex set $\boldsymbol{V}$

$\mathbb{P} = \{1, 2, 3, \ldots\}$

$\boldsymbol{\sigma} : V \to 2^{\mathbb{P}}$ such that $\sigma(v) < \infty$, $\forall v \in V$

$\boldsymbol{\chi_{G,\sigma}(q)}$, $q \in \mathbb{P}$: number of proper colorings $f : V \to \{1, 2, \ldots, q\}$ such that

$$f(v) \notin \sigma(v), \ \forall v \in V$$

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Each f is a **list coloring**, but the definition of $\chi_{G,\sigma}(q)$ seems to be new.

The arrangement $\mathcal{A}_{G,\sigma}$

$\boldsymbol{d} = \#V = \#\{v_1, \ldots, v_d\}$

$\mathcal{A}_{G,\sigma}$: the arrangement in $\mathbb{R}^d$ given by

$$\begin{aligned} x_i &= x_j, \text{ if } v_i v_j \text{ is an edge} \\ x_i &= \alpha_j, \text{if } \alpha_j \in \sigma(i) \end{aligned}$$

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\end{aligned}
$$

Theorem (easy). $\chi_{\mathcal{A}_{G,\sigma}}(q) = \chi_{G,\sigma}(q)$ for $q \gg 0$

Consequences

Since $\chi_{G,\sigma}(q)$ is the characteristic polynomial of a hyperplane arrangement, it has such properties as a **deletion-contraction recurrence**, **broken circuit theorem**, Tutte polynomial, etc.

$\mathrm{vis}(\mathcal{O}(P))$ and $\mathcal{A}_{H,\sigma}$

Theorem (easy). *Let H be the Hasse diagram of P, considered as a graph. Define* $\boldsymbol{\sigma} : H \to \mathbb{P}$ *by*

$$\sigma(v) = \begin{cases} \{1,2\}, & v = \text{ isolated point} \\ \{1\}, & v \text{ minimal, not maximal} \\ \{2\}, & v \text{ maximal, not minimal} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\mathrm{vis}(\mathcal{O}(P)) = \mathcal{A}_{H,\sigma}$.

Rank one posets

Suppose that P has rank at most one (no three-element chains).

$\boldsymbol{H(P)}$ = Hasse diagram of P, with vertex set $\boldsymbol{V}$

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Theorem.

$$v(\mathcal{O}(P)) = (-1)^{\#P} \sum_{W \subseteq V} \chi_{H_W}(-3)$$

Supersolvable and free

Recall that the following three properties are equivalent for the usual graphic arrangement $\mathcal{A}_G$.

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- G is a **chordal** graph, i.e., can order vertices $v_1, \ldots, v_d$ so that v_{i+1} connects to previous vertices along a clique. (Numerous other characterizations.)

Generalize to (G, σ)

Theorem (easy). *Suppose that we can order the vertices of G as $v_1, \ldots, v_p$ such that:*

- v_{i+1} *connects to previous vertices along a clique (so G is chordal).*
- *If $i < j$ and v_i is adjacent to v_j, then $\sigma(v_j) \subseteq \sigma(v_i)$.*

Then $\mathcal{A}_{G,\sigma}$ is supersolvable.

Open questions

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THE END