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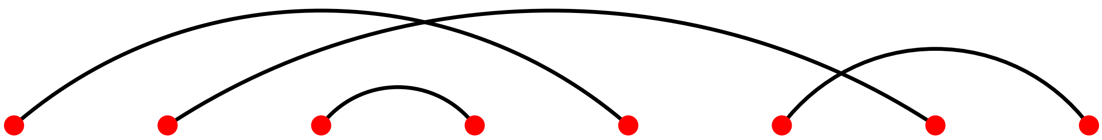
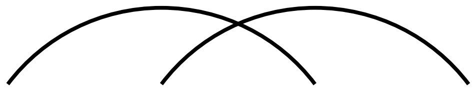
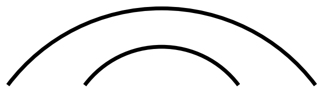
(complete) matching:



crossing: 

nesting: 

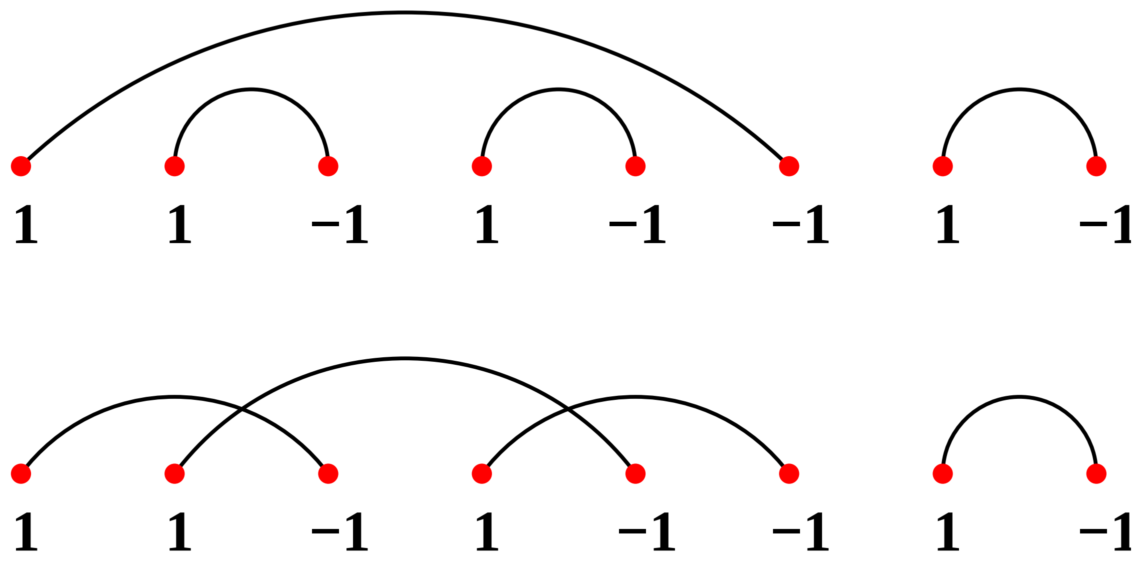
Theorem. *The number of matchings on $[2n]$ with no crossings (or with no nestings) is*

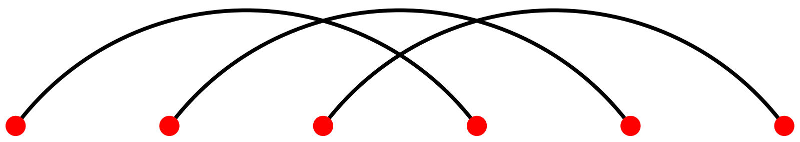
$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Recall:

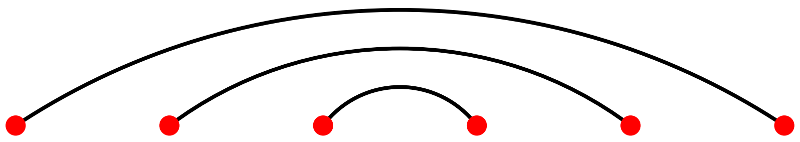
$$C_n = \#\{a_1 \cdots a_{2n} \ : \ a_i = \pm 1,$$
$$a_1 + \cdots + a_i \geq 0, \ \sum a_i = 0\}$$

(**ballot sequence**).





3–crossing



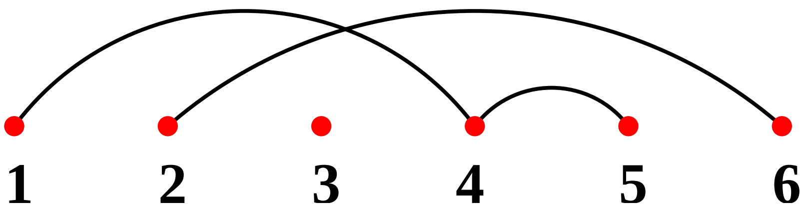
3–nesting

$$\begin{aligned} \boldsymbol{M} &= \text{matching} \\ \mathbf{cr}(M) &= \max\{k : \exists\ k\text{-crossing}\} \\ \mathbf{ne}(M) &= \max\{k : \exists\ k\text{-nesting}\}. \end{aligned}$$

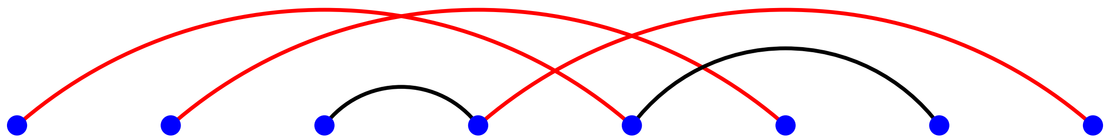
Theorem. *Let* $f_n(i,j) = \#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

Corollary. $\#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = k$ *equals* $\#$ *matchings* M *on* $[2n]$ *with* $\mathrm{ne}(M) = k$.

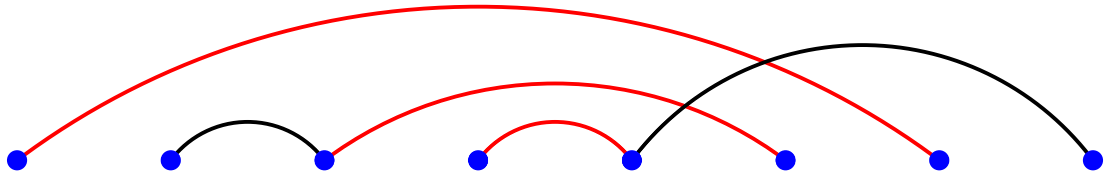
Partitions (of the set $[n]$).



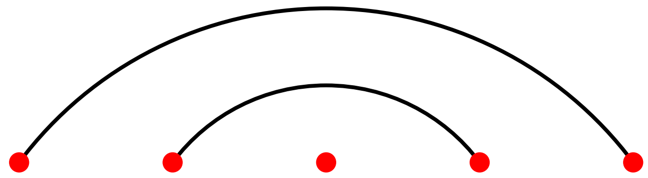
$$\pi = 145 - 26 - 3$$



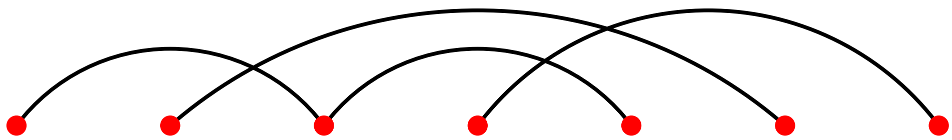
3–crossing



3–nesting



2–nesting



2–crossing

$$\begin{aligned} \boldsymbol{\pi} &= \text{set partition} \\ \mathbf{cr}(\pi) &= \max\{k : \exists\ k\text{-crossing}\} \\ \mathbf{ne}(\pi) &= \max\{k : \exists\ k\text{-nesting}\}. \end{aligned}$$

Theorem. *Let* $g_n(i,j) = \#$ *partitions* π *of* $[n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then*

$$\boldsymbol{g_n(i,j) = g_n(j,i)}.$$

A common generalization. Given $\pi \in \Pi_n$, define:

$$\mathbf{min}(\boldsymbol{\pi}) = \{\text{minimal block elements of } \pi\}$$
$$\mathbf{max}(\boldsymbol{\pi}) = \{\text{maximal block elements of } \pi\}$$

$$\min(135-26-4) = \{1,2,4\}$$
$$\max(135-26-4) = \{4,5,6\}.$$

Note. $(\min(\pi), \max(\pi))$ determines number of blocks of π, number of singleton blocks, whether π is a matching, ...

Fix $S, T \subseteq [n]$, $\#S = \#T$.

$$\boldsymbol{f_{n,S,T}(i,j)} = \#\{\pi \in \Pi_n : \min(\pi) = S, \max(\pi) = T, \operatorname{cr}(\pi) = i, \operatorname{ne}(\pi) = j\}.$$

Theorem. $\boldsymbol{f_{n,S,T}(i,j) = f_{n,S,T}(j,i)}$

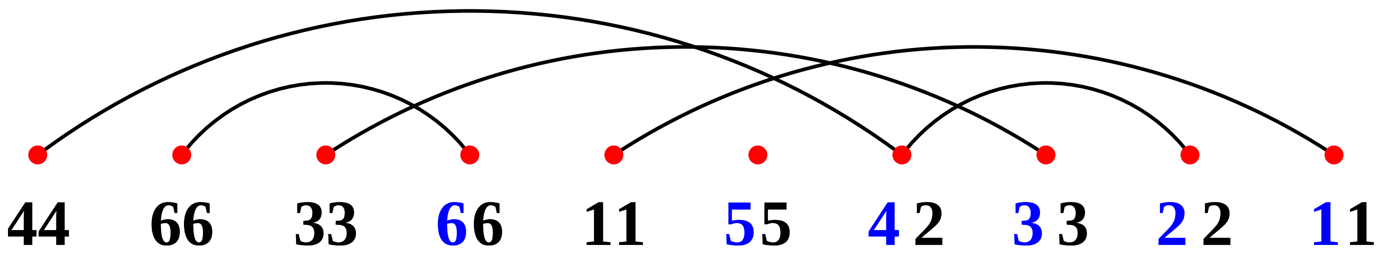
Main tool: vacillating tableaux.

Label points i with a pair $\boldsymbol{a_i b_i}$ from right-to-left.

For arcs or singletons ij with $i \leq j$, $a_j = 1, 2, \ldots, n$ in order from right-to-left.

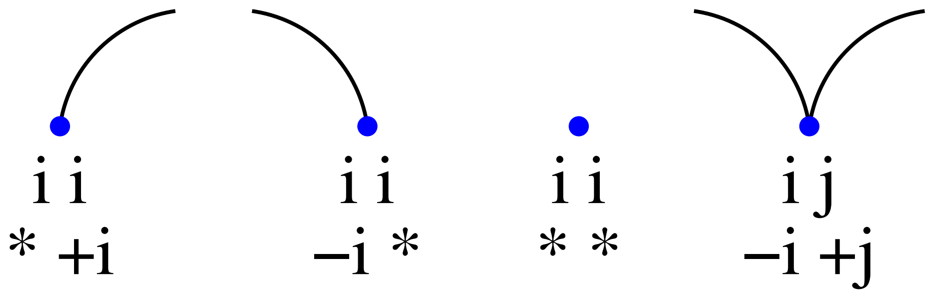
$b_i = a_j$

Otherwise $a_i = b_i$.

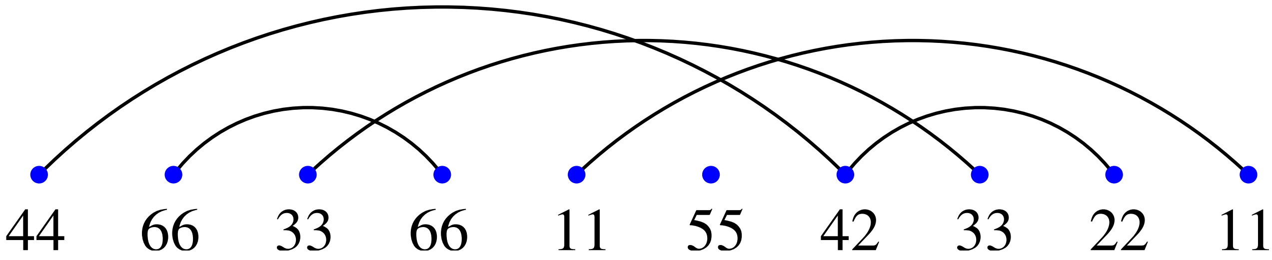


Begin with empty tableaux $\boldsymbol{T_0} = \emptyset$.

Scan numbers $a_1 b_1 a_2 b_2 \cdots a_n b_n$ left-to-right. At each step either RSK-insert, delete, or do nothing:



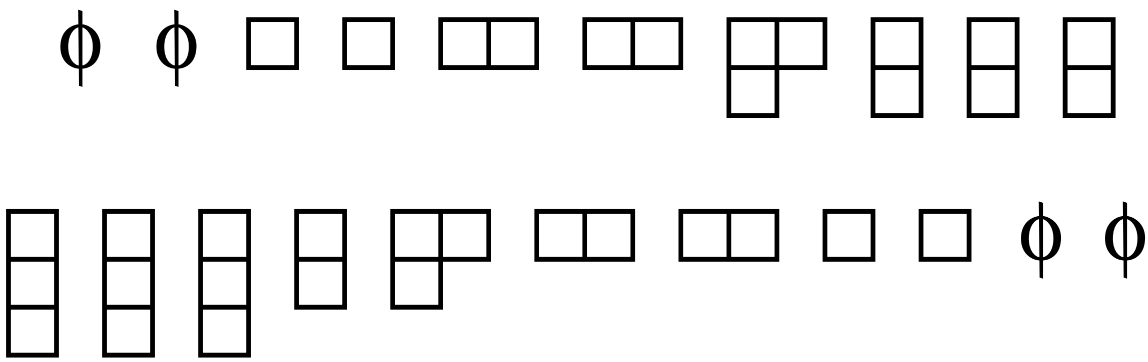
$*$: do nothing



∅	∅	4	4	4 6	4 6	3 6 4	3 4	3 4	3 4

1 3 4	1 3 4	1 3 4	1 3	1 2 3	1 2	1 2	1	1	∅	∅

Remember only the shapes:



This gives a **vacillating tableau** or **gently enhanced Sunday tableau** of length $2n$ and shape $\emptyset$, viz., a sequence

$$(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$$

of shapes such that

- $\lambda^{2i+1} = \lambda^{2i}$ or $\lambda^{2i} - \square$
- $\lambda^{2i} = \lambda^{2i-1}$ or $\lambda^{2i-1} + \square$

(Always $\lambda^1 = \lambda^{2n-1} = \emptyset$.)

Theorem. *The above correspondence is a bijection from partitions of $[n]$ and vacillating tableaux of length $2n$ and shape $\emptyset$.*

Note. Let $\boldsymbol{P(n)}$ be the **partition algebra** (Martin, Doran, Wales, Halverson, Ram, ...), a semisimple $\mathbb{C}$-algebra satisfying

$$\dim P(n) = B(n),$$

the number of partitions of $[n]$ (**Bell number**).

Implicit in theory of $P(n)$: Irreps I_n of $P(n)$ indexed by λ for which there is a vacillating tableaux

$$\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset$$

with $\lambda^n = \lambda$, and $\dim I_n$ is the number of such vacillating tableaux.

$\boldsymbol{U}$ = "add a square" operator

$\boldsymbol{D}$ = "remove a square" operator.

standard Young tableaux: U

oscillating tableaux: $U + D$

vacillating tableaux: $(U + I)(D + I)$

Theorem. *Let* $\pi \in \Pi_n$ *and*

$$\pi \longrightarrow (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset).$$

Then $\mathrm{cr}(\pi)$ *is the most number of rows in any* λ^i*, and* $\mathrm{ne}(\pi)$ *is the most number of columns in any* λ^i*.*

Compare: (*) if $w \in \mathfrak{S}_n$ and

$$w \xrightarrow{\text{RSK}} (P, Q),$$

then the number of columns of P is the length of the longest increasing subsequence of w (easy), and the number of rows of P is the length of the longest decreasing subsequence of w (harder).

In fact, proof of above theorem uses (*).

Corollary to previous theorem:

Theorem. $\boldsymbol{f_{n,S,T}(i,j) = f_{n,S,T}(j,i)}$

Proof. Let

$$\begin{aligned} \pi &\to (\lambda^0, \lambda^1, \dots, \lambda^{2n}) \\ \pi' &\to ((\lambda^0)', (\lambda^1)', \dots, (\lambda^{2n})'). \end{aligned}$$

Then $\mathrm{cr}(\pi) = \mathrm{ne}(\pi')$, $\mathrm{ne}(\pi) = \mathrm{cr}(\pi')$, $S(\pi) = S(\pi')$, $T(\pi) = T(\pi')$, etc. $\square$

Enumeration of k-noncrossing matchings (or nestings).

Recall: The number of matchings M on $[2n]$ with no crossings, i.e., $\mathrm{cr}(M) = 1$, (or with no nestings) is $\boldsymbol{C_n} = \frac{1}{n+1}\binom{2n}{n}$.

What about the number with $\mathrm{cr}(M) \leq k$?

Let $M \rightarrow V$, where V is a vacillating tableau. Remove all steps that do nothing. We obtain an **oscillating tableau**

$$(\emptyset = \mu^0, \mu^1, \ldots, \mu^{2n} = \emptyset)$$

of length $2n$ and shape $\emptyset$, i.e.,

$$\mu^0 = \mu^{2n} = \emptyset, \ \mu^{i+1} = \mu^i \pm \square.$$

This gives a (well-known) bijection between matchings on $[2n]$ and oscillating tableaux of length $2n$ and shape $\emptyset$.

$$\mathrm{cr}(M) \leq k \Leftrightarrow \ell(\mu) \leq k \ \forall i$$

Regard $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{N}^k$.

Corollary. *The number* $\boldsymbol{f_k(n)}$ *of matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) \leq k$ *is the number of lattice paths of length* $2n$ *from* $\mathbf{0}$ *to* $\mathbf{0}$ *in the region*

$$\mathcal{C}_n := \{(a_1, \ldots, a_k) \in \mathbb{N}^k \ : \ a_1 \leq \cdots \leq a_k\}$$

with steps $\pm e_i$ *(*e_i *=* i*th unit coordinate vector).*

$\mathcal{C}_n \otimes \mathbb{R}_{\geq 0}$ is a fundamental chamber for the Weyl group of type B_k.

Grabiner-Magyar: applied **Gessel-Zeilberger reflection principle** to solve this lattice path problem (not knowing connection with matchings).

Theorem. *Define*

$$\boldsymbol{F_k(x)} = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}.$$

Then

$$F_k(x) = \det \left[I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^{k}$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

*(**hyperbolic Bessel function** of the first kind of order m).*

Example. $k = 1$ (noncrossing matchings):

$$F_1(x) = I_0(2x) - I_2(2x) = \sum_{j\geq 0} C_j \frac{x^{2j}}{(2j)!}.$$

Compare:

$$\boldsymbol{u_k(n)} := \#\{w \in \mathfrak{S}_n : \text{longest increasing subsequence of length } \leq k\}.$$

$$\sum_{n\geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det\left[I_{i-j}(2x)\right]_{i,j=1}^{k}.$$

Many similar formulas involving RSK for classical groups.

$$\boldsymbol{g_{j,k}(n)} := \#\{\text{matchings } M \text{ on } [2n],$$
$$\text{cr}(M) \le j,\ \text{ne}(M) \le k\}$$

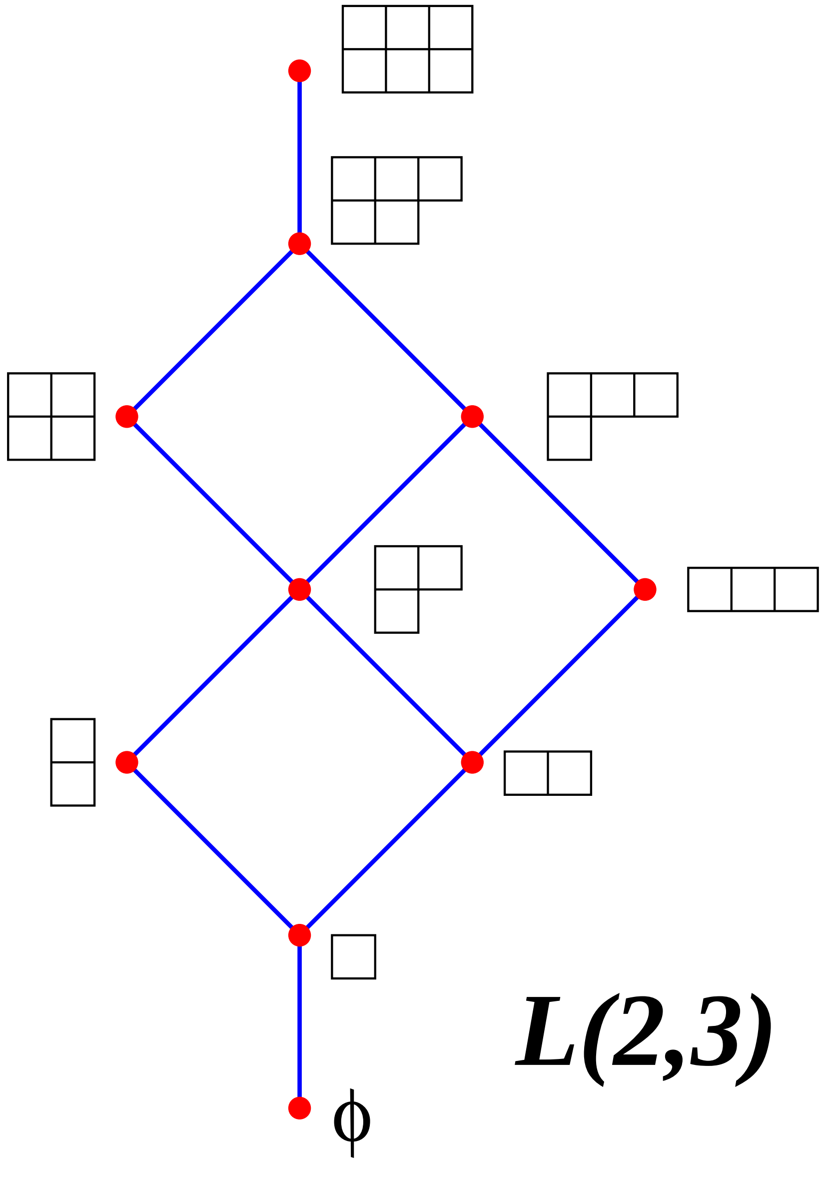
Now

$$g_{j,k}(n) = \#\{(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset) :$$
$$\lambda^{i+1} = \lambda^i \pm \square,\ \lambda^i \subseteq j \times k \text{ rectangle}\},$$

a walk on the Hasse diagram $\boldsymbol{\mathcal{H}(j,k)}$ of

$$\boldsymbol{L(j,k)} := \{\lambda \subseteq j \times k \text{ rectangle}\},$$

ordered by inclusion.


L(2,3)
ϕ

$$\boldsymbol{A} = \text{adjacency matrix of } \mathcal{H}(j,k)$$
$$\boldsymbol{A_0} = \text{adjacency matrix of } \mathcal{H}(j,k) - \{\emptyset\}.$$

Transfer-matrix method $\Rightarrow$

$$\sum_{n\geq 0} g_{j,k}(n)x^{2n} = \frac{\det(I - xA_0)}{\det(I - xA)}.$$

Conjecture. $\det(I - xA)$ factors into polynomials of "small" degree over $\mathbb{Q}$.

Example. $j = 2$, $k = 5$:

$$\det(I - xA) = (1-2x^2)(1-4x^2+2x^4)$$
$$(1-8x^2+8x^4)(1-8x^2+8x^3-2x^4)$$
$$(1-8x^2-8x^3-2x^4)$$

$j = k = 3$:

$$\det(I - xA) = (1-x)(1+x)(1+x-9x^2-x^3)$$
$$(1-x-9x^2+x^3)(1-x-2x^2+x^3)^2$$
$$(1+x-2x^2-x^3)^2$$

Variations. Can modify the insertion-deletion algorithm for vacillating tableaux so that:

- Isolated points can belong to a nesting.
- Arcs touching at their endpoints can be part of a crossing.